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I. S. SOKOLNIKOFF

ANTENNAS

THEORY AND PRACTICE

APPLIED MATHEMATICS SERIES

The Applied Mathematics Series is devoted to books dealing with mathematical theories underlying physical and biological sciences, and with advanced mathematical techniques needed for solving problems of these sciences.

ANTENNAS

THEORY AND PRACTICE

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To our friends

in

The Bell Telephone Laboratories

PREFACE

This book is about antennas: about the physical principles underlying their behavior, the theory needed in sound antenna design and in planning meaningful experiments, the applications of theory to antennas in various frequency ranges. We have written it primarily for students and practitioners of radio engineering. But no specialized engineering background is required for understanding it, and the book should be intelligible to students of applied mathematics and physics and to mathematical consultants in industry. For them this book will illustrate the type of problems of interest to radio engineers, the nature of solutions sought by them, the ways in which applicable mathematics is actually put to work by practical men.

The contents of this book express our idea of what a college textbook on antennas should contain. It is not a book of descriptive type, nor is it concerned with engineering details. Such details are learned best on the job, and college time should not be consumed by studying books that can be read at any time, and with profit, without the instructor's assistance. But a serious study of fundamental ideas and theory requires time and often the help that only college can provide. Good understanding of fundamental theory will enable the student to acquire confidence in himself, and will develop his ability to evaluate what he finds in technical periodicals. Thus he will be prepared to make full and appropriate use of available information. To make the book more usable as a reference book, an extensive index and a list of symbols have been added.

We have not evaded derivations of important formulas and proofs of important conclusions. The student should know them. He has a right to know them. But we have made a serious effort to make them as simple as possible. Anyone who has a working knowledge of general physics and calculus is adequately prepared for this book provided he possesses scientific courage. He should not be intimidated by a partial differential equation simply because he has never had a course in such equations. We do not use vector analysis because in the type of problems with which we are concerned it impedes rather than promotes understanding. We use the vector concept and, in a very few cases, shorthand notations for scalar and vector products; but these notations are normally acquired in studying physics and do not really constitute vector analysis.

The book begins with a chapter introducing various problems which arise in studying antennas and which are examined more thoroughly in subsequent chapters. Concurrently we present physical pictures of radiation and explain how electric phenomena in space are related to those in circuits. Thus we try to close the gap between circuit and field theories which exists when they are developed independently. We show how some important formulas, such as the formula for the distant field of a current element, can be obtained from physical considerations. This should enable the reader to understand a substantial part of antenna theory without more serious analytical study.

A thorough and complete understanding of antenna theory, however, requires a knowledge of Maxwell's laws of electromagnetic interaction. We explain their meaning in simple terms, illustrate them by simple applications, and then translate them into equations particularly convenient in antenna analysis. These equations are immediately used to examine waves guided by cones, wires, and cage structures. From these equations we derive the most fundamental formulas of antenna theory: the formulas for the field of an electric current element from which the field of any given current distribution may be obtained by integration. The traditional method of deriving these formulas employs advanced theorems of vector analysis and auxiliary mathematical functions which have no simple physical significance. This method was devised by mathematicians for mathematicians and is satisfactory only for those readers who are on such familiar terms with mathematics that they hardly notice it. For most students this method is only an exercise in mathematical manipulation which diverts their attention from the essential physical aspects of the problem. And it is easy to lose both mathematical rigor and physical sense in the mire of symbolism. In fact, many current versions of the method prove nothing. For these reasons we have abandoned this classical approach in favor of a simpler and physically more direct derivation of the fundamental formulas for the field of an oscillating charge. We start with the static electric field of the charge. As the charge begins to oscillate slowly, the varying electric field generates a magnetic field. In this way we obtain the principal components of the local field. This field is then joined to the distant wave arising from the interaction of electric and magnetic fields. Given all the details, this method is just as rigorous mathematically as the complete traditional method, and it is more convincing from the physical point of view. It exhibits the mechanism of excitation of electromagnetic waves. We believe that this method has something to offer even to accomplished mathematicians.

We give considerable attention to methods for obtaining the radia-

tion patterns of, and the power radiated by, given current distributions. In nondissipative media the radiated power may be calculated either by the Poynting vector method, based on power flow at large distances, or by the induced electromotive force method, based on the power contributed to the field. We explain both methods and develop the second into the "method of moments" which is particularly useful for approximate calculation of power radiated by complicated current distributions. Then we obtain several radio transmission formulas for free space. In order to make the student "ground-conscious" we consider some effects of ground on radio transmission. But space considerations have not permitted a fuller account of ground effects. Atmospheric influences had to be ignored completely. In view of several recent books on propagation this omission is not serious.

Half a century ago Pocklington demonstrated that the current and charge on thin perfectly conducting wires are propagated approximately with the velocity of light and that between any two points of monochromatic excitation the current distribution is approximately sinusoidal. Much of practical antenna theory has been based on this fundamental result. The author of this approximation was apparently forgotten. Some radio engineers call it "a practical engineering approximation," and some theoreticians once called it "a colossal fraud." Engineering books made no effort to present the theory back of it, and it is not surprising that this approximation was sometimes misused. Slender poles have often been placed on the tops of broadcasting towers to increase their effective heights, apparently without the realization that, although the currents in the towers and the poles are approximately sinusoidal, the sinusoids do not constitute one sinusoid, and that some further theorizing would have indicated that the poles are almost ineffective. The importance of phase in deviations of antenna current from the sinusoidal form has sometimes been overlooked and measured results have been misinterpreted. We devote a full chapter to the sinusoidal approximation, to various factors causing deviations from it, and to the relative importance of these deviations.

At radio transmitting and receiving terminals antennas are circuit elements. In the case of arrays they are coupled circuits. The actual values of antenna impedances and mutual impedances can be either measured or calculated by solving Maxwell's equations. But some properties such as resonance, antiresonance, reciprocity, and circuit equivalence are properties of linear dynamical systems and are independent of Maxwell's equations. Circuit equivalence theorems enable one to fit antennas into circuit diagrams. Reciprocity theorems make it unnecessary to analyze both the transmitting and the receiving

properties — one analysis is sufficient. We give considerable attention to these topics.

To illustrate the application of fundamental theory we discuss all types of antennas: "short" or long-wave antennas, half-wave and full-wave antennas, general dipole antennas, rhombic antennas, slot antennas, horns, reflectors, and lenses.

In this book we present for the first time an elementary theory of dipole antennas, based on the "sweeping off" process. We start with an approximate antenna current, determine the electric intensity tangential to the antenna, and try to sweep it off the antenna by applying equal and opposite electric intensity. We also discuss some of the results of a more advanced linear antenna theory developed by one of us; but its details as well as Hallén's method of antenna analysis are published in a separate book.

The number of papers on antennas is so great that a complete bibliography would have added perceptibly to the size and cost of this book. We have thus selected a reasonable number of references which may be consulted with profit for further information on special topics. Some of these references are given in footnotes, others at the ends of chapters. Partly to provide ready accessibility to the greatest number of students and partly because we found it difficult to undertake the task of scanning world-wide literature on antennas, most of these references are to papers published in American periodicals. We hope that our lists are adequate for beginners.

We have prepared a large number of problems; but we have left it to the instructor to make his own numerical exercises. The preparation of the latter will take little of his time, and we feel that he is the best judge of the extent to which his students need them.

Theoretically this book is suitable as a text for an undergraduate course on antennas since calculus is the only essential mathematical prerequisite, but in practice many undergraduate students will probably find some chapters too difficult. A fairly easy and short undergraduate course can be made of the following chapters: 1, 5, 6, 11, 15. This course can be augmented by adding Chapters 2, 3, and 7. In both options the student should be asked to accept equation 4-82 without derivation. This equation, however, is used only in the last few sections of Chapter 5 and in some sections of Chapter 6. These few sections in Chapter 5 can easily be omitted since they deal with an alternate method of computing the radiated power, and the sections in Chapter 6 can be replaced by the instructor with his own analysis, based on the methods presented in the early part of Chapter 5. Other options will readily occur to the instructor after he has examined the book. A graduate

course can begin with Chapter 2, or even with Chapter 4. Chapter 1 can then be assigned as collateral reading, and later in the course be a subject of a seminar.

We are very grateful to Miss Marion C. Gray who has checked the text and equations in the original manuscript and in proof, and who has prepared the extensive index. We thank Miss Dorothy T. Angell for her assistance in proofreading. To Mr. B. A. Clarke the credit should go for advice in connection with illustrations and to Mr. H. P. Gridley for the drawings.

S.A.S.
H.T.F.

New York, December 1951

SYMBOLS USED IN TEXT

(References are to page numbers)

a	linear, radius of wire antenna
b	linear
d	linear
f	frequency
g	conductivity
	directive gain, directivity, 33
h	linear, height of vertical antenna
	depth of penetration, 86
j	imaginary unit
k	constant introduced in field equations on p. 19 (evaluated p. 44)
	form factor, 193
	normalized impedance, 92
	polarization loss factor, 392
	separation constant, 113
l	linear, length of one arm of center-fed antenna
n	index of refraction, 204, 206
	polarization vector, 392
	standing wave ratio, 93
p	moment of current element
	oscillation constant, 271
q	charge
	(with subscript) reflection coefficient, 92
r	linear
(r, θ, φ)	spherical coordinates
s	linear, length of gap
t	time
v	velocity, phase velocity
w	linear
(x, y, z)	Cartesian coordinates
A	area
	effective area, 35
	magnetic vector potential, 221
A, B, S, T	functions used in defining radiation influence coefficients, 163
B	magnetic displacement density
	susceptance
C	capacitance; distributed capacitance
	Euler's constant, 235
D	electric displacement density
E	electric field intensity
$\mathcal{E}$	work, energy
F	complex space factor, 146
	dynamic component of electric intensity, 218
	force
G	conductance, distributed conductance
	directivity in decibels, 180
	quasistatic component of electric intensity, 218
H	magnetic field intensity
I	electric current
J	electric current density

K	characteristic impedance, wave impedance
	(with subscripts) radiation influence coefficients, 162
L	inductance, distributed inductance
	radiation vector of magnetic current distribution, 542
M	magnetic current density
M, N	correction functions in impedance formulas for nonconical antennas, 431
N	radiation vector of electric current distribution, 362
P	power, power flow, radiated power
Q	quality factor, 206, 287
R	resistance, radiation resistance, surface resistance
	radial wave function, 111
$\mathcal{R}$	intrinsic resistance
S	area
	space factor, 33
T	period
	free-space transmission factor, 366
	terminating resistance loss, 466
U	magnetomotive force
V	electromotive force, potential, voltage
	quasistatic potential, 216
W	power flow per unit area
	Poynting vector, 22
X	reactance
$\mathcal{X}$	intrinsic reactance
Y	admittance, shunt admittance, transfer admittance
Z	impedance, series impedance
α	attenuation constant
β	phase constant
δ	linear, effective lengthening of antenna
ε	dielectric constant
η	intrinsic impedance, 19
λ	wavelength
μ	permeability
ξ	phase angle of array, 145
	real part of oscillation constant, 271
(ρ, φ, z)	cylindrical coordinates
σ	intrinsic propagation constant, 83
τ	volume
φ	phase angle
χ	polarizability, 576
ψ	cone angle, angle between two directions
	radial wave function, 222
ω	angular velocity
	imaginary part of oscillation constant, 271
Γ	propagation constant, 91
Δ	angle of elevation, 202
	bandwidth, 278
Θ	angular wave function, 111
Φ	magnetic displacement, magnetic flux
	radiation intensity, 141
Ψ	complex Poynting vector, complex power, 78
Ω	solid angle, 79
	impedance parameter in Hallén's theory, 448

CONTENTS

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APPENDIXES

1

PHYSICAL PRINCIPLES OF RADIATION

1.1 Radio communication

Electric communication is transmission of speech, music, pictures, and other information by means of electric signals. Across short distances speech and music are transmitted directly from their source to the listeners by means of acoustic waves. A speaker or a musical instrument causes small variations in air pressure and small, coherent displacements of air particles. This disturbance of the air is propagated away from its source and reaches the ear, the receiving instrument of the listener. The acoustic disturbance spreads in all directions and becomes attenuated with increasing distance. The range of transmission of speech and music may be increased as follows. The listener within easy reach of the speaker or the musical instrument is replaced by a microphone, a device by which the variations in air pressure are translated into corresponding variations in electric current. The electric signals are then transmitted to the desired place where they are reconverted into air vibrations within easy reach of the listener. In *wire communication* the signals are transmitted with the aid of wires connecting the sending and receiving terminal equipments. *Radio communication* is transmission of electric signals *without* such connecting wires. Radio communication is theoretically possible because electric charges exert forces on other charges no matter how distant. Variable electric currents in electric circuits at the sending end act on " free electrons " (that is, easily movable electrons) in electric circuits at the receiving end, thereby producing variations in electric current corresponding to those at the sending end. Radio communication is made practicable by the designing of special circuits, *antennas*, which are particularly effective in creating sufficiently strong electric forces at large distances and, reciprocally, are particularly sensitive to the electric forces impressed on them

externally. Even between these special circuits the coupling is so small that the signals have to be amplified at the sending and receiving terminals; but it would be almost hopeless to attempt radio communication without such circuits.

As ordinary circuits are modified to make them effective antennas, their theory loses much of its original simplicity. New physical ideas are needed for understanding the behavior of the modified circuits and new mathematical methods for their quantitative analysis. Much mathematical detail appears to be unavoidable, even though in some cases the final conclusions are fairly simple. In this chapter we introduce various problems which arise in connection with antennas and which are considered in detail in subsequent chapters. We also present some physical ideas which explain the behavior of antennas in a general way and which we can keep in mind during the subsequent mathematical development. We shall even obtain some quantitative answers which will later be confirmed by rigorous analysis. But the main purpose of this chapter is to present a broad view of the entire subject before attacking seriously the various aspects of the electric behavior of antennas.

1.2 From circuits to fields

There is a major difference between the transmission media employed in wire and those in radio communication which affects our ways of think-

Fig. 1.1 A two-wire transmission line connecting the transmitting and receiving terminal equipment.

ing about them. In wire communication the signals are transmitted via transmission lines consisting of either parallel conductors external to each other (Fig. 1.1) or coaxial conductors, which directly connect the transmitting and receiving terminals. The transmission lines remind us of ordinary circuits in all respects except their length. At any point along the line we can break into the circuit and measure either the current in each conductor or the voltage between the conductors. The difference between the ordinary circuits at the terminals of the wire transmission system and the long circuit connecting them is in the distribution of capacitance and inductance. In a local circuit the capacitance is confined primarily to specially designed capacitors and the

inductance to specially designed coils. Except at high frequencies the capacitance and inductance of the leads connecting the circuit elements are usually negligible. But in a long circuit we have nothing but the "connecting leads." Since these leads are long, we have to take their capacitance and inductance into consideration. For the same reason we should take into account the distributed character of these circuit parameters. The current needed to charge those sections of the line that are near the transmitting end alters the current in the more distant sections. The voltage drop across the inductance of the near-by sec-

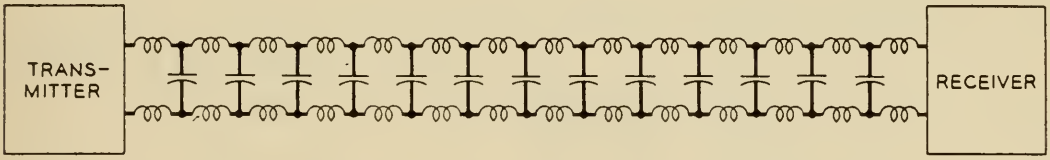


FIG. 1.2 A long network of infinitesimal inductors and capacitors having the electrical properties of a two-wire transmission line.

tions affects the voltage which appears across the distant sections. It is useful to visualize long circuits as shown in Fig. 1.2. Thus no new concepts are needed for understanding long circuits; ordinary circuit concepts are adequate.

On the other hand, in radio communication the transmission medium is free space, where we have no terminals to which we might connect voltmeters and ammeters in order to study the electric effects. What we know from basic physical experiments is that electric charges and magnets are subject to forces when brought into the vicinity of charged and current-bearing conductors. The customary way of expressing this fact is to say that an electric charge is surrounded by an electric field of force and that an electric current is surrounded by a magnetic field of force. For a quantitative study we need practical means of measuring the strength or the *intensity* of the electric and magnetic fields. Since an electric charge placed at a given point in an electric field is subject to a force proportional to the charge, it is natural to define the electric intensity, usually denoted by the letter E, as the force per unit charge. In the ratio of the force to the test charge, the magnitude of the charge is eliminated, and we have a measure of a property of the field. The direct measurement of the force per unit charge is not very practical, particularly in a field that varies with time as during the transmission of electric signals. Besides, in radio communication we are not interested in these forces as such; we are interested in their effect on electric circuits. It would be convenient if we could measure the electric intensity with a voltmeter. To do this we provide

the free space with a pair of terminals by inserting a probe consisting of two wires, each of length l (Fig. 1.3). To these terminals we can now connect a voltmeter. It is easy to see that there will be a voltage across these terminals. Metals contain free electrons, that is, electrons easily detachable from atoms, which can thus be displaced by weak forces. Hence, the free electrons will be displaced in the probe, and we should have an induced voltage. The magnitude of this voltage will depend on the intensity of the field and on the probe. To determine the relative distribution of the electric intensity we merely move the probe from

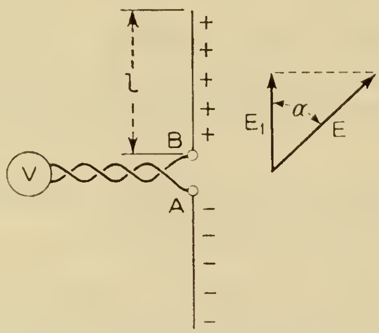


FIG. 1.3 A probe for measuring electric intensity.

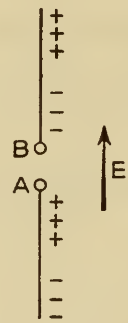


FIG. 1.4 A probe with floating terminals in an electric field.

place to place and record the voltages. We shall find that even at the same place the induced voltage depends on the direction of the probe. This is as it should be, for, if the probe is perpendicular to the force exerted by the field on a typical free electron in the probe, this electron is merely drawn to the surface of the probe. Only the component of the force tangential to the probe moves the electron from one arm to the other and thus gives rise to the voltage between the arms.

Of course, before the voltmeter is connected to the probe, the electrons cannot pass from one arm to the other; they are merely displaced between the ends of each arm* (Fig. 1.4). Thus there will be a voltage between A and B.

To determine the absolute electric intensity, we should eliminate the properties of the probe. This can be done by calibrating the probe, that is, by observing its indication in a field of known intensity. For instance, the field between two infinite parallel plates, with equal charges of opposite sign, is uniform and calculable from the voltage V_0 between the plates (Fig. 1.5). If the distance between the plates is h,

* In Figs. 1.3, 1.4, and 1.5 the charges are shown merely to indicate a possible instantaneous distribution; it is assumed that the field is alternating.

then the voltage per unit length, that is, the intensity E_0, is V_0/h. If the voltage induced in the probe is V_1, then each induced volt represents the intensity V_0/V_1h volts per meter (where h is measured in meters).

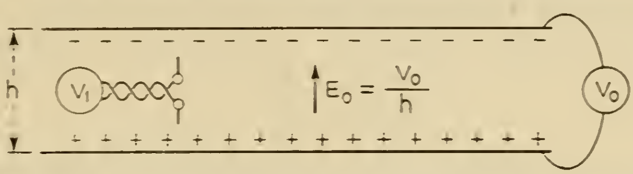


FIG. 1.5 A cross section of a parallel-plate capacitor.

If at a given point of another electric field the induced voltage is V, the electric intensity at that point is

$$E = \frac{VV_0}{V_1h}. \tag{1}$$

Observation shows that the voltage induced in a thin probe is substantially independent of the radius and proportional to the length of the probe. In fact, in the field between the parallel plates we find

$$V_1 = \frac{V_0l}{h}. \tag{2}$$

Substituting in equation 1, we have

$$E = \frac{V}{l}. \tag{3}$$

Theory discloses that this equation is exactly true only when the probe is infinitely thin. We should also bear in mind that all these equations give only the average intensity at points occupied by the probe; hence, l should be as small as possible.

The probe for measuring electric intensity is the simplest example of a receiving antenna. The name *antenna* was probably suggested by the resemblance of the thin arms of the probe to the feelers or antennae of an insect; at present, however, this name is applied to all circuits designed for effective long-distance coupling.

An electric current is said to be surrounded by a magnetic field of force because a magnet brought into the vicinity of the current is subject to a torque. Opposite forces act on the ends of the magnet. These forces are proportional to the current. The magnetic field between two parallel gratings consisting of wires carrying equal and opposite currents (Fig. 1.6) is more or less uniform if the width l of the gratings is large compared with their separation h, and if the distance between the wires is small. Between two infinite parallel plates carrying equal and opposite steady currents the field is exactly uniform. When we explore the

intensity of the field with a test magnet, we find that, if the current per unit length perpendicular to the lines of flow I/l is kept constant, the intensity is independent of the separation between the gratings and of the number of wires. This enables us to use the ratio

$$H = \frac{I}{l} \tag{4}$$

as a measure of the *intensity of the magnetic field* between the gratings. The magnetic intensity at any other place may then be obtained by

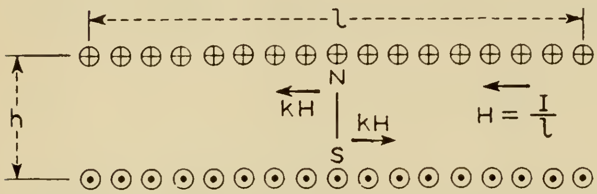


Fig. 1.6 Gratings of parallel wires carrying equal and opposite currents.

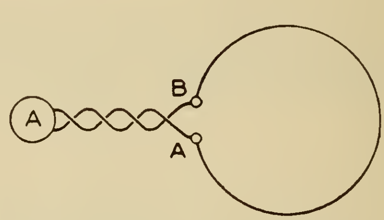


Fig. 1.7 A loop for measuring magnetic intensity.

comparing the torques on the test magnet. In an alternating field it is more practicable to measure the current in a test loop (Fig. 1.7) or a test coil of several turns.

It appears that at a point in free space the electric intensity E and the magnetic intensity H take the place of the voltage and current at the terminals of a circuit. The physical dimensions of E and H are those of voltage and current per unit length. When we wish to refer to either E or H, we shall employ the more general term *field intensity.**

In the above discussion of field measurements we assumed that a voltmeter was connected to the test probe and an ammeter to the test loop. What would happen if we connected an ammeter to the probe and a voltmeter to the loop? We would have to use more sensitive instruments since the effects would be smaller. We would also discover a difference between free space and other homogeneous media, such as water. We would observe the same relative readings as long as they were made in the same medium; but certain factors of proportionality would appear if the media were changed. In this way we would discover the dielectric constant ε and the permeability μ of a medium, and two new field quantities $D = \varepsilon E$ and $B = \mu H$; we would learn to associate these physical constants of homogeneous media with the corresponding capacitance and inductance in circuits. Thus we would gradually dis-

* The *intensity of a wave* is usually defined as the square of the field intensity. Another term, the *radiation intensity*, is defined as the power radiated per unit solid angle (see Chapter 5).

cover a greater similarity between the electric effects in an infinite medium and those in ordinary circuits than at first appeared to be possible.

The above discussion shows the importance to radio communication of formulating a field theory. The next chapter is devoted to this problem.

1.3 Maxwell's equations

The next major problem is to develop equations for calculating field intensities. For static electric fields the intensity may be calculated from Coulomb's law of force between two point charges q_1, q_2,

$$F = \frac{q_1 q_2}{4\pi\varepsilon r^2}, \tag{5}$$

where r is the distance between the charges and ε is the dielectric constant of the medium. Thus the intensity of the field produced by q_1 is radial, and its magnitude is

$$E_1 = \frac{q_1}{4\pi\varepsilon r^2}. \tag{6}$$

Any given charge distribution may be subdivided into volume elements, and the intensity of the field produced by the aggregate may be calculated by adding vectorially the intensities of the fields created by the individual elements. Similarly there are rules for calculating static magnetic fields from the currents creating them. There is no interaction between static electric and magnetic fields; one can exist without the other. On the other hand, electric and magnetic fields that vary with time interact, and we have electromagnetic fields. In order to calculate such fields we have to know the charges and currents producing the fields and the laws of interaction as well. In their final form the laws of interaction were formulated by Maxwell and are known as Maxwell's equations.

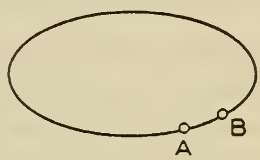


FIG. 1.8 A closed curve.

These equations form the cornerstone of field theory and play the role of Kirchhoff's equations in circuit theory. Maxwell's equations may be stated as follows. Imagine a closed curve (Fig. 1.8) in an electromagnetic field and a surface of which this curve is the edge. Let $E_{\tan}$ be the component of the electric intensity tangential to a typical element of the curve. We shall call the product $E_{\tan}\,\Delta s$, where Δs is the length of the element, the voltage along the element. Adding these voltages around the curve and dividing the total voltage by the length l of the curve we obtain the average tangential in-

tensity $E_{\text{tan}}^{\text{av}}$. The total voltage around the curve can thus be expressed as the product $E_{\text{tan}}^{\text{av}}\, l$. Similarly we divide the surface of area S into elements, and obtain the average normal component $H_{\text{nor}}^{\text{av}}$ of the magnetic intensity. One of Maxwell's equations may now be written as follows:

$$E_{\text{tan}}^{\text{av}}\, l = \mu \frac{\partial H_{\text{nor}}^{\text{av}}}{\partial t} S, \tag{7}$$

where t is time and the coefficient of proportionality, μ, is the permeability of the medium. The other equation is

$$H_{\text{tan}}^{\text{av}}\, l = \varepsilon \frac{\partial E_{\text{nor}}^{\text{av}}}{\partial t} S + I, \tag{8}$$

where ε is the dielectric constant and I is the current crossing the surface. In the equations above we have ignored the relative directions of the normal to the surface and the tangent to its edge. When these are agreed upon, one of the equations will have a negative sign. The usual convention is such that the negative sign appears in equation 7.

These equations can sometimes be used as they stand; but they become much more useful if expressed in terms of the Cartesian, cylindrical, or spherical components of E and H. In each coordinate system we obtain six partial differential equations connecting the set of three E components with the set of three H components. It should be stressed that equations 7 and 8 are exact and that they apply to any *closed* curve, either large or small, lying either in free space or partly or wholly on the surface of a conductor. From these equations we can derive Kirchhoff's equations for electric circuits by imposing suitable restrictions on the size and geometry of the circuits and on the time rate of change of the electromagnetic field.

1.4 Lines of force

Even though Maxwell's equations can be stated rather simply, they cannot be solved easily. They are very general. They must have solutions applicable to electric circuits, wire transmission lines, wave guides, antennas, waves in the medium between antennas, radar reflection, etc.; hence they must possess widely different solutions: some simple and others exceedingly complex. It is much easier to discover some of their simple solutions and then find physical problems to which these solutions supply an answer than to start with a physical problem and solve it in a straightforward manner. Any aid to our thinking about electromagnetic fields will prove to be very helpful. The con-

cept of *lines of force* is particularly useful.* An electric line of force is a line tangential to E at every point through which it passes. Similarly a magnetic line of force is tangential to H.

Electric lines may start on a positive charge and end on a negative charge; they may start on a positive charge and go to infinity or come from infinity and end on a negative charge; they may be closed curves neither starting nor ending on any charge; but they may not just start at some point in the medium where there is no charge or end at a point with no charge. The density of the lines of force indicates the relative magnitude of the electric intensity. Consider, for example, a uniformly charged metal sphere. The lines of force are radial. They spread out as the distance from the charged sphere increases. If we imagine A lines emerging from the sphere, the number of lines per unit area at distance r from the center is $A/4\pi r^2$. From Coulomb's law we found that the intensity of the field is given by equation 6; hence, it is proportional to the density of the lines of force. In the case of two equal and opposite charges all the lines emerging from the positive charge must end on the negative charge (Fig. 1.9). Hence, the number of lines crossing the upper hemisphere of radius r must decrease as r increases, and more and more lines cross over below the equatorial plane. Hence, the electric intensity decreases more rapidly than $1/r^2$.

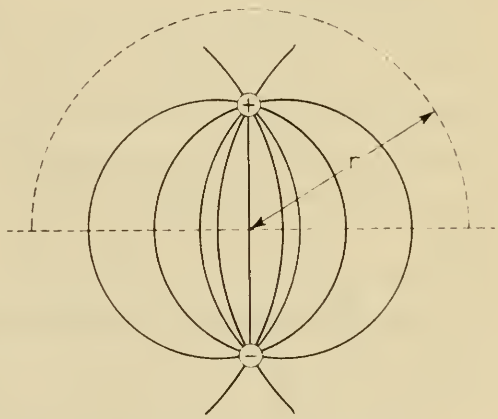


FIG. 1.9 Electric lines of force in the field around equal and opposite charges.

Magnetic lines are always closed because there are no magnetic charges. For theoretical purposes it is sometimes convenient to consider hypothetical magnetic charges, in which case the magnetic lines behave just as the electric lines do.

Elementary books on electricity and magnetism display prominently electric lines beginning and ending on charges or at infinity, as well as magnetic lines going round conductors carrying electric currents, and we shall not consider them further. But closed electric lines and magnetic lines that do not surround conductors may be a novelty to the reader. What are the conditions for their existence? Let us consider a closed electric line. We may apply equation 7 to it. Since E is

* We are assuming a homogeneous dielectric, the case of interest to the radio engineer. Otherwise, we should consider " tubes of displacement."

tangential to the closed curve, the left-hand side may not vanish. Therefore, the right-hand side does not vanish, and the number of magnetic lines crossing a surface bounded by the electric line must vary with time. Closed electric lines must thus be linked with magnetic lines. Similarly from equation 8 we conclude that magnetic lines must be linked either with electric current or with electric lines or with both. As we shall see in Section 1.8, radiation of energy into free space depends on the existence of closed electric lines.

1.5 Waves

Leonardo da Vinci gave what seems to us the best characterization of waves: "The impetus is much quicker than the water, for it often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place."* The initial disturbance creating a wave is often localized, and the impetus travels from it in straight lines or radii; or we might say that the impetus radiates from the focus of the disturbance as the spokes of a wheel from its hub — hence the term *radiation.* The energy carried by a wave is called radiant energy.

That the radiant energy leaves its source is easy to understand, when the initial disturbance is of short duration. The disturbance is gone but the impetus created by it travels outwards; the energy associated with this traveling wavelet moves farther and farther from its source unless the wavelet encounters an obstacle which may send it back or "reflect" it.

To understand the mechanism of wave propagation let us consider a mass attached to a spring (Fig. 1.10*a*). If we pull the mass to the left and let it go, oscillations take place. At first the stretched spring pulls the mass to the right and thus accelerates it. Then the moving mass compresses the spring and gradually produces an opposite force eventually reversing the motion of the mass. The pull or the push exerted by the spring on the point of attachment varies with the phase of the motion; but, if the wall is rigid, this varying force produces no visible effect. Next let us consider a chain of masses and springs (Fig. 1.10*b*). As before, a leftward pull on the mass at the left will start accelerating it. Since the first spring is initially unstretched, it exerts no opposite pull on the moving mass and no pull to the left on the second mass. The full force applied to the first mass is not immediately transmitted to the second mass. There is a delay in the transmission of

* From a lecture by T. Levi-Civita on "What are Waves?", *Rice Institute Pamphlets*, **24**, October 1938, pp. 168–205.

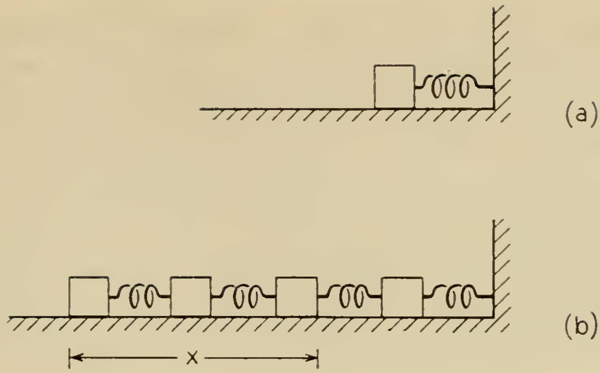


FIG. 1.10 (*a*) A mass attached to a rigid wall by a spring; (*b*) a chain of masses and springs.

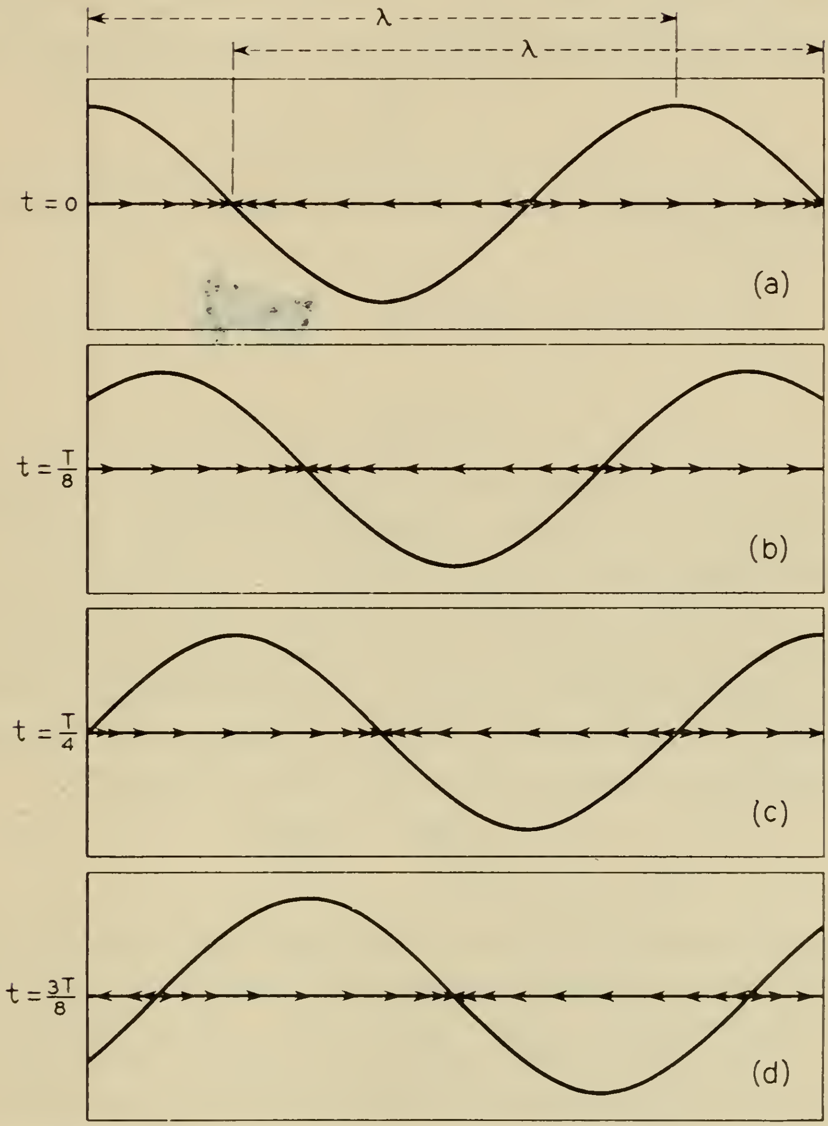


FIG. 1.11 Distribution of motion along a long chain of masses and springs at various times showing the movement of the wave from left to right.

force and movement along the chain. Hence, if the mass at the left is forced to move sinusoidally with time, the other masses also move sinusoidally, but with a phase delay increasing with the distance from the origin of motion. In Fig. 1.11 the arrows show the distribution of motion in a long chain at four instants one eighth of a period apart. If all masses and springs are identical we expect the *wave profile*, that is, the distribution of motion along the chain, to be sinusoidal.* Inequalities in masses and springs cause a deformation in the profile; but at present we are not interested in the fine quantitative details of wave propagation.

What we observe in Fig. 1.11 is the movement of the wave profile from left to right. The distance from crest to crest is called the *wavelength* and is denoted by λ. The wave moves through this distance in time T equal to the period of oscillations; hence, the wave velocity is

$$v = \frac{\lambda}{T}. \tag{9}$$

The frequency f in cycles per second is the number of complete oscillations in one second,

$$f = \frac{1}{T}. \tag{10}$$

Hence,

$$v = \lambda f. \tag{11}$$

If the wave velocity is independent of the frequency, the wavelength and the frequency vary inversely. Low-frequency waves are *long waves*, and high-frequency waves are *short waves*.

At $x = 0$ the phase of the wave is $2\pi t/T$. The phase is delayed by 2π radians in one wavelength or by $2\pi/\lambda$ radians per unit length; hence, the phase delay in distance x is $2\pi x/\lambda$. The actual phase at distance x from the source of oscillations is $(2\pi t/T) - (2\pi x/\lambda)$, and the magnitude of the oscillations is proportional to the cosine of the phase,

$$\psi \propto \cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right). \tag{12}$$

When the motion reaches the fixed end of the chain, the reaction of the wall starts a backward wave. In the steady state this reflected wave is superimposed on the original wave, and *standing waves* are formed. When two sinusoidal profiles are added, the new profile is also sinusoidal; but the profile must be such that there is no motion at the

* Except when the oscillations are so fast that the full force on one mass is *never* transmitted to the next, and then the motion must be attenuated along the chain.

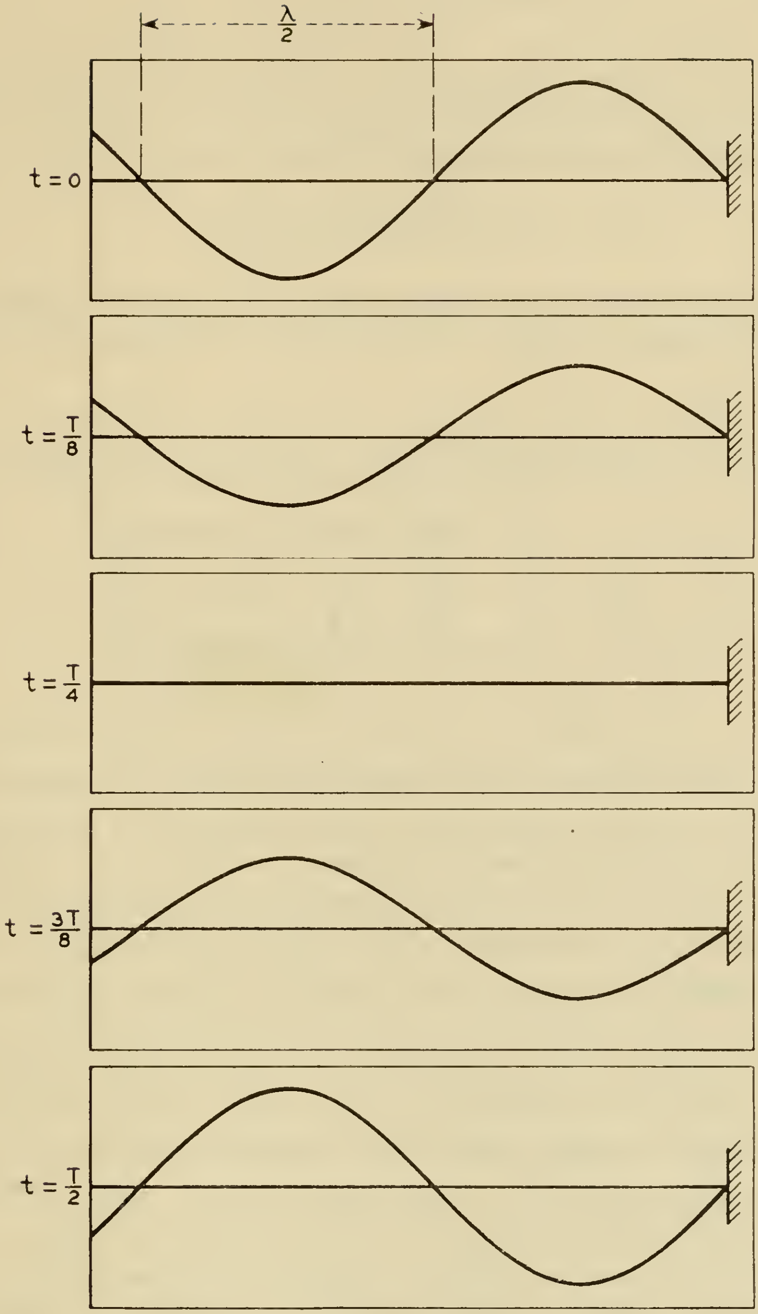


FIG. 1.12 Stationary waves.

fixed end $x = l$ at any time. Therefore, the magnitude of the oscillations must be proportional to a sine function of the form

$$\psi \propto \sin \frac{2\pi(l - x)}{\lambda}. \tag{13}$$

We cannot have a constant multiple of t between the parentheses since that would make it impossible for ψ to be zero at $x = l$ at *all* times. Since ψ is still varying sinusoidally with time, we must have

$$\psi \propto \sin \frac{2\pi(l - x)}{\lambda} \cos \frac{2\pi t}{T}. \tag{14}$$

Hence, the time factor affects the motion of all points equally. Figure 1.12 shows the distribution of the motion at five instants one eighth of the period apart.

Electric oscillations in simple circuits (Fig. 1.13*a*) and their propagation in long circuits consisting of a succession of simple circuits

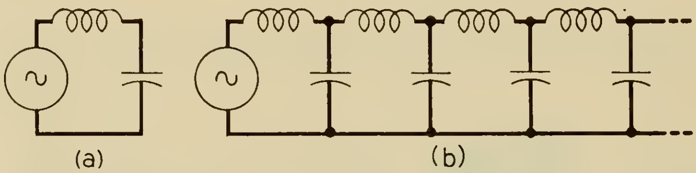


FIG. 1.13 (*a*) An electric circuit and (*b*) a chain of electric circuits.

(Fig. 1.13*b*) are analogous to mechanical oscillations and waves. If an electric charge is placed on one plate of the capacitor in the simple circuit, a voltage appears across the capacitor *and the coil.* Hence, an electric current must flow. This current increases gradually until half the charge has been transferred to the other plate. At that moment the driving voltage disappears, but the current through the coil cannot stop suddenly. It should take just as much time to stop this current as was required to bring it to a maximum; in this time the entire charge from the first plate is transferred to the second, and the sequence of movements begins to repeat itself in the opposite direction. A voltage applied at the beginning of the long chain of circuits (Fig. 1.13*b*) forces an electric current through the first coil and starts to charge the first capacitor. It takes time for the full voltage to appear across the first capacitor. Hence, there is a delay in the transmission of voltage and current in a long circuit; that is, the voltage and current are propagated with a finite velocity.

1.6 Electric waves on wires and in free space

Parallel conductors may be considered as chains of infinitely small inductors and capacitors in tandem (Fig. 1.2). We thus expect a delay

in the transmission of voltage across the conductors and in the accumulation of charge density on one or the other conductor and of electric current in either conductor. Electric lines of force help to exhibit the distribution of charge (Fig. 1.14). We have clusters of dense lines of force where the charge density is high. In the successive clusters the lines are oppositely directed. The distance between these clusters is $\lambda/2$; the distance between the successive similarly directed clusters is λ. In a progressive wave the entire succession of clusters moves steadily in one direction. In a standing wave the clusters remain stationary, but their density varies sinusoidally with time.

Long circuits provide a conceptual link between ordinary circuits and fields. It is easy to understand that there is a capacitance between

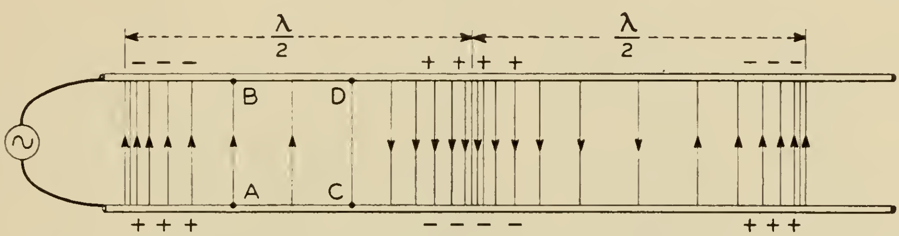


FIG. 1.14 Electric lines of force between parallel conductors.

two opposite sections of long conductors. There is also an inductance associated with a " loop " formed by such sections, provided the sections are fairly long. In short sections it is hard to see the " loop " in its physical sense. We have to close our eyes a little and pretend that there is no difficulty in applying Kirchhoff's equations to very short sections of parallel conductors. Applying Maxwell's equations, on the other hand, presents no difficulty, for the closed loops contemplated in them may be partly or entirely in free space. We can choose the loop $ACDB$ (Fig. 1.14) and apply equation 7. As we add successive voltages from A to C to D and back to A, we should find a quantity that differs from zero and depends on the time rate of change of the average H normal to the area of the loop. The voltage from A to C along the conductor is small; it would equal zero if the conductor were perfect. Similarly the voltage between B and D is small compared to that between A and B. Hence, there is a difference between the voltages V_{AB} and V_{CD}, and the magnitude of the voltage varies along the two-conductor transmission line.

From waves on parallel wires we may pass to waves on diverging wires (Figs. 1.15 and 1.16); then we may expand each wire into a large cone (Fig. 1.17). One cone may further be expanded into a plane (Fig. 1.18). Finally we may dispense with the upper cone except for a

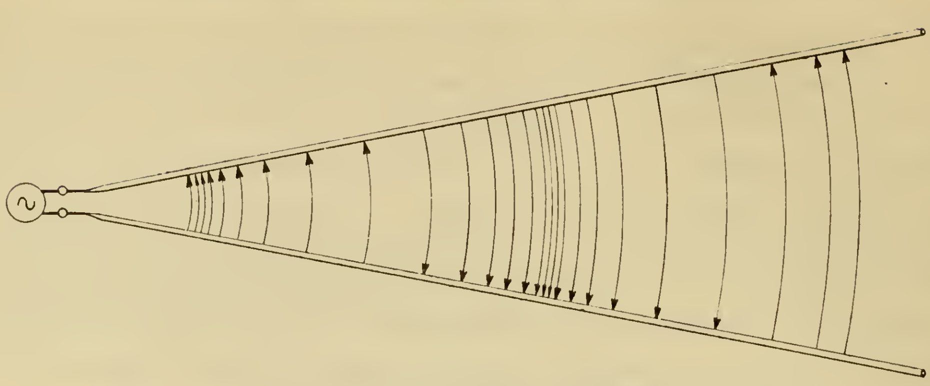


FIG. 1.15 Electric lines between divergent conductors.

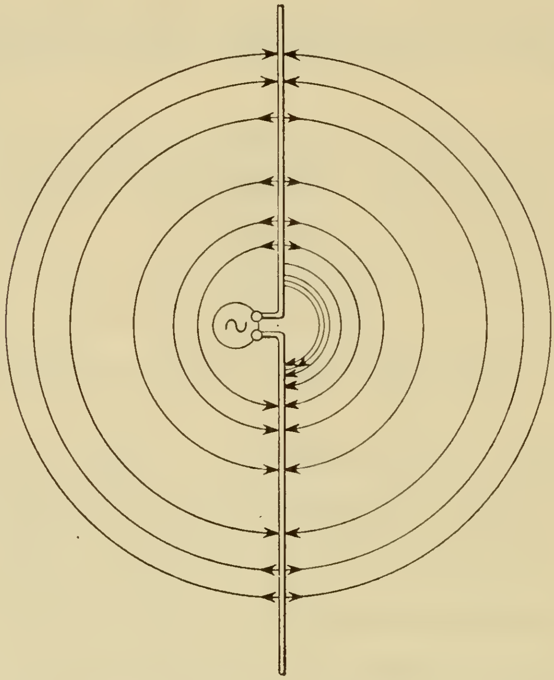


FIG. 1.16 Electric lines between divergent conductors.

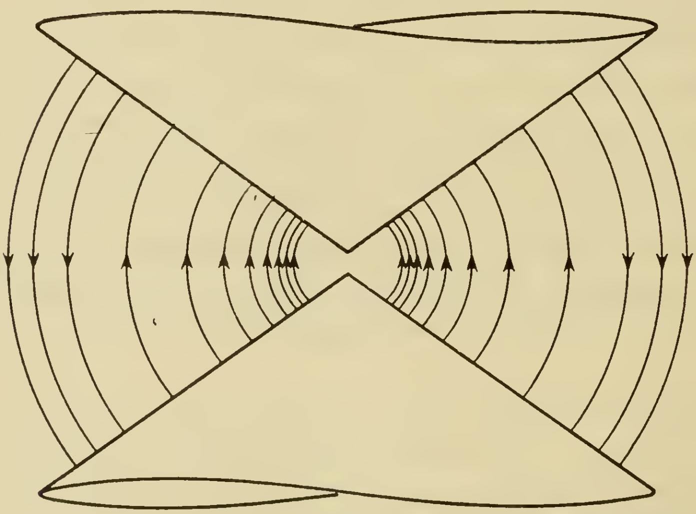


FIG. 1.17 Electric lines between conical conductors.

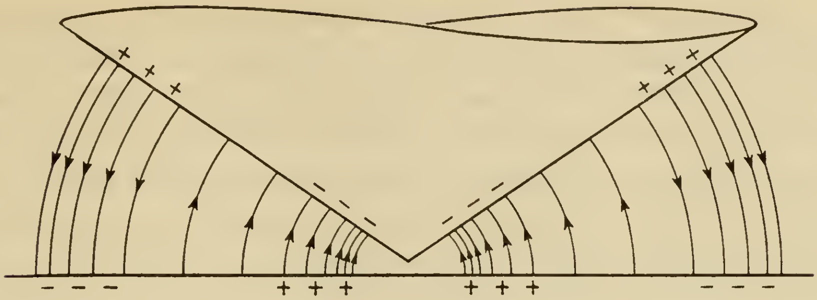


FIG. 1.18 Electric lines between a cone and the ground plane.

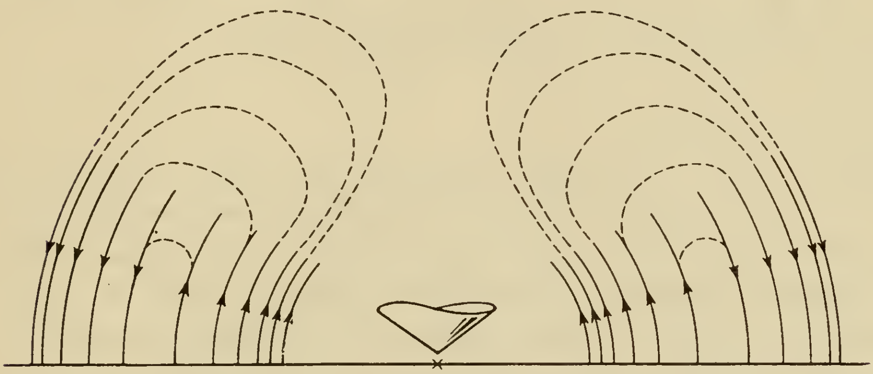


FIG. 1.19 Electric lines that have detached themselves from a vestigial cone.

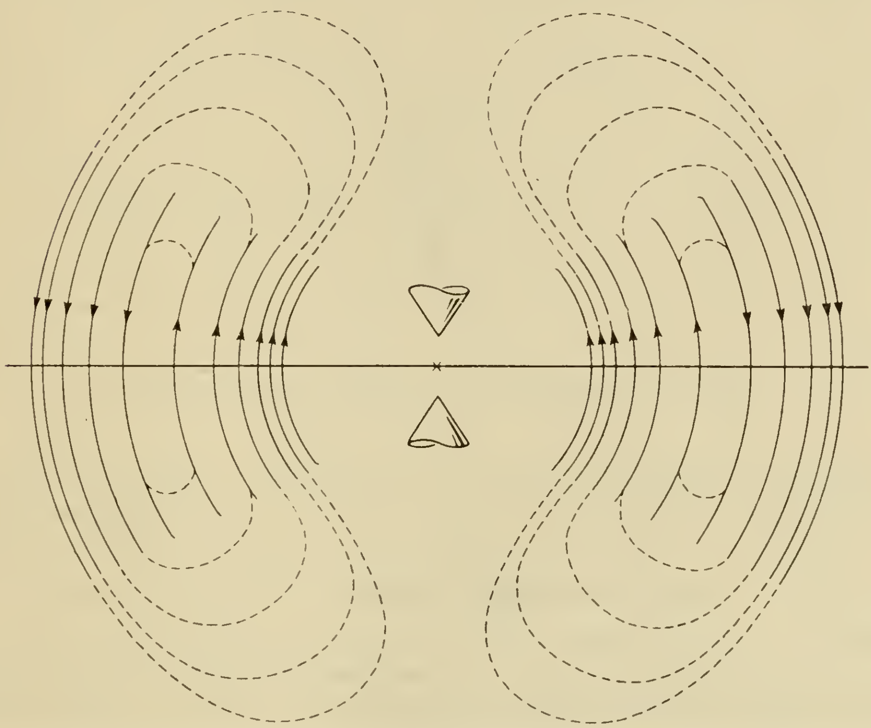


FIG. 1.20 Electric lines that have detached themselves from vestigial cones.

section near the apex (Fig. 1.19) and obtain a picture of electric waves spreading above ground. Similarly we can dispense with both cones in Fig. 1.17 except for sections near their apices, and draw a picture of waves in free space (Fig. 1.20). If the conductors are removed, the lines of force close on themselves. In Section 1.4 we found that such closed lines of force, detached from conductors, are permitted by Maxwell's equations. However, we should ask ourselves the question: Why should the lines of force detach themselves from the vestigial cones? This brings us to the consideration of short antennas. In Chapters 3 and 4 we shall return to a fuller examination of plane and spherical waves, on conductors and in free space. Here we shall only add that free space is a kind of "radial transmission line."

1.7 Short antennas

Antennas are said to be small when their dimensions are much smaller than λ. They are said to be short if one dimension is much greater than the other two and yet much smaller than λ. In short antennas the charge passing through the generator reaches the ends in a very small fraction of a period. A short antenna is primarily a capacitor. The electric lines of force look as shown in Fig. 1.21, that is, if the charge remains constant. Suppose, however, that we connect the antenna to an a-c generator at some instant $t = 0$. During the first quarter period

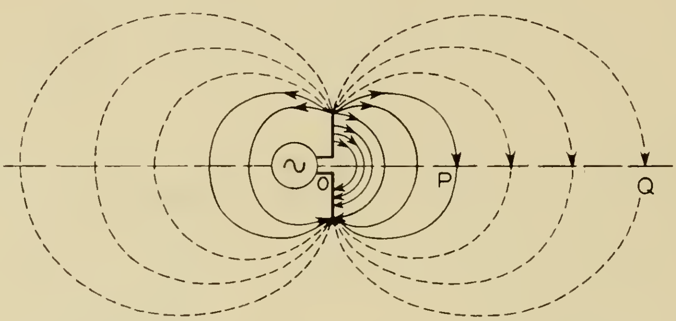


FIG. 1.21 Electric lines around a short antenna.

the charge on the upper arm reaches a maximum. Let N be the number of electric lines issuing from the upper arm and terminating on the lower arm. The lines cross the equatorial plane within the circle of radius OP equal to $\lambda/4$. Then the discharge begins. We can think of it as a flow of charge of opposite sign which eventually neutralizes the previous charge. Thus during the next quarter period we have N opposite lines of force spread over the circle of radius OP. But in this interval a fraction kN of the original lines of force spreads to the circle of radius $OQ = \lambda/2$. Only $(1 - k)N$ lines remain within the smaller circle;

adding to these the N opposite lines, we have kN lines within the circle of radius OP directed oppositely to those between the circles OP and OQ. At the end of the first half period we have no charge on the antenna; hence the kN lines of force have become detached from the antenna and must have joined the lines of force between P and Q, as shown in Fig. 1.20. Figure 4.22 shows the exact shape of the lines at an instant of complete discharge $3T/2$ sec after the antenna is connected to the generator. Figure 4.23 shows the lines one quarter period later.

The kN detached lines move on, and new lines of force take their place. As the lines move outward, the width of the cluster remains $\lambda/2$. Hence, the same lines spread over an area between two circles whose radii are $r - (\lambda/4)$ and $r + (\lambda/4)$, where r is the distance from the antenna to the center of the cluster. This area equals $2\pi r$ times $\lambda/2$; that is, $\pi\lambda r$. Therefore, for large r, where the relative distribution of the lines of force within the cluster is independent of r, the relative density is proportional to $kN/\pi\lambda r$. In accordance with our conception of lines of force this density is proportional to the electric intensity. The number of lines N issuing from the fully charged antenna is proportional to the charge and therefore to the maximum current in the generator. The fraction k of this total number which becomes detached from the antenna is proportional to the length $2l$ of the antenna. The reason is that the fields of two neighboring equal and opposite charges nearly cancel each other at distances $r \gg 2l$ and the residue should, for small separations between the charges, be proportional to the separation. Hence, the electric intensity in the equatorial plane is proportional to $2Il/\pi\lambda r$. The dimensions of this quantity are amperes per meter, whereas the dimensions of the electric intensity are volts per meter. Thus the dimensions of the ratio are those of an impedance. Now the only quantity having these dimensions in the part of the field that has become completely detached from its source is the ratio $\eta = E/H$ of the electric to the magnetic intensity. This ratio is called the *intrinsic impedance* of the medium;* the intrinsic impedance of free space turns out to be 376.7 ohms, which is very nearly 120π or 377 ohms. Our formula for the electric intensity in the equatorial plane of a short antenna becomes

$$E = k_1 \frac{2\eta Il}{\pi\lambda r}, \tag{15}$$

where k_1 is a dimensionless factor.

The electric lines in the cluster are not uniformly distributed since

* The intrinsic impedance may be obtained experimentally by measuring E with a probe and H with a loop.

they are oppositely directed on the two sides of the center and the density must vanish at the center. At large distances where the relative distribution is independent of r, the density must be proportional to the cosine of the phase (equation 12). Hence, in the equatorial plane,

$$E = k_2 \frac{2\eta Il}{\pi\lambda r} \cos\left(\frac{2\pi t}{T} - \frac{2\pi r}{\lambda} + \varphi_0\right). \tag{16}$$

We have included a constant phase φ_0 to allow for the fact that our present argument is limited to distances that are large compared with λ, so that we cannot determine the absolute phase difference between the

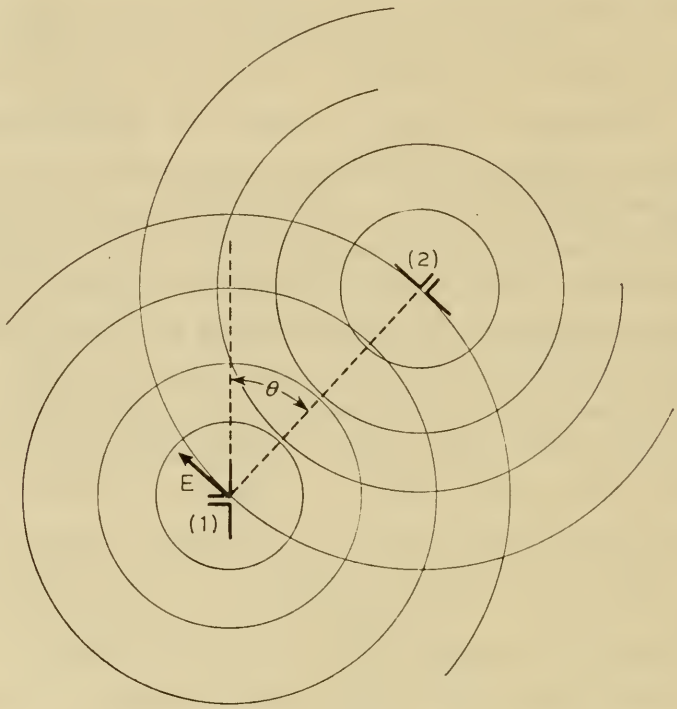


FIG. 1.22 Two short antennas.

current in the antenna and the field intensity at some distant point. This is as far as we can go at present; the numerical factor k_2 may be determined either as in Section 1.19 or by solving Maxwell's equations. Thus we shall find that $k_2 = \pi/4$ and $\varphi_0 = -\pi/2$. However, we shall continue to use the indefinite factor until we actually determine its value.

All detached electric lines cross the sphere of radius r passing through the center of the cluster. Hence, the radial electric intensity is proportional to $kN/4\pi r^2$, that is, to $2Il/4\pi r^2$. As before, we have to include the intrinsic impedance of the medium and the phase factor. Hence, the radial intensity varies inversely as the square of the distance from the antenna, whereas the meridian intensity varies inversely as the first

power; the ratio diminishes as λ/r, and *at large distances the electric intensity is substantially perpendicular to the radius drawn from the antenna to the point under consideration.*

To extend equation 16 to directions other than those in the equatorial plane consider two identical antennas (Fig. 1.22). When antenna 2 is used as a transmitting antenna, the electric intensity in its equatorial plane is given by equation 16. The voltage induced in antenna 1 is proportional to the tangential component of this intensity, that is, to the product of equation 16 and $\sin\theta$. If now we excite antenna 1 with the same current, the voltage induced in antenna 2 should equal that previously induced in antenna 1. This is a very special example of reciprocity which needs no proof; for the antennas are identical and, in free space, indistinguishable. Hence, the meridian component of the electric intensity of antenna 1 is

$$E_\theta = k\,\frac{2\eta Il}{\pi\lambda r}\cos\left(\frac{2\pi t}{T} - \frac{2\pi r}{\lambda} + \varphi_0\right)\sin\theta, \tag{17}$$

where we have dropped the subscript associated with k in equation 16.

The above equations have been obtained on the assumption that r is large compared with λ. If r is very small compared with λ, we have a substantially electrostatic distribution of E. The lines of force reach these points in a small fraction of the period, and the changes in the intensity reflect the changes in the charge on the antennas. The electric intensity around a single charged particle varies inversely as r^2; around two neighboring equally and oppositely charged particles it varies inversely as r^3. If this law of variation of the field intensity applied to all distances, radio communication would be impracticable. The fields would become too weak to be detected. But, as equation 17 shows, at large distances the field intensity diminishes only as $1/r$.

1.8 Radiation

The existence of closed electric lines has important implications which may best be brought out by contrast with lines terminating on charges. Imagine two long conductors connected to a generator by means of a switch. If the switch is turned on and off for a brief interval of time, an electric disturbance is created. This disturbance continues to exist after the power is turned off, that is, after the initial cause has ceased to exist. Nevertheless the traveling disturbance is associated with the charges on the conductors, and it may be thought that the existence of these charges is essential to the existence of the field. Now the existence of closed electric lines implies that an *electromagnetic field may exist without the charges.* Electric charges are required to excite the field;

but they are not needed for its continued existence. This phenomenon is analogous to the excitation of water waves by dropping a pebble in a shallow pool or by immersing and then withdrawing the end of a stick. The wavelet is seen moving even though the pebble that caused it rests on the bottom or the stick is nowhere near the water. Electromagnetic waves are just as real as the charges that cause them. Radar is a practical method of demonstrating the existence of free electromagnetic waves as well as their application to the detection of objects.

Any traveling wave carries some energy with it. A blow to the first mass in the chain shown in Fig. 1.10*b* transfers to it a certain quantity of energy, which at first appears as the kinetic energy of the mass. Successively this energy is transformed into the potential energy of the first spring, the kinetic energy of the second mass, the potential energy of the second spring, etc. Thus the energy travels on. Similarly, energy can travel through a succession of coils and capacitors or along continuous electric transmission lines. Each cluster of lines of force detached from the antenna contains a certain amount of energy. This follows from the fact that the electric charge introduced into the cluster must move under the action of E and, hence, acquire energy. The only source of this energy is the field, at least after the oscillations in the antenna have ceased. By such considerations as these we pass from the idea of electromagnetic energy associated with charges and currents to the idea of free or *radiated* energy.

When a wave is guided by a pair of conductors, the power carried by it is VI, where V is the voltage between the conductors and I is the current in one of them. From this expression we can obtain a corresponding formula for the power flow per unit area in a free space wave. For this purpose, let us consider a pair of wide-angle coaxial cones (Fig. 1.17). The electric lines follow the meridians, and the magnetic lines are circles coaxial with the cones, that is, lines of parallel. At distance r from the apex, the length of such a line is $2\pi r \sin\theta$, where θ is the polar angle. Between the cones of wide angle, θ is nearly 90° and the length of a magnetic line does not vary much with θ. From Maxwell's equation 8 we find that the product of the magnetic intensity H and the length of the circle equals the electric current I in the cone at distance r from the apex, there being no radial electric lines linked with the circle. The voltage V between the cones equals the electric intensity E times the length of the meridian intercepted between the cones. Hence, the power VI carried by the wave between the cones equals EHS, where S is the area of the sphere of radius r intercepted between the cones. The power per unit area is thus

$$W = EH. \tag{18}$$

Both E and H are tangential to the sphere; they are perpendicular to each other; and the flow of power is radial. Hence, we can express the flow of power per unit area as the vector product,

$$\vec{W} = \vec{E} \times \vec{H}, \tag{19}$$

which gives the magnitude and direction of flow. This vector is called the *Poynting vector*. In the next chapter we shall examine the concept of energy storage and power flow in more detail and in relation to electric circuits. The present derivation is simple and direct; but it requires a knowledge of the formula for the power transmitted by electric transmission lines.

We have seen that at large distances from the antenna the ratio E/H is a constant η, called the intrinsic impedance of the medium. Hence, the power flow per unit area may be expressed in terms of E alone,

$$W = \frac{E^2}{\eta} \cdot \tag{20}$$

Since E varies sinusoidally with time, and since the average value of the square of a sine (or cosine) per cycle is 1/2, we have

$$W_{\text{av}} = \frac{E_a{}^2}{2\eta}, \tag{21}$$

where E_a is the amplitude of E.

The electric intensity of a wave excited by a short antenna is given by equation 17. Using equations 20 and 21, we have

$$W = k^2 \frac{4\eta I^2 l^2}{\pi^2 \lambda^2 r^2} \cos^2\left(\frac{2\pi t}{T} - \frac{2\pi r}{\lambda} + \varphi_0\right) \sin^2 \theta, \tag{22}$$

and

$$W_{\text{av}} = k^2 \frac{2\eta I^2 l^2}{\pi^2 \lambda^2 r^2} \sin^2 \theta.$$

To obtain the total power flowing through the sphere we should multiply this equation by an element of area $r^2 \sin \theta \, d\theta \, d\varphi$ and integrate,

$$P = k^2 \frac{2\eta I^2 l^2}{\pi^2 \lambda^2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \sin^3 \theta \, d\theta \, d\varphi = k^2 \frac{16\eta I^2 l^2}{3\pi\lambda^2} \cdot \tag{23}$$

We can now express the electric intensity of the wave excited by a short antenna in terms of the radiated power. From equation 23 we find

$$kIl = \left(\frac{3\pi P}{16\eta}\right)^{1/2} \lambda. \tag{24}$$

Substituting in equation 17, we have

$$E_\theta = \left(\frac{3\eta P}{4\pi}\right)^{\frac{1}{2}} \frac{1}{r} \cos\left(\frac{2\pi t}{T} - \frac{2\pi r}{\lambda} + \varphi_0\right) \sin \theta. \tag{25}$$

Note that the unknown factor k has disappeared.

If P is the power radiated by *any* antenna, the average power flow per unit area must be $P/4\pi r^2$, since the same radiated power must pass through every sphere concentric with the antenna. Since E is proportional to the square root of this quantity, the average E varies inversely as r for *any* antenna; that is, far enough from the antenna so that the lines of force have become detached from the antenna. This remote region of the field around the antenna is often called the *radiation field* to distinguish it from the local field, where we are concerned with direct forces between the charges and currents.

1.9 Heat loss

We are now in a position to consider the effectiveness of various circuits as antennas. The object of a transmitting antenna is to create as strong a distant field as possible, and the object of a receiving antenna is to respond as well as possible to an externally impressed field. Equation 17 shows that the strength of the distant field created by a short antenna is proportional to the length of the antenna in wavelengths. On the other hand, when expressed in terms of the radiated power, the electric intensity (equation 25) is independent of the length of the antenna and of the frequency of oscillations. Should we conclude that the effectiveness of the antenna increases with its length and with the frequency of oscillations, or should we assert that it is independent of these factors? It is true that, if the currents in two antennas are equal, the longer antenna creates a stronger field; but what is there to prevent us from using a transformer and thus raising the current in the shorter antenna if the same power is available? Higher current in a practical circuit means higher dissipation of power in heat in the conductors on account of their resistance. The resistance of a short antenna is proportional to its length; hence, the dissipated power is proportional to the length. The radiated power, on the other hand, is proportional to the square of the length. Hence, the power delivered to an extremely short antenna will largely be dissipated in heat. As the length of the antenna increases, the ratio of the radiated to the dissipated power also increases. Of two antennas, the longer one will be a more effective radiator.

The antenna should be as "open" as possible. If we bend the arms of the antenna in Fig. 1.21 so that they become parallel, as in Fig. 1.23, we shall find that for the same current input the distant field

will be considerably weaker than before. The fields of two equal and opposite currents nearly cancel each other. The resistance of the wires, on the other hand, is not affected by bending. Similarly a square loop is a more effective antenna than a narrow rectangular loop of the same perimeter.

Ordinary electric circuits are deliberately designed so that the associated fields are confined to small regions and are very weak at large distances from these regions. Capacitors, for instance, are made of closely spaced metal sheets, and strong fields are confined to the regions between the plates. This design increases the capacitance without increasing the size of the capacitor and at the same time diminishes the direct coupling between capacitors in different parts of the network. The capacitor becomes an effective antenna only when it is open as in Fig. 2.1*b*.

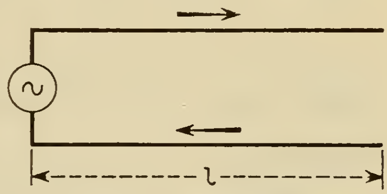


FIG. 1.23 An ineffective dipole antenna.

There are no perfect conductors in nature; but in theory it is often desirable to consider perfectly conducting antennas. This does not mean that we neglect their resistance altogether; it means merely that we intend to consider it separately. Equation 23, for example, gives the radiated power irrespective of the antenna resistance. This is the entire power delivered to the antenna if the antenna is perfectly conducting and only a fraction of it otherwise. Perfectly conducting short antennas of different lengths are equally effective.

1.10 Antenna impedance

Power is delivered from a generator to a transmitting antenna or conveyed from a receiving antenna to a load by means of a *feeder* transmission line. The terminals of the antenna where it is connected to the feeder may be either as clearly defined as those of an ordinary circuit or as ill defined as those in some microwave antennas, which appear to be inseparable from their feeder lines. In any case, however, if we go along the feeder toward the generator (or the load), we shall usually find a place where it is possible to measure the voltage and current and thus determine their ratio, the impedance. As far as these measurements are concerned, the antenna, or the antenna together with a part of the feeder, is indistinguishable from a network of resistors, capacitors, and inductors. To an observer ignorant of the details of the physical structure, the antenna appears as a concealed circuit whose electrical characteristics are given completely by its impedance. The conventional diagram for a concealed circuit is a box (Fig. 1.24) with a pair of

exposed or *accessible terminals.* In the presence of an external field there is an open-circuit voltage at the terminals of the receiving antenna analogous to the open-circuit voltage at the terminals of any electric network containing generators. As far as the voltage and current at the

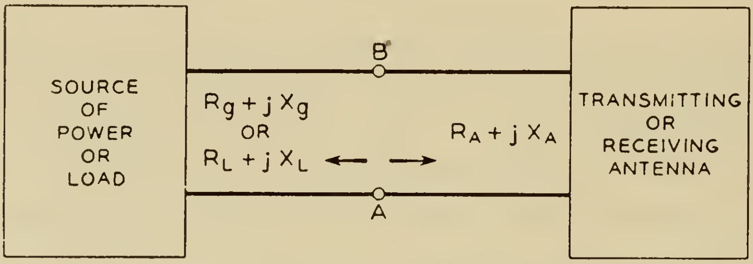


FIG. 1.24 An antenna as a concealed electric network.

terminals are concerned, we cannot distinguish the receiving antenna, or any other circuit, from an electric generator with a certain internal impedance.

Let $R_A + jX_A$ be the impedance of the antenna and V be the amplitude of the open-circuit voltage. Let $R_L + jX_L$ be the impedance of the load. The current through the load is then

$$I = \frac{V}{|(R_A + R_L) + j(X_A + X_L)|}. \tag{26}$$

The power transferred to the load is

$$P = \tfrac{1}{2}R_L I^2 = \frac{R_L V^2}{2[(R_A + R_L)^2 + (X_A + X_L)^2]}. \tag{27}$$

This power increases as $X_A + X_L$ decreases in magnitude. Let us make

$$X_L = -X_A; \tag{28}$$

then,

$$P = \frac{R_L V^2}{2(R_A + R_L)^2}. \tag{29}$$

This power vanishes when $R_L = 0$ and when $R_L = \infty$; hence, for some R_L the power transferred to the load is maximum. To obtain this maximum we differentiate P with respect to R_L and equate the derivative to zero. Thus we find the second condition for maximum power transfer from the antenna to the load,

$$R_L = R_A. \tag{30}$$

Substituting in equation 29, we have

$$P_{\max} = \frac{V^2}{8R_A}. \tag{31}$$

To summarize: *For maximum transfer of power the impedance of the load should be the conjugate of the impedance of the antenna,*

$$R_L + jX_L = (R_A + jX_A)^* = R_A - jX_A,$$

or

$$R_L = R_A, \qquad X_L = -X_A. \tag{32}$$

If the antenna is perfectly conducting, the resistive component of its impedance is due solely to the loss of power by radiation and for this reason is called the *radiation resistance.* For a short antenna, for instance, the radiated power is given by equation 23. On the other hand, this power is given by

$$P = \tfrac{1}{2}R_A I^2; \tag{33}$$

hence,

$$R_A = k^2 \frac{32\eta l^2}{3\pi\lambda^2} \cdot \tag{34}$$

The reactance of a short antenna comes primarily from the capacitance. As l increases, however, the inductance begins to play an increasingly important role.

Short antennas will be considered further in Chapter 10. Here we shall add only one concluding remark. Equation 34 gives the radiation resistance. To obtain the total input resistance we must include the *ohmic resistance* representing the loss of power in heat. The maximum power transferred to the load is then

$$P_{\max} = \frac{V^2}{8(R_{\text{rad}} + R_{\text{ohm}})} \cdot \tag{35}$$

The ohmic resistance is proportional to l. The induced voltage V is proportional to l. When l is so small that the radiation resistance is negligible compared with the ohmic resistance, the maximum received power is proportional to l; but, when l is so large that the radiation resistance is the controlling factor, then the maximum received power is independent of l.

1.11 Current distribution in thin antennas

We have seen (Section 1.5) that in a mechanical chain of masses and springs a progressive wave reaching a fixed point is reflected. When added to the incident wave, it forms a standing wave with a node for the displacement (and velocity) at the fixed point. Similarly, we may have standing waves in a finite chain of inductors and capacitors. The current node will be at the open end of the chain. In the case of an antenna the situation is complicated by the fact that the field is distributed in three dimensions and there is no place where the electrical

disturbance "terminates." The current must vanish at the open end of a wire, and, if we assume that the reflected current wave is similar to the incident current wave, we shall have standing waves (Fig. 1.25). Near its nodes, the sine wave is nearly straight; hence, if $l \ll \lambda$, the current distribution is approximately linear (Fig. 1.25*a*). If $l = \lambda/4$, the current distribution is given by a full half-wave (Fig. 1.25*b*). If $l = \lambda/2$, we have two half-waves (Fig. 1.25*d*). These distributions are gradually approached as the radius of the antenna is diminished; for then the electric and magnetic fields are concentrated more and more in the immediate vicinity of the wires, and we have a more nearly one-dimensional form of wave motion.

No matter how thin the antenna is, the current distribution deviates from the perfect sine wave. This is particularly evident if we consider

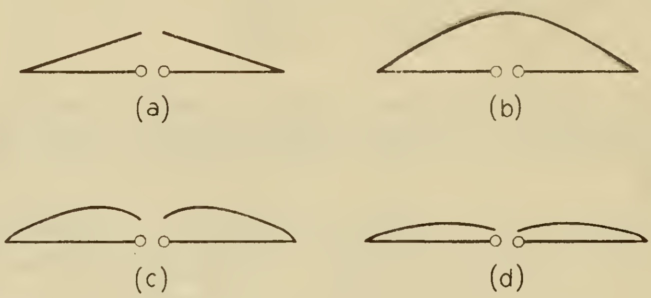


FIG. 1.25 Approximate forms of current distribution in dipole antennas: (*a*) short antenna, (*b*) resonant antenna, (*c*) intermediate between resonance and antiresonance, (*d*) antiresonant antenna.

the case shown in Fig. 1.25*d* in which the input current apparently vanishes. Since some power is radiated, it must be delivered to the antenna. This power is given by equation 33 where I is the amplitude of the input current; hence, the input current cannot vanish. The actual current distribution in a thin antenna of this length is shown by the solid line in Fig. 11.13. The deviation from the perfect sine wave is greatest near the input terminals. The various factors affecting current distribution in antennas are considered in Chapter 8. The simple sine-wave approximation, however, gives the magnitude of the distant field with sufficient accuracy for many practical purposes. It is only in impedance calculations that we need various corrections, and even then only in the input current.

1.12 Calculation of radiation fields

In order to calculate the radiation field we subdivide the antenna into elements of length dz (Fig. 1.26). An element differs from a short antenna in that the current in the element is substantially constant

along its length, whereas in the antenna it tapers off linearly toward the ends. The electric intensity of the wave generated by the current element is therefore of the form of equation 17, but with a different value of k. We can, however, obtain an expression with the same numerical factor k by properly interpreting the product $2lI$ of the length of the antenna and the input current in relation to the corresponding product for the element. In the element the current is constant, and, hence, twice as effective as in the short antenna; we compensate for this by letting the length dz of the element be equal to one half of the length $2l$ of the short antenna, so that $lI = I(z)\,dz$. The electric intensity of the wave generated by a current element is, therefore,

$$dE_\theta = k\frac{2\eta\, I(z)\, dz}{\pi\lambda r} \times \cos\left(\frac{2\pi t}{T} - \frac{2\pi r}{\lambda} + \varphi_0\right)\sin\theta. \qquad (36)$$

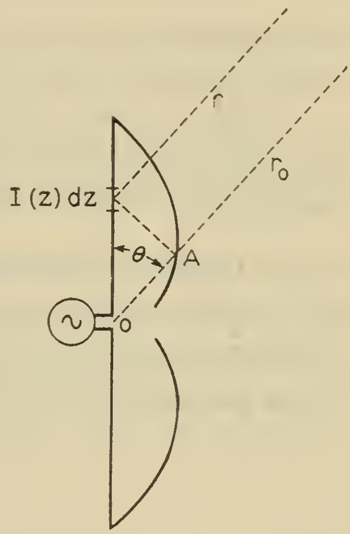


FIG. 1.26 A linear antenna and a typical current element of moment $I(z)\,dz$.

Let r_0 be the distance from the center of the antenna to some distant point; then the distance r from a typical current element to the same point equals r_0 minus the projected distance between the center and the element,

$$r = r_0 - z\cos\theta. \qquad (37)$$

Substituting in equation 36, we have

$$dE_\theta = k\frac{2\eta\, I(z)\, dz}{\pi\lambda(r_0 - z\cos\theta)}\cos\left(\frac{2\pi t}{T} - \frac{2\pi r_0}{\lambda} + \varphi_0 + \frac{2\pi z\cos\theta}{\lambda}\right)\sin\theta. \qquad (38)$$

The product $z\cos\theta$ in the denominator of the amplitude factor is negligible compared with r_0. In the phase factor, however, the product of $2\pi z/\lambda$ and $\cos\theta$ is not negligible, for, if the antenna is long, this product may be comparable to π or even be greater than π. The remainder of the phase

$$\frac{2\pi t}{T} - \frac{2\pi r_0}{\lambda} + \varphi_0$$

is the same everywhere on the sphere of radius r_0 and for all current elements. Hence, if we are not interested in the absolute phase of the

field, we may omit this part of the phase and write the following expression for the element of electric intensity,

$$dE_\theta = k \frac{2\eta\, I(z)\, dz}{\pi\lambda r_0} \cos \frac{2\pi z \cos\theta}{\lambda} \sin\theta. \tag{39}$$

If we substitute the sine wave approximation for $I(z)$ and integrate, we obtain the total electric intensity.

If the current elements are not all situated on the same straight line, their fields must be added vectorially. As shown in Chapter 12 the calculation of distant fields may be reduced to a simple routine.

1.13 Directive radiation

Equation 36 shows that, if the distances of two elements to a distant point differ by $\lambda/2$, the phases of their fields differ by π and the amplitudes are substantially the same. Hence, the two fields cancel in that particular direction. In some other direction the waves may arrive at a given point in phase and reinforce each other. The greatest reinforcement takes place when the phase difference is an integral multiple of 2π, and there is total annihilation when the phase difference is an odd multiple of π. In between there is either partial reinforcement or partial cancelation. We assume, of course, that the currents in the two elements are in phase; if they are not, we should allow for the difference.

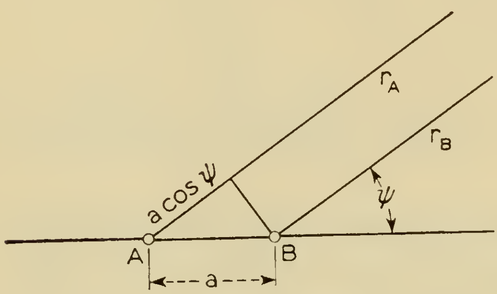


FIG. 1.27 An array of two antennas.

In this way we can direct radiation in some preferred directions. Suppose, for example, that we have two current elements or two short antennas perpendicular to the plane of the paper, distance a apart (Fig. 1.27). Each antenna radiates uniformly in all directions in the plane of the paper. If the currents are the same, the amplitudes of the waves arriving at some distant point are the same and the resultant depends on the relative phases. If the phases of the antenna currents are the same, the phase factors are

$$\cos\left(\frac{2\pi t}{T} - \frac{2\pi r_A}{\lambda} + \varphi_0\right) \quad \text{and} \quad \cos\left(\frac{2\pi t}{T} - \frac{2\pi r_B}{\lambda} + \varphi_0\right)\cdot \tag{40}$$

The total intensity is proportional to the sum of these factors, which is

$$2 \cos \frac{\pi(r_B - r_A)}{\lambda} \cos\left(\frac{2\pi t}{T} - \frac{2\pi}{\lambda}\,\frac{r_A + r_B}{2} + \varphi_0\right)\cdot \tag{41}$$

Since $r_B - r_A = a \cos \psi$, where a is the distance between the antennas and ψ is the angle between a typical direction and the line joining them, the field intensity is proportional to

$$2 \cos\left(\frac{\pi a}{\lambda} \cos \psi\right) \cdot \tag{42}$$

If $a = \lambda/2$, this vanishes when $\psi = 0$ and equals 2 when $\psi = \pi/2$; that is, the line of zero radiation is AB and the line of maximum radiation is perpendicular to it.

As we have just seen, the relative distribution of the field at some great distance r from the sources of radiation does not depend on the actual phases of the waves arriving from separate sources but only on the differences in phase. If E_A is a complex number representing the electric intensity of the wave from one source (Fig. 1.27), the intensity of the wave arriving from the other source is

$$E_B = E_A e^{j(2\pi a/\lambda) \cos \psi}, \tag{43}$$

where the exponential function expresses the phase lead arising from the shorter distance traveled by the wave from B compared with the distance traveled by the wave from A. Figure 1.28 expresses this relationship graphically as is frequently done in a-c problems. The diagonal of the parallelogram represents the resultant field intensity. Field intensities from any number of sources can thus be added graphically.

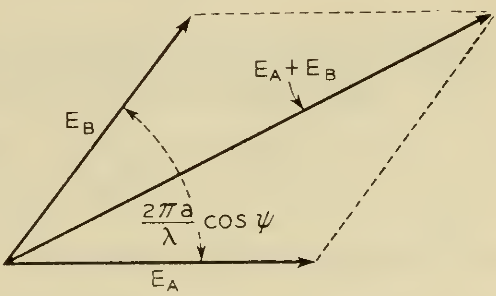


FIG. 1.28 Addition of distant fields of two antennas.

The second factor in the expression 41 shows that, as far as the phase is concerned, the mid-point between the radiating sources at A and B is acting as an effective point of radiation; for $\frac{1}{2}(r_A + r_B)$ is the distance from the mid-point. We can illustrate this graphically as shown in Fig. 1.29. The phase of the wave originating at A and arriving at some distant point is retarded with reference to a similar wave originating at O while the phase of the wave originating at B is advanced by the same amount. If the sources at A and B are of the same strength and are operating in phase, the resultant electric intensity at a distant point is given by the diagonal of a rhombus (Fig. 1.29b). The phase is the same as if the source of the wave were located at O. Clearly, the magnitude

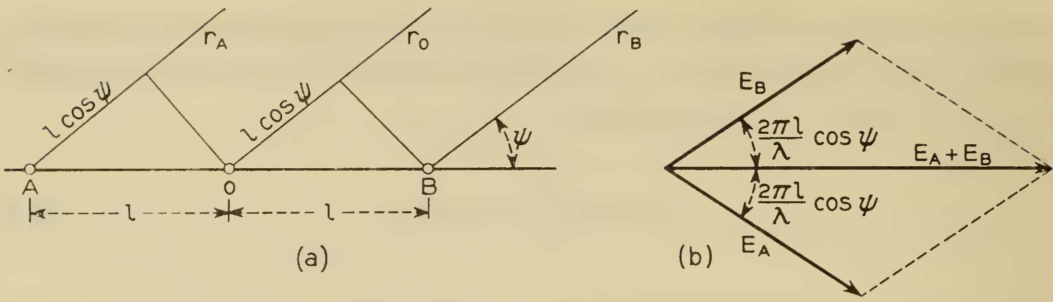


FIG. 1.29 As far as the distant field is concerned, any system of sources may be replaced by an appropriate effective source of radiation at any point somewhere within the system. If two sources are identical, their mid-point is a particularly convenient location for the effective source.

of the resultant intensity is twice that of the projection of either component on the diagonal,

$$|E_A + E_B| = 2|E_A| \cos\left(\frac{2\pi l}{\lambda} \cos \psi\right) \cdot \tag{44}$$

For a continuous linear radiator (Fig. 1.30) we assume that the field intensity from a differential element is $E_1(s)\, ds$. If the source is

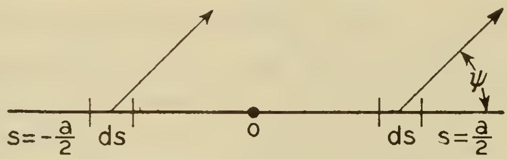


FIG. 1.30 A continuously distributed source of radiation.

symmetric about the mid-point O, $E_1(-s) = E_1(s)$ and the resultant field intensity from two elements equidistant from O is

$$2E_1(s)\cos\left(\frac{2\pi s}{\lambda} \cos \psi\right) ds.$$

For the entire source, we have

$$E = 2\int_0^{a/2} E_1(s) \cos\left(\frac{2\pi s}{\lambda} \cos \psi\right) ds. \tag{45}$$

For example, if our source is uniformly distributed and if E_0 is the intensity that would exist at a distant point when the entire strength of the source is concentrated at O, then $E_1 = E_0/a$, and equation 45 gives

$$E = E_0 \frac{\sin[(\pi a/\lambda) \cos \psi]}{(\pi a/\lambda) \cos \psi} \cdot \tag{46}$$

The solid curve in Appendix II shows E/E_0.

The calculation of distant fields of two-dimensional arrays is facilitated by the *product rule.* Suppose that we have six identical sources arranged in a rectangular pattern (Fig. 1.31). As far as distant points are concerned, we can replace the rectangular array by either a linear vertical array in which the elements represent the radiations from the horizontal rows or a linear horizontal array in which the elements represent the radiations from the vertical columns. Hence, $E = (E_0S_h)S_v$ or

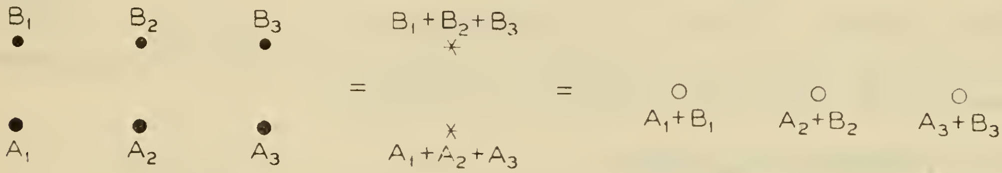


FIG. 1.31 A rectangular array is a linear array of linear arrays.

$E = (E_0S_v)S_h$, where S_h is the multiplying factor when we gather each horizontal row in a single point of effective radiation and S_v is a similar "space factor" for the elements in the vertical columns; that is,

$$E = E_0S_hS_v. \tag{47}$$

For example, for a uniform rectangular array of area ab, we have

$$E = E_0 \frac{\sin[(\pi a/\lambda) \cos \psi_a]}{(\pi a/\lambda) \cos \psi_a} \frac{\sin[(\pi b/\lambda) \cos \psi_b]}{(\pi b/\lambda) \cos \psi_b}, \tag{48}$$

where ψ_a and ψ_b are the angles made by a typical direction in space with the sides of the rectangle indicated by the subscripts.

Clearly, with several antennas at our disposal we can arrange them so that their fields add in some particular direction and, to a large extent, interfere destructively in directions outside a certain main beam. Nonuniform distribution of radiation means a gain in the signal strength for a given radiated power, since the power that is not radiated in some directions is available for other directions. We can define the *directivity g* as the ratio of the maximum power flow per unit area to the average power flow at the same distance,

$$g = \frac{W_{\text{max}}}{W_{\text{av}}}. \tag{49}$$

Hence, if P is the radiated power, then,

$$W_{\text{av}} = \frac{P}{4\pi r^2} \quad \text{and} \quad W_{\text{max}} = g\frac{P}{4\pi r^2}. \tag{50}$$

If $g = 1$, $W_{\text{max}} = W_{\text{av}}$. The power is radiated uniformly in all directions. Such a radiator is called the *isotropic* radiator. The direc-

tivity g of an antenna is the gain in power over an isotropic radiator in the sense that the radiated power P/g is sufficient to obtain the same power flow per unit area at a given distance from the antenna as that from an isotropic radiator emitting the power P. A uniformly heated sphere is an example of an isotropic radiator of electromagnetic waves. However, such a sphere is not a source of coherent radiation, that is, radiation characterized by a definite phase at a given instant at a given place. Heat is emitted by various molecules in wavelengths with random phases. There are no isotropic sources of coherent radiation. In theory, however, it is convenient to employ such radiators as reference standards in defining the directivity of antennas.

1.14 Directive reception

In the case of two receiving antennas at A and B (Fig. 1.27) the wave arrives at these points with different phases. The phase difference depends on the projected distance $a \cos \psi$ between the antennas. The outputs of the antennas can be combined in phase or in some other mutual phase relationship. For a given connection between the antennas the outputs tend to reinforce each other in some directions and cancel in others. Thus several nondirective antennas can be arranged for directive reception.

One obvious advantage of directive reception is the *signal-to-noise gain* when noise is arriving more or less uniformly from all directions. If the antenna receives poorly from waves arriving in some directions, it does not receive much noise arriving from those directions.

There is an obvious reciprocity between directive radiation and directive reception. If the antennas at A and B (Fig. 1.27) are connected by lines of equal length to a common generator, the antenna currents are in phase. Hence, the maximum radiation takes place at right angles to AB; and, if $a = \lambda/2$, there is no radiation along AB. If now we replace the generator by a load, then the outputs of the antennas arrive with equal retardations. Hence, for waves arriving perpendicularly to AB the outputs reinforce each other, and for waves parallel to AB the outputs cancel. Thus the *directivity of a given antenna used as a transmitting antenna equals its signal-to-external-noise gain when used as a receiving antenna.*

1.15 Transmission of power between antennas in free space

The ratio of the power received by the antenna when its impedance is *matched* to the load (that is, when their impedances are conjugate complex) to the power per unit area in the incoming wave is called the

effective area of the antenna,

$$A = \frac{P_{\text{rec}}}{W}. \tag{51}$$

It is usually understood that the receiving antenna is oriented for best reception with respect to the incoming wave.

If the transmitting antenna is so oriented that its main beam is directed toward the receiving antenna, $W = W_{\max}$ in equation 50; hence,

$$P_{\text{rec}} = P_{\text{tr}} \frac{g_{\text{tr}} A_{\text{rec}}}{4\pi r^2}. \tag{52}$$

The directivity g can always be determined from the relative distribution of the field for various directions in space since the factor of proportionality cancels from equation 49. The power radiated by a short antenna, for example, is given by equation 23; therefore,

$$W_{\text{av}} = k^2 \frac{4\eta I^2 l^2}{3\pi^2 \lambda^2 r^2}. \tag{53}$$

To find $W_{\max}$ we let $\theta = \pi/2$ in equation 17, substitute in equation 20, and take the time average,

$$W_{\max} = k^2 \frac{2\eta I^2 l^2}{\pi^2 \lambda^2 r^2}. \tag{54}$$

Hence,

$$g = \frac{W_{\max}}{W_{\text{av}}} = 1.5. \tag{55}$$

To obtain the effective area of a perfectly conducting short antenna we note that the amplitude of the induced voltage is $E_a l$, where E_a is the amplitude of the electric intensity and l is the length of the antenna arms. By equations 31 and 32 the maximum received power is

$$P = \frac{E_a{}^2 l^2}{8R_A} = \frac{3\pi\lambda^2 E_a{}^2}{256\eta k^2}. \tag{56}$$

The incoming power per unit area is given by equation 21,

$$W = \frac{E_a{}^2}{2\eta}. \tag{57}$$

Hence, the effective area is

$$A = \frac{P}{W} = \frac{3\pi\lambda^2}{128k^2}. \tag{58}$$

Thus, we cannot find the effective area without first calculating k. In Section 1.19 we shall find that k^2 equals $\pi^2/16$ so that

$$A = \frac{3\lambda^2}{8\pi}, \tag{59}$$

and, for two short perfectly conducting antennas,

$$P_{\text{rec}} = P_{\text{tr}}\left(\frac{3}{8\pi}\right)^2\left(\frac{\lambda}{r}\right)^2. \tag{60}$$

Equation 59 shows that, irrespective of its length and radius, a short perfectly conducting antenna *is capable* of collecting from a plane wave the power passing through a rectangle whose sides are equal approximately to $\lambda/4$ and $\lambda/2$. This power is collected *only* if the load is matched to the antenna; otherwise the received power may be very small. The ohmic resistance of practical short antennas tends to reduce the received power still further.

1.16 Large radiating, reflecting, and absorbing surfaces

In the preceding sections we have considered small radiators and systems of small radiators. Since a large radiator may always be subdivided into small parts, we have a general method for obtaining the radiation fields of known current distributions. In some respects, however, large radiators are much simpler than the small ones, and we can learn much if we consider them from a different point of view. Let us take an infinite plane sheet of electric current of density C amperes per meter perpendicular to the flow lines (Fig. 1.32). Let us assume that the current is distributed uniformly so that C has the same value at all points. According to the Ampère-Maxwell law, magnetic lines of force encircle the current. In the present case the magnetic intensity must have equal and opposite values on the opposite faces of the current sheet. The magnetomotive force around a narrow rectangular circuit just encircling a strip of current of unit width is $2H_0$, where H_0 is the magnetic intensity

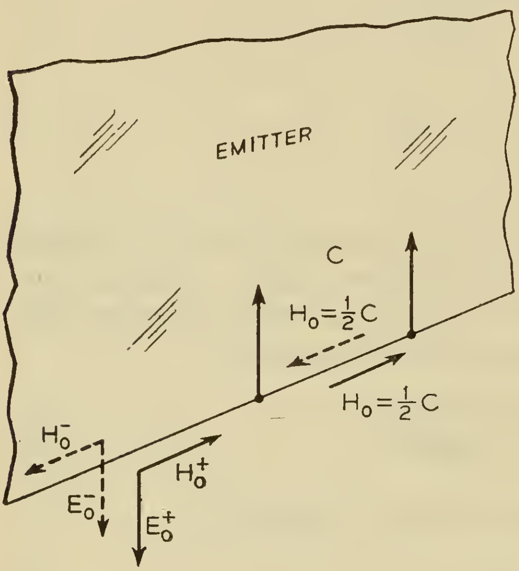


FIG. 1.32 A section of an infinitely large radiating electric current sheet.

at either face of the sheet. By the Ampère-Maxwell equation 8, this mmf should equal the encircled current C. Hence, $H_0 = \frac{1}{2}C$. If C is alternating with some frequency f, H_0 will also alternate, and these alternations will be propagated in both directions from the sheet. We shall have the usual phase retardation proportional to the distance from the sheet. In order to support the current against the reaction of the wave it originates, we have to apply an electric intensity equal and opposite to the electric intensity of the wave. Since the applied intensity must be in the direction of the current, the electric intensity E_0 of the wave must be in the direction opposite to C (Fig. 1.32). Note that on either side of the sheet the waves travel in the direction of advance of a right-handed screw whose handle is turned through 90° from E_0 to H_0. We have already noted that the ratio of E to H in a progressive wave with straight lines of force is a constant η, called the intrinsic impedance of the medium, and that in free space this constant equals approximately 120π ohms. Hence $E_0 = \frac{1}{2}\eta C$. The instantaneous work per unit area done by the applied electric intensity is $\frac{1}{2}\eta C_a^2 \cos^2(2\pi t/T)\, dt$, where C_a is the amplitude of the current density. The average of the square of the cosine is $\frac{1}{2}$; hence, the average emitted power is $\frac{1}{4}\eta C_a^2$. Half this power travels one way and half the other way. The radiation from an area S is $\frac{1}{4}\eta C_a^2 S$.

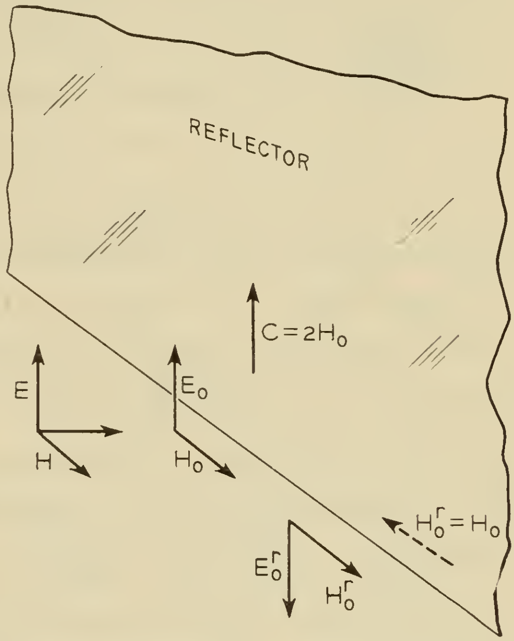


Fig. 1.33 A section of an infinitely large perfectly conducting surface serving as a reflector.

Consider now a perfectly conducting plane surface and a uniform plane wave (such as that generated by the current sheet) traveling perpendicularly to it (Fig. 1.33). The electric intensity E_0 impressed on the conductor will start a current of density C. We have just found that $E_0 = \frac{1}{2}\eta C$ so that $C = 2E_0/\eta = 2H_0$. The magnetic intensity produced by the induced current is seen to be such that behind the conductor it annihilates the magnetic intensity of the incident wave, while in front of the conductor the intensity is doubled. The annihilation of H is complete at all distances behind the conductor because the wave generated by the induced current is traveling in the same direction as the incident wave and with the same phase retardation. Hence, if

these waves annihilate each other just behind the conducting sheet, they will continue to annihilate each other at all distances. In front of the conductor the wave generated by the induced current is traveling in the direction opposite to that of the incident wave, and a standing wave is formed. Since H is maximum just in front of the plane, it vanishes at the distance $\lambda/4$ from it; E vanishes at the plane and is maximum where H vanishes.

What we have just said seems to imply that behind the conducting sheet there exist two waves, the incident wave and the wave generated by the induced current, and that these waves annihilate each other because they are of equal intensity and 180° out of phase. We have been forced to make this implication because we have been thinking in terms of steady state. If we were to consider a thin plane wavelet impinging on the conducting plane, we should come to the conclusion that the wavelet does not penetrate the plane but is merely reflected by it. Any wave is a succession of wavelets; hence, any wave is just reflected by the conducting plane. We can use this picture to obtain the intensity of the reflected wave in the steady-state case considered in the preceding paragraph. A perfect conductor is so defined that the electric intensity tangential to it vanishes. Hence, at the conducting plane the electric intensity of the reflected wave must be equal and opposite to that of the incident wave. Using the rule of the right-handed screw, we find that the magnetic intensities of the incident and reflected waves are equal at the surface of the conducting plane. This is the picture we would normally use when considering sound waves impinging on a rigid barrier. However, in making calculations it is permissible (and sometimes convenient) to consider the rigid barrier as a system of sources of sound waves which, behind the barrier, annihilate the waves incident on it. Physically this is not a natural way of thinking; but mathematically it is permissible. On the other hand, we do not have the same feeling against considering conducting reflectors as systems of secondary sources because electric currents actually flow in them and we are used to the idea that such currents always produce fields.

Let us now return to the current sheet which generates the wave and place a reflecting sheet a quarter wavelength behind it (Fig. 1.34). We have seen that H vanishes at a quarter wavelength in front of a reflecting plane. In the present arrangement this means that H vanishes just behind the generating current sheet, or the emitter, irrespective of the density of its current. Therefore, the magnetic intensity H_0 just in front of the emitter equals the current density C. Hence, $E_0 = \eta H_0 = \eta C$. The average work done by the impressed electric intensity and, hence, the power radiated per unit area is $\frac{1}{2}\eta C_a{}^2$. The power radiated by an

area S is

$$P = \tfrac{1}{2}\eta C_a{}^2 S. \tag{61}$$

The phase of the wave arriving at the reflector from the emitter is retarded by 90°, and, if it is to be annihilated in its further course by the wave generated by the current in the reflector, the current density C^r must be 180° out of phase with the incident H, that is, 90° ahead of C. Hence, in front of the emitter the waves from the emitter and reflector are in phase.

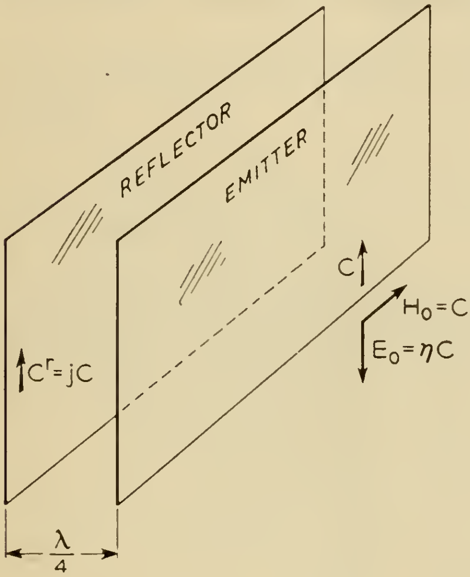


FIG. 1.34 A radiating current sheet backed by a reflector.

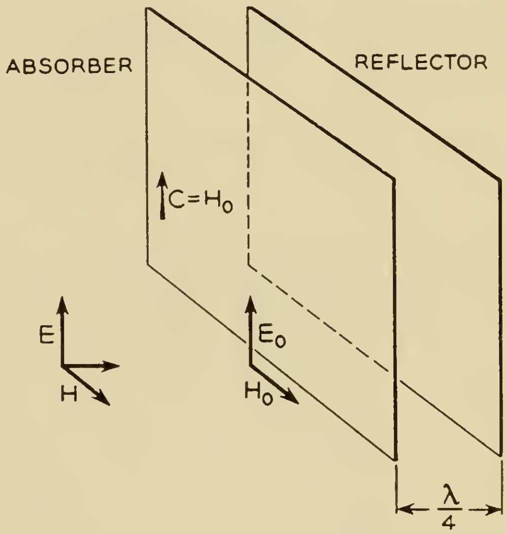


FIG. 1.35 A resistive sheet backed by a reflector and serving as a perfect absorber.

Lastly, let us consider a resistive sheet backed by a reflector (Fig. 1.35). If the distance between these sheets is $\lambda/4$, H vanishes just behind the resistor. Let the surface resistance of the front sheet be such that the incident wave is completely absorbed. In this case the current density C in the sheet must equal the incident magnetic intensity H_0. Hence, the surface resistance is $E_0/C = E_0/H_0 = \eta$ (120π ohms in free space). Such a resistive sheet absorbs all incident power, and its effective area as a receiving antenna equals its actual area.

Throughout this section we have been assuming infinitely large emitting, reflecting, and absorbing plane sheets. But we expect that the properties we have been discussing are shared by large sheets. Thus we expect that the effective area of a receiving antenna consisting of a resistive sheet with surface resistance equal to 120π ohms and a reflector behind it at a distance of a quarter wavelength will be approximately equal to its actual area. There will be an edge effect since the current perpendicular to the edge must certainly vanish, while in the ideal case

we are assuming that it is the same throughout; but the edge effect will depend on the perimeter of the resistive sheet and the main effect on the area.

1.17 Pine-tree antennas

The waves emitted by various elements of a large uniform radiating current sheet add in phase in the direction perpendicular to the sheet, and almost in phase in directions making small angles with the perpendicular. In other directions there is substantial destructive interference.

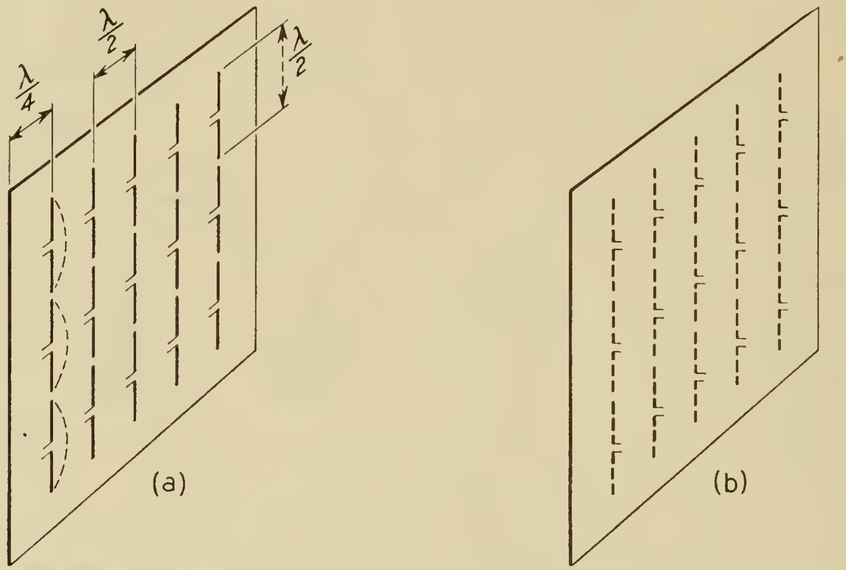


FIG. 1.36 Transmitting and receiving curtains of dipole antennas backed by reflectors, or "pine-tree" antennas.

Hence, such sheets are highly directive antennas. Practical antennas with properties similar to those of large current sheets are curtains of half-wave antennas, the so-called "pine-tree antennas" (Fig. 1.36). In these arrays, however, the distances between the feed points of the adjacent antennas (either in a row or in a column) should not greatly exceed $\lambda/2$. Otherwise the waves emitted by the individual dipole antennas will add constructively in some directions other than the perpendicular to the plane of the array, and there will be several large beams.

Let the distance between the feed points be $\lambda/2$, and imagine that the current in each dipole is spread uniformly over the area $\frac{1}{2}\lambda \times \frac{1}{2}\lambda$. If the maximum amplitude of the current in each dipole is I_0, the average value along the dipole is $(2/\pi)I_0$, and the average density over the area $(2/\pi)I_0$ divided by $\lambda/2$,

$$C = \frac{4}{\pi\lambda} I_0. \tag{62}$$

The power radiated by the area allotted to each antenna is

$$P = \tfrac{1}{2}\eta C^2 \left(\frac{\lambda}{2}\right)^2 = \frac{2\eta}{\pi^2} I_0^2. \tag{63}$$

Equating this to

$$P = \tfrac{1}{2} R I_0^2, \tag{64}$$

where R is the input resistance of the dipole, we find

$$R = \frac{4\eta}{\pi^2} = \frac{480}{\pi}. \tag{65}$$

Near the edges of the pine-tree antenna the resistance will naturally be different.

1.18 The directivity of a large radiating surface backed by a reflecting plane

To obtain the directivity of a large radiating current sheet, we have to calculate the distant field, the maximum power flow per unit area, and the average power flow. Let the radiating current sheet be in the xy

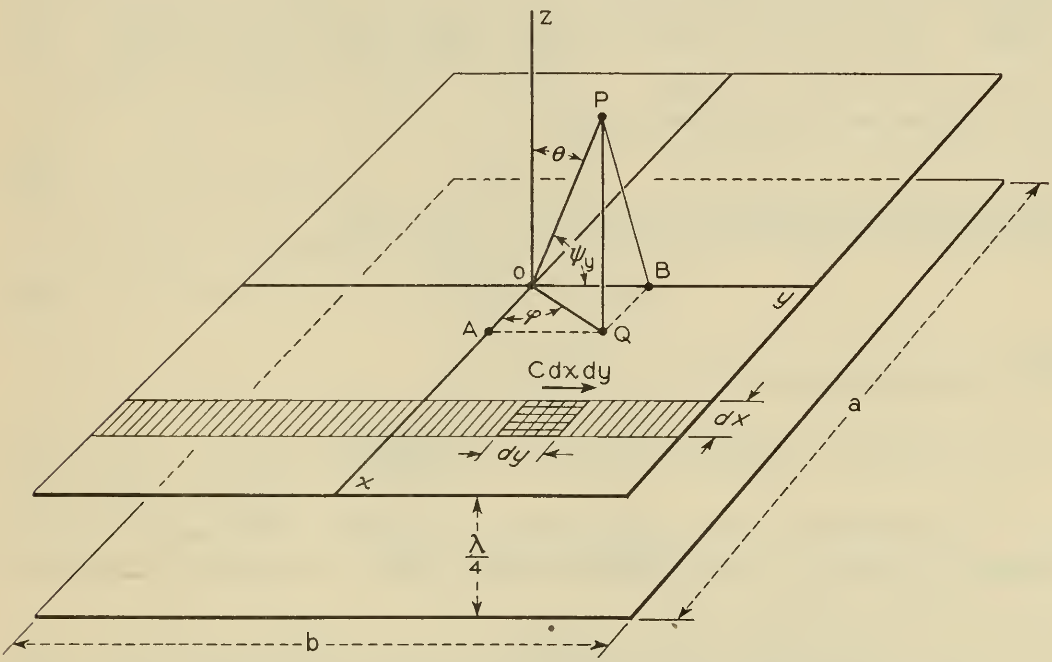


FIG. 1.37 Illustrating the calculation of the directivity of a large radiating surface.

plane (Fig. 1.37). Equation 48 gives the distant field. In this equation E_0 is the field we would have if the entire current were concentrated at the center O. If C is the current density, the total current is Ca. If we multiply this by b, we obtain the moment of the current distribution Cab. Equation 36 gives the electric intensity of a current element

of moment $I(z)\,dz$. The angle θ in this equation is the angle between a typical direction in space and the axis of the element. In the present case this is the angle ψ_y between the y axis and direction (θ, φ). Hence, if we drop the phase factor, we can write equation 48 as follows:

$$E_1 = \frac{2k\eta Cab \sin\psi_y}{\pi\lambda r} \frac{\sin[(\pi a/\lambda)\cos\psi_x]}{(\pi a/\lambda)\cos\psi_x} \frac{\sin[(\pi b/\lambda)\cos\psi_y]}{(\pi b/\lambda)\cos\psi_y}. \tag{66}$$

We are interested only in the region where E_1 is large, that is, where θ is small. Let u be the angle made by OP with the yz plane. This angle is complementary to ψ_x; therefore, $\cos\psi_x = \sin u \simeq u$. Similarly $\cos\psi_y = \sin v \simeq v$, where v is the angle made by OP with the xz plane. Hence, in the region where E_1 is significant, we have

$$E_1 = \frac{2k\eta Cab}{\pi\lambda r} \frac{\sin(\pi au/\lambda)}{\pi au/\lambda} \frac{\sin(\pi bv/\lambda)}{\pi bv/\lambda}. \tag{67}$$

The radiating current sheet and the reflector a quarter wavelength below it form an array of two sources. We have seen that the phase of the current density in the reflector is 90° ahead of that in the emitter. Hence, in the forward direction the direct and reflected waves arrive in phase. In directions making small angles with the z axis the waves arrive very nearly in phase. Thus the electric intensity produced by the emitter backed by the reflector is

$$E = 2E_1. \tag{68}$$

The maximum radiation is along the z axis, and

$$E_{\max} = \frac{4k\eta Cab}{\pi\lambda r}. \tag{69}$$

The maximum power flow per unit area is

$$W_{\max} = \frac{E_{\max}{}^2}{2\eta} = \frac{8k^2\eta C^2a^2b^2}{\pi^2\lambda^2r^2}. \tag{70}$$

To obtain the total power flow we integrate $(E^2/2\eta)\,dS$ over the sphere of radius r. In the region where u and v are small the element of area is $dS = r^2\,du\,dv$, and

$$P = \frac{8k^2\eta C^2ab}{\pi^2\lambda^2} \int_{-u_0}^{u_0} \frac{\sin^2(\pi au/\lambda)}{(\pi au/\lambda)^2}\,du \int_{-v_0}^{v_0} \frac{\sin^2(\pi bv/\lambda)}{(\pi bv/\lambda)^2}\,dv, \tag{71}$$

where the limits of integration are small and yet large enough to include most of the radiation. We now change the variables of integration. In the first integral we let $t = \pi au/\lambda$, and in the second $t = \pi bv/\lambda$. Hence,

$$P = \frac{8k^2\eta C^2ab}{\pi^4} \int_{-\pi au_0/\lambda}^{\pi au_0/\lambda} \frac{\sin^2 t}{t^2}\,dt \int_{-\pi bv_0/\lambda}^{\pi bv_0/\lambda} \frac{\sin^2 t}{t^2}\,dt. \tag{72}$$

As the size of the current sheet increases, the limits of integration tend to infinity. Hence, the limit value of P is

$$P = \frac{8k^2\eta C^2 ab}{\pi^4}\left[\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt\right]^2. \tag{73}$$

The bracketed integral equals* π; hence,

$$P = \frac{8}{\pi^2} k^2\eta C^2 ab. \tag{74}$$

Hence, the average power flow per unit area of the sphere of radius r is

$$W_{\mathrm{av}} = \frac{2k^2\eta C^2 ab}{\pi^3 r^2}. \tag{75}$$

To obtain the directivity we divide equation 70 by equation 75,

$$g = \frac{4\pi ab}{\lambda^2}. \tag{76}$$

1.19 Transmission of power between a large radiating surface and a large absorbing surface

We have seen that a large resistive sheet, backed by a reflector, is capable of receiving all power incident on it. Therefore, the effective area of our sheet as a receiving antenna is $A = ab$. Consequently, we have the following relationship between the directivity (and the signal-to-external-noise gain) and the effective area,

$$g = \frac{4\pi A}{\lambda^2}. \tag{77}$$

Hence, for large antennas of the type considered, equation 52 for the transfer of power becomes

$$P_{\mathrm{rec}} = P_{\mathrm{tr}} \frac{A_{\mathrm{tr}} A_{\mathrm{rec}}}{\lambda^2 r^2}. \tag{78}$$

This equation is symmetric in the effective areas, and we have the reciprocity theorem for transmission of power between two large antennas.

If we assume† that the reciprocity theorem applies to all antennas, then from equation 52 we find

$$g_1 A_2 = g_2 A_1, \qquad \frac{A_2}{A_1} \doteq \frac{g_2}{g_1}. \tag{79}$$

* For details of integration see Section 5.17.

† Later in our discussion we shall be able to prove it.

Hence, the *effective areas are proportional to the directivities, and equation* 77 *is general.* Therefore, the transmission formula 78 is also general.

The reader will not fail to notice that it was easy to obtain the first transmission formula 52 and rather difficult to obtain the second formula 78. The reason is that the directivity is a natural concept for transmitting antennas and the effective area is natural for receiving antennas. In the microwave antenna art, however, antennas are large, and the second formula is more useful. One might even say that it is as useful as Ohm's law in circuit engineering.

For an isotropic radiator $g = 1$, and from equation 77 we obtain its effective area,

$$A_0 = \frac{\lambda^2}{4\pi} \cdot \tag{80}$$

Having made the foregoing calculations, we find ourselves able to determine the unknown constant of proportionality k in the expression for the distant field of an electric current element. Thus, the power radiated by the current sheet is given by equation 74. It is also given by equation 61, where in the present notation $C_a = C$. Hence,

$$k = \tfrac{1}{4}\pi. \tag{81}$$

Had we known this value in the first place, we could have obtained the directivity (equation 76) more simply from equations 61 and 69. The latter equation can be derived without obtaining the complete radiation pattern (equation 67), since in the forward direction the fields of all current elements add in phase and thus act as if they were all in the center of the current sheet. But we know of no other methods of obtaining k except that above and that of solving Maxwell's equations (see Chapter 4). It is really remarkable that so many results are obtainable from elementary physical ideas and from mathematical calculations requiring only relatively simple integration. It should be noted, however, that these results have been obtained from physical considerations expressed by Maxwell's equations. In a sense we found partial solutions of these equations by direct reasoning rather than by formal mathematical manipulation. Some of our conclusions depend on general properties of waves (of any kind) and thus should be obtainable without any reference to Maxwell's equations. We shall consider these properties in Section 1.22.

1.20 Magnetic current sheets

An infinitely large uniform electric current sheet generates uniform plane waves traveling perpendicularly to the sheet. Let us now take two such

sheets (Fig. 1.38) with equal and opposite currents. If the distance s is very small compared with λ, the magnetic intensity between the sheets is distributed uniformly. Each current sheet contributes $\frac{1}{2}C$ to the magnetic intensity; hence, $H = C$. The magnetic flux density is μH, where μ is the permeability, and the magnetic flux per unit length along the vertical edges in Fig. 1.38 is μHs. The time rate of change of this flux we shall call the density M of the magnetic current. Hence, $M = j\omega\mu Hs$. Since $H = C$, we have $M = j\omega\mu Cs$. Let us assume that

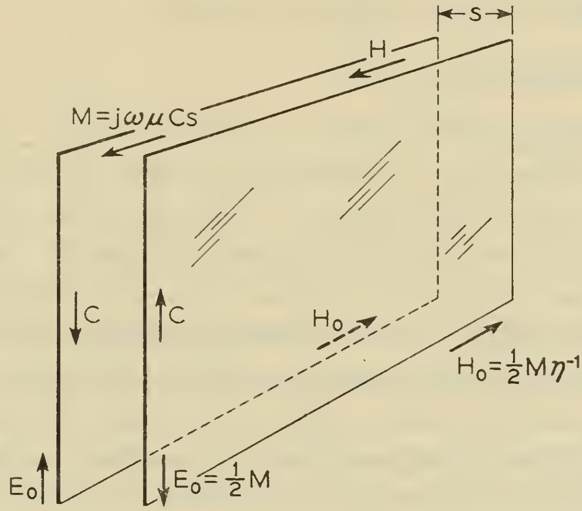


FIG. 1.38 A double electric current sheet or a simple magnetic current sheet.

s is infinitely small and C infinitely large while the product Cs is finite. According to the Faraday-Maxwell equation 7, the electromotive force around a narrow rectangular circuit encircling the magnetic current M, as we have just defined it, is M. Therefore, the electric intensities at the external faces of the double electric current sheet are oppositely directed but equal in magnitude to $E_0 = \frac{1}{2}M$. The magnetic intensities are similarly directed and are equal to $H_0 = E_0/\eta = M/2\eta$.

The magnetic current between the electric current sheets has the same direction everywhere. It is analogous to a plane electric current sheet. That is, a double electric current sheet may be considered as a simple magnetic current sheet. The concept of magnetic current sheet is particularly convenient because its ability to generate waves depends not only on the electric current in the current sheets but on several other factors; it depends, in fact, on the voltage required to produce the given time rate of change of magnetic flux M.

Comparing simple electric and magnetic current sheets, we find that in passing across the former the electric intensity is continuous but the

magnetic intensity reverses its direction, while in passing across the latter the electric intensity reverses its direction but the magnetic intensity is continuous. Hence, if we combine two such sheets, we can reduce the field on one side to zero. We have only to choose C and M perpendicular to each other and make $M = \eta C$. Then on one side we shall have $E_0 = M$, $H_0 = C$, and on the other $E_0 = H_0 = 0$. The field-free half-space is indicated by the direction in which a right-handed screw will advance if its handle is turned through 90° from M to C. We shall leave it to the reader to verify these conclusions by superposing the fields generated by the two current sheets.

1.21 Wave propagation

If we drop a pebble in the center of a large, shallow pool of quiescent water, we shall observe an expanding circular wavelet long after the pebble comes to rest on the bottom. The size of the pebble determines the intensity of the wavelet; but, once the pebble comes to rest, all further connection between it and the wavelet is terminated. We can forget about the pebble. Now, if we start dropping pebbles in rapid succession, our attention is once more drawn to them as the source of waves. In the past the radio engineer has been thinking so habitually in terms of static fields and steady-state oscillations that he naturally came to feel that electromagnetic waves were " attached " to the charges and currents producing them. He has tended to think more about the charges and currents than about the waves. It was only with the coming of pulse techniques that the waves began to attract his attention. And in designing microwave antennas it is often preferable to think more about the waves than about the charges and currents producing them.

From this point of view a perfectly conducting plane is a barrier to the forward progress of waves impinging on it. Reflected electromagnetic waves originate at this barrier as reflected water waves do at a rigid boundary. A perfectly conducting plane is a convenient abstraction approximating a metal plate. There is a great difference between waves in free space (and other perfect dielectrics) and waves in media of high conductivity. In free space the ratio of the electric intensity to the magnetic is large; for waves at large distances from conductors the ratio is $120\pi = 377$ ohms. On the other hand, in good conductors a weak electric intensity produces a strong current and, hence, a strong magnetic field. The conductivity of copper is 5.8×10^7 mhos per meter. Hence, one volt produces a current density of 5.8×10^7 amperes per square meter. In a plate one millimeter thick the linear current density produced by one volt is 5.8×10^4 amperes per meter. The magnetic intensity on each side of such a plate is half this value. The

intensity of one ampere per meter will thus be produced by about a third of 10^{-4} volt. This is to be compared with 377 volts in free space. This disparity in the relative magnitudes of E/H ratios causes the reflection of waves from a metal plate (Fig. 1.39*a*). At the surface of the plate the electric intensity of the reflected wave must be opposite to that of

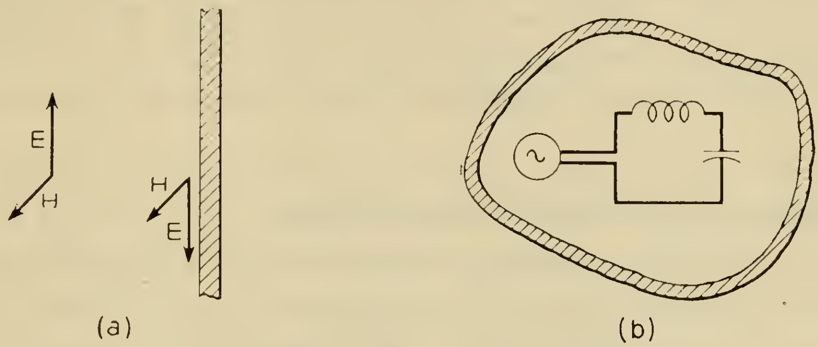


Fig. 1.39 Shielding effect of metals.

the incident wave and must be nearly equal to it in order that the ratio of the total E to the total H be consistent with the small value in the conductor.

A very small fraction of the incident energy enters the conducting plate. As it progresses further into the plate, this energy is transformed into heat, and a very insignificant amount is left for generating waves on

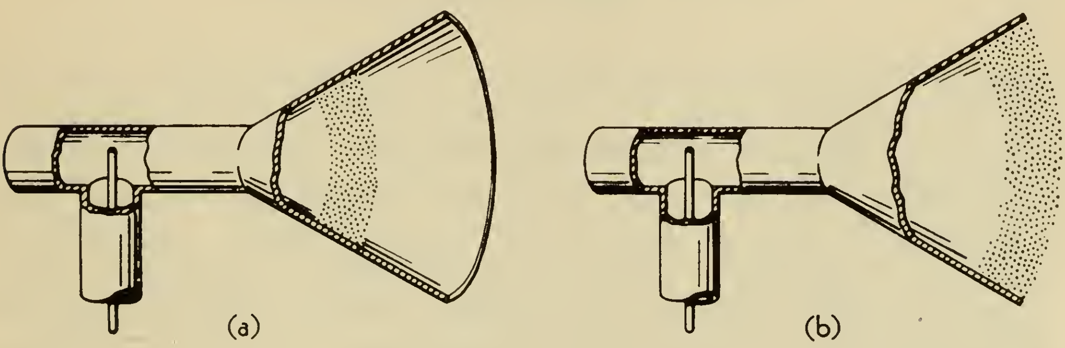


Fig. 1.40 A pulse traveling through a shielded transmission system can reach the external space only through an open aperture.

the opposite side of the plate. This is the shielding principle familiar to circuit engineers. Of course, once we have calculated the electric currents in the shield (Fig. 1.39*b*), we can "explain" the weak field outside the shield as due to the neutralization of the original field of the circuit by the induced currents. But, in general, there is no way of determining the induced currents except by following mathematically the progress

of the wave from its source through the successive parts of the surrounding media.

Thus, as we follow the progress of an electromagnetic pulse through the transmission system into a horn (Fig. 1.40), we conclude that the energy escapes through the aperture of the horn and not through the walls. The field in the aperture excites the waves in the external space. In Chapter 16 we shall consider more thoroughly the mechanism of wave propagation through apertures. However, in the important practical case of large apertures we need not know this mechanism precisely.

1.22 Radiation through large apertures

Suppose that we have a source of power P_1 and a means of expanding the wavefront over an area A_1 in such a way that the electric intensity has the same magnitude and phase at all points of this area just before the wave escapes into the unlimited medium (Fig. 1.41). Huygens

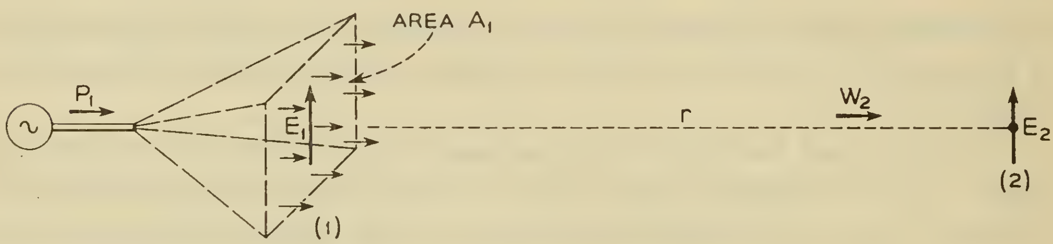


FIG. 1.41 Radiation from an aperture.

originated the idea that each element of the wavefront of any wave acts as a new source of waves. In our case it is the electric and magnetic forces in each element of the aperture that excite waves in the surrounding medium. The wave from each point source is spherical, but there is no reason to suppose that the pattern of radiation is uniform in all directions. Since waves tend to progress forward, the pattern of each elementary source must be heart-shaped. If the aperture is large, its radiation pattern is determined primarily by the interference pattern of waves arriving from different points in the aperture and not by the pattern of the elementary source. This pattern is given by the factors depending on ψ_a and ψ_b in equation 48 or by the corresponding factors depending on u and v in equation 67. The power flow per unit area at distance r is proportional to the square of the electric intensity; hence,

$$W = W_2 \frac{\sin^2(\pi a u/\lambda)\,\sin^2(\pi b v/\lambda)}{(\pi a u/\lambda)^2(\pi b v/\lambda)^2}, \tag{82}$$

where u and v are the angles made with the edges of the aperture and W_2

is the power flow on the axis, where we have a receiving antenna. The total power flow is

$$P_1 = \iint Wr^2\,du\,dv = \frac{\lambda^2 r^2 W_2}{\pi^2 ab} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx, \quad (83)$$

where the infinite limits of integration have been substituted for very large limits. The value of each integral is π, and

$$P_1 = \frac{\lambda^2 r^2 W_2}{ab} = \frac{\lambda^2 r^2 W_2}{A_1}. \quad (84)$$

If the aperture of the receiving antenna is also large and such that it permits a uniform inphase distribution of E, all power incident on the aperture is directed to the load, and the received power is

$$P_2 = A_2 W_2. \quad (85)$$

Hence,

$$\frac{P_2}{P_1} = \frac{A_1 A_2}{\lambda^2 r^2}. \quad (86)$$

To obtain the power flow W_2 from an isotropic radiator we must supply to it the power

$$P_0 = 4\pi r^2 W_2. \quad (87)$$

Hence, the gain in power of a large aperture is

$$g = \frac{P_0}{P_1} = \frac{4\pi A_1}{\lambda^2}. \quad (88)$$

The ideal large-aperture antenna of the type assumed in the preceding derivation may be closely approximated by a sector of a large biconical antenna (Fig. 1.42*a*), with a lens (Fig. 1.42*b*), producing a uniform phase distribution. The magnitude of E over the aperture is constant without the lens, but the wavefront is curved, and, unless it is straightened, there will be a reduction in the power flow W_2 at the receiving antenna and a consequent decrease in the effective area of the aperture.

Another possible cause for the decrease in the effective area of the aperture is a nonuniformity of distribution of the aperture field as in a pyramidal horn with four metal faces. At a metal surface the tangential electric intensity must be very small; hence, there is little radiation through parts of the aperture in the vicinity of the edges parallel to E. In the case of a dominant wave in the horn, the aperture field is distributed uniformly in one direction, sinusoidally in the other. Since the average value of one half-cycle of a sine wave equals $2/\pi$ times its maxi-

mum value, the distant electric intensity on the axis of such a horn with a correcting lens is $2/\pi$ times that of an ideal aperture with uniform illumination. The power flow per unit area is $(2/\pi)^2$ times the power

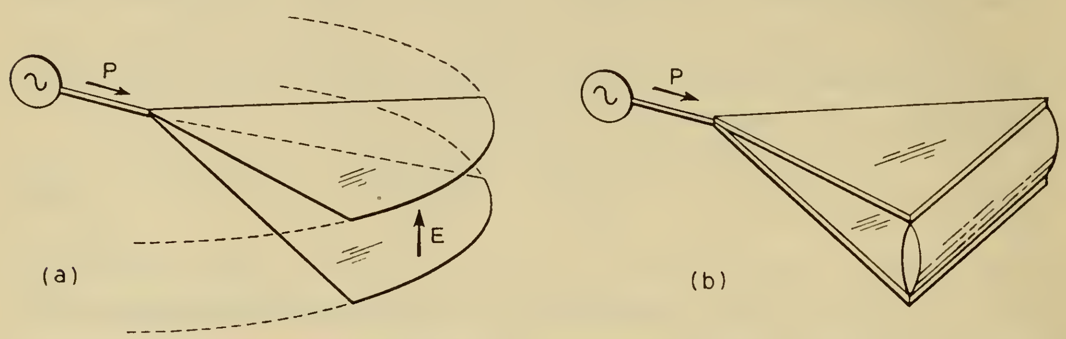


FIG. 1.42 An antenna with a uniform distribution of E over the aperture is obtained if we (*a*) take a sector of a large biconical horn and (*b*) insert a lens to straighten the wavefront.

flow from the uniform aperture. On the other hand, the power flow through the aperture is proportional to the square of the sine, and thus is half of the flow through the uniformly illuminated aperture. Hence, the effective area is only $8/\pi^2$ times the actual area.

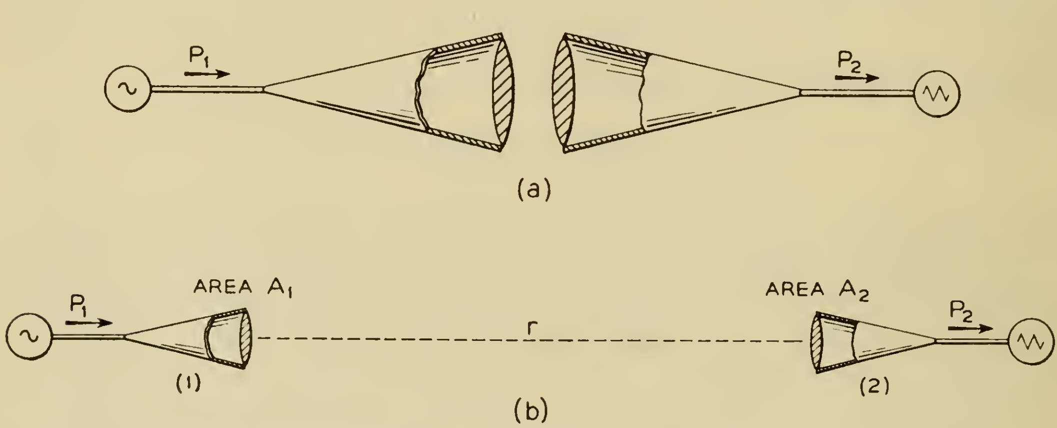


FIG. 1.43 Two ideal antennas: (*a*) facing each other; (*b*) far apart.

Let us compare the transmission of power between two ideal antennas facing each other (Fig. 1.43*a*), with that between a well-separated pair (Fig. 1.43*b*). In the vicinity of the aperture the wavefront is substantially plane; hence, in the first case most of the power will pass into the second aperture. A transmission loss of less than 1 db has been measured for large horn-reflector antennas facing each other. The reader may easily construct a graphic picture of a change from a plane wavefront at the aperture to a spherical front at large distances by taking a segment of a straight line and drawing circles of ever-increasing

radius with the centers at closely spaced points on the line. Thus for large distances between two antennas in free space the transfer of power becomes inversely proportional to the square of the distance. When the antennas are near the earth, reflections from it may greatly increase the transfer loss.

1.23 Parabolic reflectors

Parabolas have a property useful for directive radiation. The distance from the focus F of a parabola (Fig. 1.44) to a point on the parabola and from there to a perpendicular to the axis of the parabola is the same for all points on the parabola. Thus $FP + PQ = FP' + P'Q'$. Suppose now that we place a point source of radiation at the focus of a large paraboloidal conducting surface (or a line source along the focal line of a large paraboloidal cylinder). The impressed field will induce electric currents in the surface in proportion to the intensity of the field at all points except near the edge. In view of the above property of the parabola the secondary waves excited by the induced currents will add in phase at any distant point in the direction of the axis AB of the parabola. In other directions the secondary waves from the various elements of the paraboloidal reflector tend to interfere with each other destructively. Thus we can obtain directive properties comparable to those of a horn with the same aperture. There will be some loss in directivity because the " illumination " of the reflector is not uniform. To prevent interference between the secondary waves and the primary wave a small reflector may be placed behind the primary source.

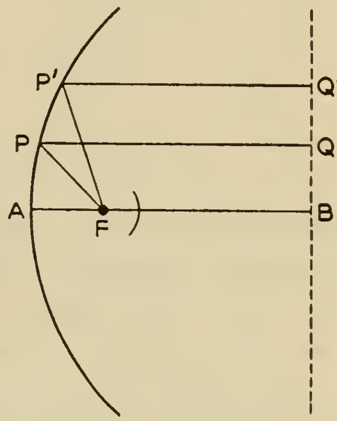


FIG. 1.44 A parabolic (or paraboloidal) reflector.

1.24 Lenses

The speed of electromagnetic waves in solid dielectrics is smaller than their speed in air. This property may be utilized to increase the directivity of a relatively nondirective source by placing a *lens* in front of it (Fig. 1.45). When a wave travels rapidly, the distance from crest to crest is large. If this wave enters a medium that forces it to move more slowly, the distance from crest to crest is diminished. Hence, in this medium the phase change in the direction of propagation is more rapid than in the medium with higher speed of propagation. Consider now a symmetric lens and a source F on its axis. We can make the total

phase change along a path $FPQR$ from the source to a distant point R equal to the phase change along the axis $FABC$ by properly shaping the lens. Since $FA + BC < FP + QR$, the phase change along $FA + BC$ is smaller than it is along $FP + QR$. We compensate for this by making $PQ < AB$, where PQ and AB are the distances traversed by the waves through the medium of more rapid phase change.

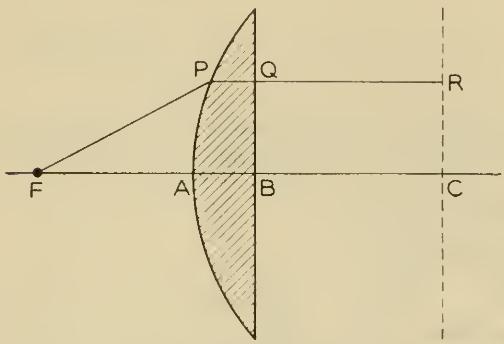


Fig. 1.45 A lens.

1.25 Wide-band linear antennas

Figure 1.25 shows approximate current distributions in thin wires. If the total length of the antenna is $\lambda/2$, the current is large at the input point, and the input impedance must be relatively small. We can calculate the radiation field as explained in Section 1.12 and obtain the power flow per unit area and the total radiated power. This power is proportional to the square of the input current. From the coefficient of proportionality we find the input resistance. To the extent to which the sinusoidal form of current distribution is a good approximation to the actual form, the input resistance of the half-wave antenna is independent of the radius of the antenna. Calculations show that this resistance equals 73 ohms.

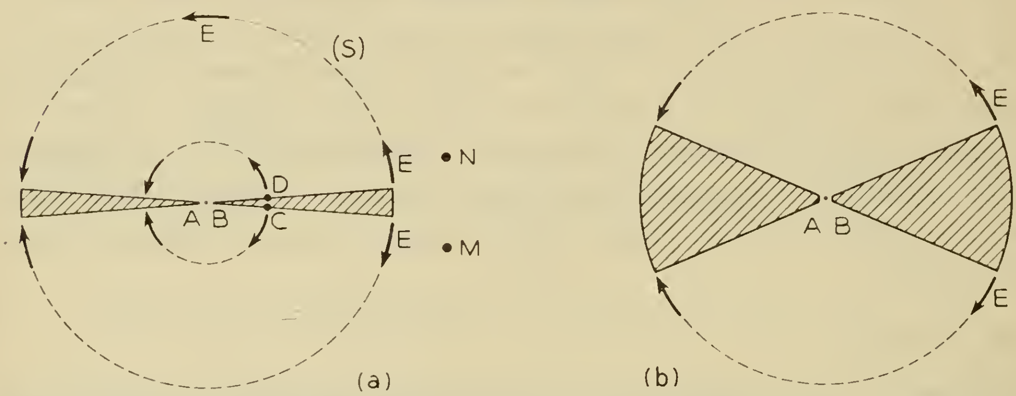


Fig. 1.46 Two biconical antennas.

To the present order of approximation the input current in a full-wave antenna is zero, and its impedance is infinite. This indicates that the actual input impedance is large. We can obtain a general idea of its dependence on the radius from the following considerations. Let us take a thin biconical antenna (Fig. 1.46a). Initially, as the waves spread from the input point, the electric lines are circles. The maximum

electric intensity is at the surface of the antenna and is normal to it. At points C, D on the opposite sides of a cone, the electric intensities are in opposite directions. However, they are shielded from each other by the cone, and they excite a strong wave in the forward direction. When the cones are terminated, the shielding is withdrawn, and at points such as M, N the wave will be weak. Only in the equatorial plane the wave beyond the spherical surface S will be comparable in intensity to that inside S. But in this plane E is not as strong as it is near the cones. If we make the cones thinner and keep the same voltage along the meridians, the greater part of this voltage will be concentrated near the cones where it is ineffective for exciting a wave in the external space. Thus less power will be radiated, and the input impedance must be greater.

Thus, while the resonant impedance does not vary much with the change in the antenna radius, the antiresonant impedance increases indefinitely as the radius approaches zero. Hence, as the frequency of oscillations is varied, impedance fluctuations in a thin antenna are greater than in a fat antenna. In a "wide-band" antenna impedance fluctuations must be relatively small, and the cone angles must be large (Fig. 1.46*b*).

1.26 Antennas for various purposes and in various frequency ranges

The earliest antenna, used by Hertz to confirm Maxwell's theory and the existence of electromagnetic waves predicted by it, consisted of two rods, each about 30 cm long, connected to two plates about 40 cm square (Fig. 1.47). The oscillations were excited by connecting the rods to an induction coil and utilizing the spark discharge between the spheres A, B at the ends of the rods. Later Hertz devised a system, shown in Fig. 1.48, in which the dipole antennas were placed along the focal lines of parabolic reflectors about 2 meters in height. The transmitting antenna on the right consisted of two cylinders, 12 cm long and 3 cm in diameter. The arms of the receiving antenna were wires 50 cm long. The frequencies of the oscillations generated by Hertz were very high, and the waves were short. Hence, he was able to demonstrate reflection and interference of electromagnetic waves within the confines of a large room. He produced standing waves and could measure the distance between the adjacent nodes in the field intensity.

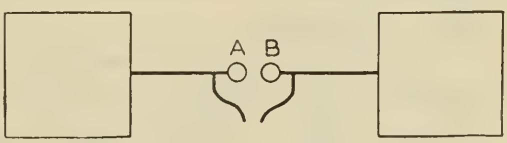


FIG. 1.47 A dipole antenna.

Much lower frequencies were originally used for long-distance radio

communication. For instance, Marconi used waves 5000 and 7000 meters long in his first experiments with transmission of signals across the Atlantic Ocean. His vertical antennas had to be very short compared with the wavelength. Horizontal antennas seemed to be impracticable, since the earth currents are in opposite direction to the current in a horizontal wire and thus tend to neutralize the field of the wire. We have already seen that short antennas are inefficient because of the relatively large heat losses. There are still greater heat losses due to earth currents, since the conductivities of soils are poor. This led to problems of improving ground conditions by means of buried wires. Additional advantages were secured by using multiple radiators (Fig. 1.49*a*), and thus reducing the density of the ground currents. Even at higher frequencies short antennas have to be used whenever space is limited, as in ships, airplanes, railroad cars, apartments, etc. Chapter 10 will be devoted entirely to short antennas.

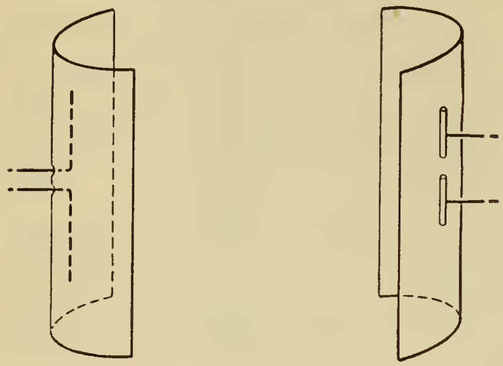


FIG. 1.48. Dipole antennas with parabolic reflectors.

The frequency range allotted to domestic broadcasting stations using amplitude modulation is between 500 and 1600 kc/sec. In round figures, λ is between 200 and 600 meters. Antennas become towers (Fig. 1.49*b*), sometimes self-supporting and sometimes not. Some of these towers are still short (in the electrical sense), whereas others are comparable to $\lambda/4$ and even to $\lambda/2$. Chapters 11, 12, and 13 are devoted to such antennas.

If the earth were perfectly conducting, horizontal antennas at heights small compared with $\lambda/10$ would be impracticable, irrespective of their length. Over a poor ground, however, there exists a fairly substantial horizontal component of E which can induce currents in wires parallel to the ground. These currents can be made to add in phase in the load. The signal will then increase with the length of the wire. A *wave antenna* (Fig. 1.49*c*) operates on this principle.

At frequencies so high that it is practicable to construct antennas with dimensions comparable to and even greater than the wavelength, the variety of antenna types is restricted only by the engineer's imagination. Horizontal antennas become useful because they can be elevated and thus freed from the influence of the ground. There are two main types of short-wave antennas: (1) *omnidirectional antennas* which

radiate equally in all horizontal directions and are used for broadcast purposes and (2) *beam antennas* for point-to-point communication. Figure 1.49*d* shows an omnidirectional loop antenna. The loop is energized at four equidistant points in order to make the current dis-

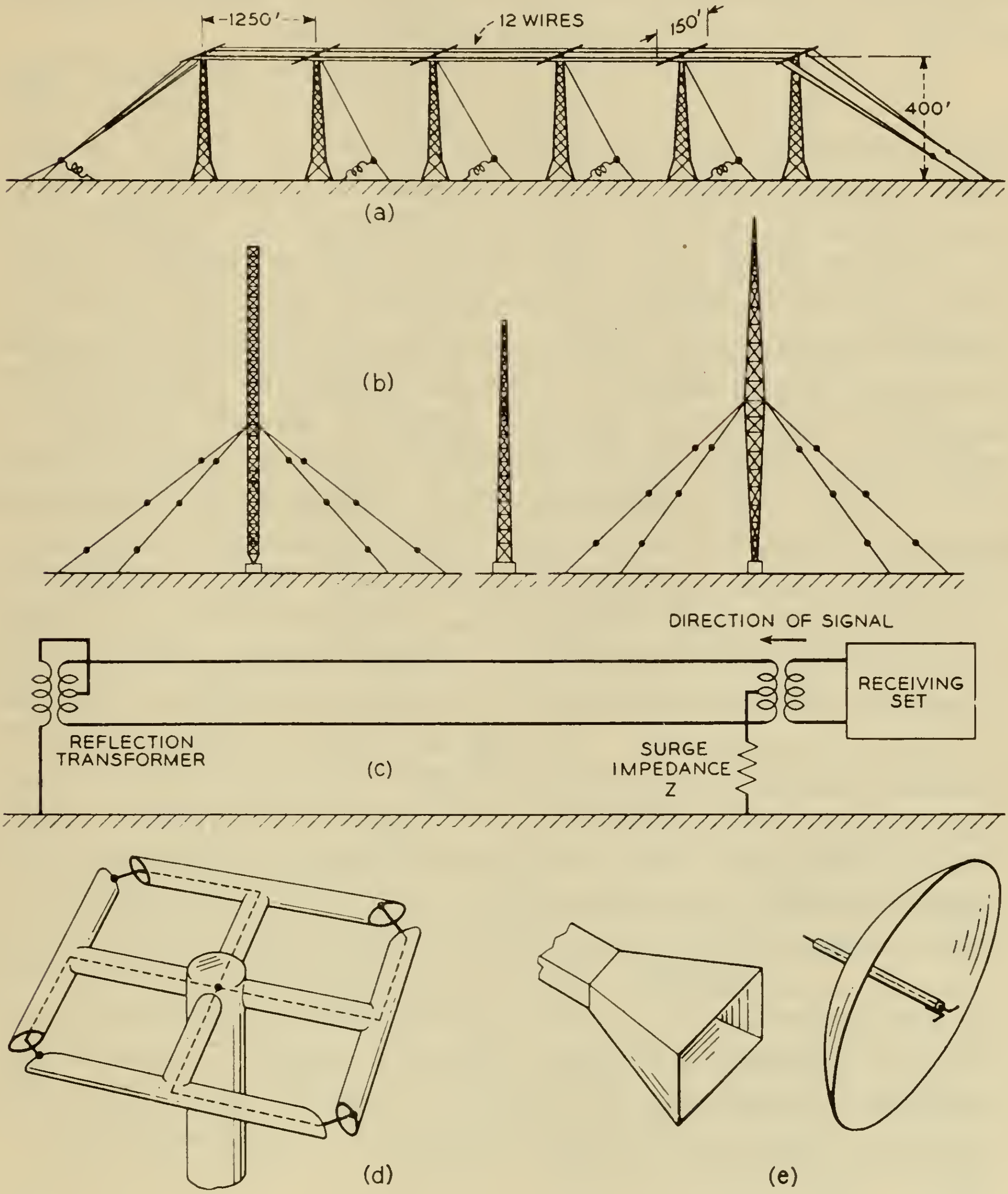


FIG. 1.49 Representative antennas.

tribution more nearly uniform round the loop and thus maintain the omnidirectional property even for a fairly large loop.* Figure 1.49*e* shows two examples of highly directional antennas: a horn and a parabolic reflector. The horn is analogous to a square curtain of dipole

* A small loop in a horizontal plane is always omnidirectional.

radiators operating in phase, whose fields add in the forward direction while interfering destructively in the directions sufficiently off this particular direction. The underlying principle is that each element of the wave in the aperture of the horn acts as a source of waves beyond the aperture. This idea was first put forward by Huygens as a physical principle; later it was substantiated mathematically.

Short-wave omnidirectional antennas may be stacked vertically in order to increase the field near the ground and conserve power. Figure 1.49 shows just a few antennas to exhibit the variety of types. These and many other antennas are discussed in Chapters 8 to 19, some in more detail than others. Space does not permit our consideration of all antennas that have been proposed at one time or another, but a sufficient number of examples will be analyzed in this book to enable the reader to understand the general principles and to appraise any antenna that he may design himself or encounter in periodic literature or in a handbook. There is no essential difference between antennas for amplitude-modulated waves and those for frequency-modulated waves, or for ordinary voice broadcasts and for television. The operational frequency, or rather the wavelength, is the most important factor affecting the practicability of a given antenna type. If $f = 1{,}000{,}000$ cps, $\lambda = 300$ meters; hence, horns, whose apertures must equal several square wavelengths in order to be effective, are not practical. The purpose for which a given antenna is to be used determines the required directional characteristics. Generally the radiation pattern should be omnidirectional in broadcasting and highly directional in point-to-point communication and should possess a sharp null in direction finding. However, the nature of the terrain affects the radiation pattern, and the radiation characteristics of the antenna may have to be made nonuniform in order to obtain a more uniform distribution of signal strength at some distance from the antenna. And, if a broadcast antenna is too far off center of the area to be covered, it may be desirable to make its radiation pattern nonuniform for the most effective coverage. In some cases, a nonuniform pattern may be required in order to reduce interference between two stations on the same frequency serving adjacent areas.

1.27 Earth and atmosphere

The earth and the atmosphere modify the performance of antennas. At large distances from a tower antenna the signal on the surface of the earth is much weaker than it would be in free space at the same distance, or in the ideal case of a perfectly conducting plane earth. The narrow beams of highly directive microwave antennas may be bent in passing through the atmosphere and may miss the receiving antenna. This

might not be very serious if the atmosphere were stable; but the atmospheric characteristics vary and thus impose an upper limit on the directivity of practicable antennas. For simplicity we have to design antennas on the assumption that they are in free space; then we have to evaluate the effects of the earth and the atmosphere and either allow for them or try to remedy them.

1.28 Theory and practice

How much theory does one need to design a practical antenna? Sometimes very little and sometimes a great deal. A city apartment dweller who is seeking a private antenna for reception in the standard broadcast range of frequencies (550 to 1500 kc/sec) has a simple problem. He should know that in this frequency range the voltage picked up by a wire antenna is nearly proportional to its length. Practical considerations and diminishing returns impose a limit on the length. It is much more sensible to determine this length, about 30 ft, experimentally than to try to calculate it. This is an example in which the experiment is simple and cheap while the calculations are impossible.

On the other hand, theory will save money in the design of broadcast towers. It may not be complete enough to give all the required information, and it may have to be supplemented by model experiments, but its value is unquestionable. Theory is needed in planning good experiments, and it can be equally useful in discouraging other types of experiment which are bound to fail.

It is difficult to overstress the importance of good judgment in theory as well as in practice. Good judgment is obviously needed in making idealizations, simplifications, and physical approximations without sacrificing the practical value of the conclusions. Wrong conclusions have sometimes been drawn even from meticulously rigorous mathematical analysis solely from a lack of good judgment.

The principal problems before us are: (1) to develop basic field concepts, (2) to obtain equations for calculating fields and to examine their principal implications, (3) to derive the expressions for the electric and magnetic intensity of the wave excited by an element of electric current in free space, (4) to examine directive effects of spatial arrangements of current elements, (5) to obtain formulas for transmission of power between antennas, (6) to derive current distributions in antennas with given connections to generators, (7) to examine the reciprocity between transmitting and receiving antennas, (8) to evaluate antenna impedances, and (9) to apply the general principles to selected examples of practical antennas.

In this book we consider these problems in the following order. In

Chapter 2 we develop the essentials of field theory and explain Maxwell's equations. In Chapter 3 we apply this theory to plane waves. In Chapter 4 we consider spherical waves on wires and in free space; there we derive the most basic formula of all: the formula for the field of a differential current element. In Chapter 5 we present the principle of directive radiation and its application to antenna arrays and also two methods for calculating the radiated power when the current distribution is given. In Chapter 6 we consider the problem of transmission of power between two antennas. In Chapter 7 we discuss the basic quantitative theory of wave propagation and reflection; we apply this theory to the problem of reflection of radio waves from a hypothetical plane earth. In Chapter 8 we discuss the current distribution in thin antennas: a simple sinusoidal approximation to it, various factors affecting it, and the relative importance of these factors in calculating radiation patterns, radiated power, and impedance. Chapter 9 is devoted to those aspects of antennas in which they are merely "concealed electric circuits," in fact, to those aspects in which antennas are merely linear dynamical systems and in which their behavior is independent of Maxwell's equations. In Chapter 10 we consider short antennas and in Chapter 11 self-resonant antennas. Chapters 12 and 13 present an elementary theory of more general linear antennas and some conclusions of the more advanced theory. The following chapters deal successively with rhombic antennas, miscellaneous systems of linear antennas, horns, slot antennas, reflectors, and lenses.

PROBLEMS

Elsewhere in this book the answers to the problems are given as an aid to the student and for future reference. However, some of the questions asked below would lose their point if the student could see the answers. These questions are intended to stimulate the student to think broadly and to encourage him in his efforts to obtain the maximum information from incomplete data. The questions are answered later in the book; but the student is asked to forbear the temptation of looking at the answers until he feels that he has done all he can.

1.7–1 How does the radial component of E vary with the angle θ between the radius and the short antenna?

1.7–2 Discuss the waves excited by an alternating current in a small loop.

1.7–3 Obtain as complete an expression as possible for the magnetic intensity of the field at large distances from a small, single-turn loop of area S, carrying an alternating current of amplitude I.

1.7–4 How does the field of a small loop depend on the number of turns?

1.7–5 Consider two identical short antennas forming a cross. Assume that the current amplitudes are equal and that the antennas are operated in phase. What is the electric intensity along the line perpendicular to the antennas?

1.7–6 What is the answer to the preceding problem if the antennas are operated in quadrature? What is the locus of the ends of the electric vector?

1.7–7 What are the answers to the questions in the preceding problem if the amplitudes of the antenna currents are unequal?

1.8–1 What is the electric intensity at large distances from a small loop?

1.8–2 Calculate the power radiated by a small loop of area S having n turns.

1.9–1 What are the factors affecting the effectiveness of a small loop as an antenna?

1.10–1 Obtain the radiation resistance of a small loop.

1.13–1 Consider two identical vertical antennas, distance $\lambda/2$ apart. Draw a polar diagram showing the variation in the electric intensity in the ground plane on the assumption that the antenna currents are equal.

1.13–2 Solve the preceding problem on the assumption that the antenna currents are equal in amplitude but 180° out of phase.

1.13–3 Solve Problem 1.13–1 on the assumption that the distance between the antennas is $\lambda/4$ and that the magnitudes of the antenna currents are equal while the phases differ by 90°.

1.13–4 What is the directivity of a small loop? *Ans.* 1.5.

1.13–5 What is the directivity of a hypothetical antenna that radiates uniformly in all directions between the cones making 45° angles with the axis of the antenna and does not radiate in other directions? *Ans.* $\sqrt{2}$.

1.13–6 Calculate the power radiated by a current element of moment $I\,dz$.
Ans. $P = 640k^2(I\,dz/\lambda)^2$, where k has the same value as in equations 17 and 36.

1.13–7 Consider an antenna of length $\lambda/2$, and assume that the current distribution is sinusoidal. Let the maximum amplitude be I_0. Make a simple approximation with regard to the pattern of radiation and find the radiated power and the radiation resistance.

Ans. $$P = \frac{640k^2I_0^2}{\pi^2}, \qquad R_{\text{rad}} = \frac{1280k^2}{\pi^2}.$$

1.13–8 Consider two colinear current elements, that is, elements of the same straight line, distance d apart. Find the ratio of the power radiated by them to the power radiated by an isolated element.
Ans.

$$6\int_0^{\pi/2} \cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin^3\theta\,d\theta = 2 + 6\left(\frac{\lambda}{2\pi d}\right)^2\left[\frac{\sin(2\pi d/\lambda)}{2\pi d/\lambda} - \cos\frac{2\pi d}{\lambda}\right].$$

Note that the second term represents the ratio of the mutual radiated power to the power radiated by an isolated element.

1.13–9 Consider a full-wave antenna, and assume the sinusoidal current distribution shown in Fig. 1.25*d*. Use the results of the two preceding problems to obtain the radiated power. *Ans.* $1670k^2I_0^2/\pi^2$.

1.13–10 Consider an array of n identical, short, vertical antennas, arranged on a horizontal line distance l apart. If the phases of the currents in the successive antennas from left to right are 0, $2\pi l/\lambda$, $4\pi l/\lambda$, $\cdots$, $2(n-1)\pi l/\lambda$, then the waves

from these antennas reinforce each other in the direction of the array (from left to right). If E_0 is the field at some distant point from one antenna, nE_0 is the field of the entire array. At some angle ψ with the array the field is $E_0 + E_0\underline{/\vartheta} + E_0\underline{/2\vartheta} + \ldots + E_0\underline{/(n-1)\vartheta}$, where $\vartheta = (2\pi l/\lambda)(\cos\psi - 1)$.

Suppose now that the antennas are fed by means of a transmission line loaded with series capacitance so that the wave velocity along the line is higher than that of light. Let λ_1 be the wavelength along the line so that the phase retardation from one antenna to the next is $2\pi l/\lambda_1$. Find the angle ψ between the line of the array and the direction in which the waves arrive in phase.

Ans.
$$\psi = \cos^{-1}\frac{\lambda}{\lambda_1}.$$

1.13–11 Equation 42 gives the radiation intensity of two identical short antennas, distance a apart, carrying equal currents in phase in the equatorial plane. Show that if each antenna is replaced by a directive radiator whose field intensity is $E(\psi)$, the field intensity of the pair is $2E(\psi)\cos[(\pi a/\lambda)\cos\psi]$. Use this result to show that the field intensity of three equidistant short antennas on the same straight line, carrying currents in phase but with their amplitudes in the ratio 1, 2, 1, is $\{2E_0\cos[(\pi a/\lambda)\cos\psi]\}^2$ where E_0 is the field of either of the end antennas. Extend this formula to four antennas with the amplitudes of the currents proportional to 1, 3, 3, 1; then to five antennas with the amplitudes of the currents proportional to 1, 4, 6, 4, 1.

1.13–12 Show that for an antenna extending from $z = -l$ to $z = l$ equation 39 may be reduced to

$$E_\theta = k\frac{2\eta}{\pi\lambda r_0}\sum_{n=0}^{\infty}(-)^n A_n\frac{(2\pi l/\lambda)^{2n}}{(2n)!}\cos^{2n}\theta\sin\theta,$$

$$A_n = \int_{-l}^{l}\left(\frac{z}{l}\right)^{2n} I(z)\,dz.$$

1.13–13 Calculate the first two terms of the series in the preceding problem for the half-wave antenna shown in Fig. 1.25*b*.

Ans. $E_\theta = k(2\eta/\pi^2 r_0)I_0[1 - (\frac{1}{8}\pi^2 - 1)\cos^2\theta]\sin\theta$, where I_0 is the maximum amplitude of the antenna current.

2

MAXWELL'S EQUATIONS

2.1 Electromagnetic field concepts and equations

Circuit and field theories are both necessary for an understanding of antennas. The first antenna used by Hertz in his famous experiments with electromagnetic waves was nothing but a capacitor, with its plates spread apart so that the " interior " of the capacitor became indistinguishable from the space around it (Fig. 1.47). The relation between the voltage and the current at the terminals of this antenna is just as important as in the case of any circuit; but we are forced to pay more attention to the electrical conditions at points in the " interior " of the circuit, which has become the exterior of the antenna including the earth and the space around it.

In the capacitor with charges $+q$ and $-q$ on its plates, the electric field is confined primarily to the region between the plates; but, around the antenna, this field is distributed more freely. Since the magnetic field is strongest near the leads, the electric and magnetic fields in the capacitor are well separated and independent, but around the antenna they penetrate each other and interact. The magnitude of this interaction increases with the frequency. In the capacitor the magnetic field is spurious and is deliberately kept small; but the antenna capacitance is inherently smaller, and the magnetic field becomes important. These factors make the antenna impedance different from the impedance of an ordinary capacitor. Still more important is the difference between the mutual capacitance between two ordinary capacitors a short distance apart and the mutual impedance between " capacitor antennas " far from each other. To evaluate these differences we have to study distributed electric and magnetic fields.

There is a close parallelism between circuit and field concepts. The differences arise from practical considerations. When considering the performance of an electric circuit as a whole, we are interested only in the quantities that can be measured at its terminals. Thus we are

interested in the *total* current I entering the circuit through one terminal and leaving it through the other. This current is defined as the quantity of electric charge passing through the terminals per unit time; thus, if a charge dq passes in time dt, then,

$$I = \frac{dq}{dt}. \tag{1}$$

On the other hand, in the interior of the circuit the conditions may vary from point to point, and the total current gives no indication of local flow of charge. Instead we have to consider the *current density* J at a typical point P, which is defined as a vector in the direction of the current at P whose magnitude equals the current per unit area perpendicular to the direction of flow. If the current I is distributed uniformly in a wire whose cross section is S, the current density is simply

$$J = \frac{I}{S}. \tag{2}$$

In dealing with the electric circuit as a whole, we are not concerned with forces acting on the charge at various points. We are interested only in the total effect. Thus the electromotive force, frequently called the " voltage " V between the terminals, is defined as the total work done by the electric forces per unit charge passing through the circuit. In the interior the work done on a moving charge may vary from one part of the circuit to the other. There we are interested in the *electric intensity* E at a given point P defined as a vector in that direction in which the voltage per unit length is maximum. The magnitude of E equals this maximum voltage per unit length. If the voltage between the ends of a wire of length l is V and if this voltage is distributed uniformly along the wire, the electric intensity at various points is simply V/l. In good metals the current density is proportional to the electric intensity,

$$J = gE, \tag{3}$$

where g is the conductivity of the medium.

The electric intensity may also be defined as the force per unit charge. This definition is equivalent to that given in the preceding paragraph. If F is the force acting on a charge q, the work done by F when the charge is displaced through a small distance s from point P to point Q is $F_{PQ}s$, where F_{PQ} is the component of force in the direction PQ. The voltage from P to Q is, by definition, the work per unit charge $V_{PQ} = F_{PQ}s/q$. The voltage per unit length is $E_{PQ} = F_{PQ}/q$, that is, the component of force (in the direction PQ) per unit charge. The maximum voltage per unit charge is F/q.

Throughout this book we shall use the meter-kilogram-second-coulomb (mksc) system of units. In this system all electric units are the practical units commonly used in laboratory measurements. The unit of work is the practical unit called the joule (equal to 10^7 ergs). The work done in raising one kilogram (2.205 lb) to a height of one meter against gravity is approximately 9.8 joules. The unit of electromotive force (" voltage ") is thus the joule per coulomb, called the volt. The unit of electric intensity is the volt per meter (that is, the joule per coulomb per meter). The mksc unit of force is the joule per meter, called the newton. The newton is equal approximately to the pull of gravity on 0.102 kg (or 0.225 lb); it is also equal to 10^5 dynes. The unit of electric intensity, the volt per meter, is also the newton per coulomb.

To describe the performance of an electric circuit we need two quantities: the current through the terminals and the voltage across them. Similarly two quantities are needed in order to specify electrical conditions at a point in an electric field. In conducting media these quantities are the current density and the electric intensity. In non-conducting media, between the plates of a capacitor, for instance, we also need two field quantities. One of these quantities is the electric intensity. Inside a capacitor formed by two large closely spaced parallel metal plates the electric field is uniform except near the edge. The electric intensity is perpendicular to the plates, and is equal to V/l. The other quantity is defined in terms of the electric charge displaced by the electric intensity from one face of a thin metal plate to the other (Fig. 2.1). The electric intensity acts on free electrons in the plate and displaces them relatively to the positively charged nuclei until the field produced by the displaced charge just cancels the original field in the space occupied by the search plate. The electrons must continue their motion until the net forces acting on them in the *interior* of the search plate vanish. Just outside the plate these forces must be perpendicular to the plate since the tangential forces would still be able to move the electrons. The charge displaced across such a plate can be measured if the plate is made of two separable halves. These component parts must be pulled apart *in the field* where the field keeps the displaced charges from recombining. For a sufficiently small area ΔS of the search plate the displaced charge is proportional to the area and, at a

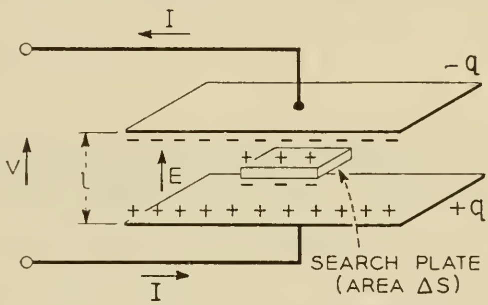


Fig. 2.1 A parallel plate capacitor.

given point P, is maximum for a particular orientation of the plate. The vector D in this direction whose magnitude equals the displaced charge per unit area is called the *electric displacement density.* In noncrystalline media D is parallel and proportional to E; thus,

$$D = \varepsilon E, \tag{4}$$

where the constant of proportionality ε is called the dielectric constant of the medium. In crystalline media the Cartesian components of D are linear functions of the Cartesian components of E.

Between the plates of the capacitor the magnitude of D equals very nearly the ratio q/S of the positive charge to the area of one plate. This relationship is exact for two infinitely large, uniformly and oppositely charged parallel planes.

If V is varying with time, E and D are also varying. Since D has the dimensions of charge per unit area, its time derivative dD/dt has the dimensions of electric current density. This derivative is called the *electric displacement current density.*

$$J_d = \frac{dD}{dt} = \varepsilon\frac{dE}{dt}\cdot \tag{5}$$

It is equal to the electronic current per unit area that will exist between the faces of the search plate placed perpendicularly to D. The importance of displacement current was first recognized by Maxwell. He was led to make the hypothesis that displacement currents generate magnetic fields just as electronic currents do. Maxwell's hypothesis is best confirmed by radar waves which continue to exist after the electronic currents, which generate them in the first place, subside.

In Section 1.2 we defined the magnetic intensity H between two large parallel current sheets, carrying equal and opposite steady currents (Fig. 1.6). It is a vector perpendicular to the current, and its magnitude equals the current per unit length. The definition depends on the experimental fact that the magnetic field between such current sheets, as indicated by its action on a thin magnetic needle, is uniform and that its intensity depends solely on the current per unit length, irrespective of the distance between the current sheets. Experiment shows that the magnetic field is also uniform inside a long closely wound solenoid (Fig. 2.2*a*). Of course, it is assumed that measurements are not taken too close to the winding where there are gaps in the current. To obtain exact uniformity of the field we should have an infinitely long solenoidal current sheet (Fig. 2.2*b*). Then, if I_c is the circulating current in a length l of the solenoid ($I_c = nI$ where I is the current in the winding and n is the number of turns in the length l), the intensity of the magnetic

field depends only on the *circulating current per unit length*, I_c/l. Neither the size nor the shape of the cross section of the solenoid affects the intensity of the field. In actual solenoids the intensity near the ends is smaller than it is in the middle. Furthermore, the intensity throughout the middle part is slightly smaller than I_c/l.

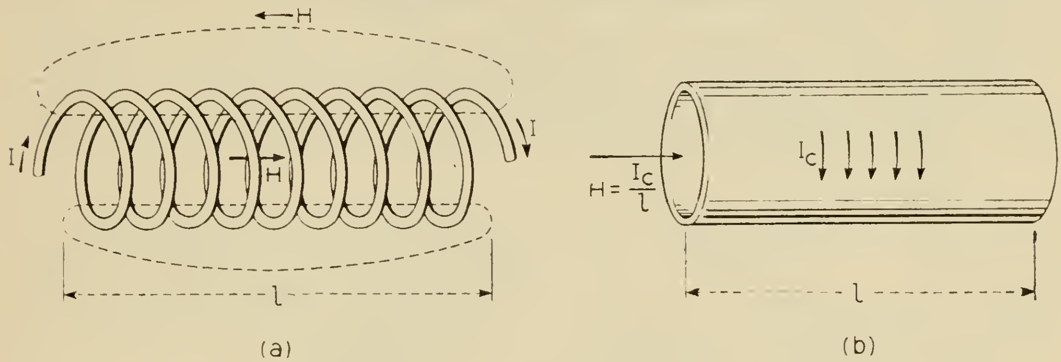


FIG. 2.2 (*a*) Inside an infinitely long, closely wound solenoid, the magnetic intensity H is parallel to the axis of the solenoid and is equal to nI/l, where n is the number of turns in a length l along the axis; (*b*) inside a sheet of circulating current, $H = I_c/l$, where I_c corresponds to nI in (*a*).

To state the exact relationship between the magnetic intensity and the current, we shall define the *magnetomotive force* along a specified curve AB,

$$U = \int_{AB} H_s\,ds. \tag{6}$$

Experiments suggest that the magnetomotive force round a closed curve equals the enclosed current,

$$U = \oint H_s\,ds = I. \tag{7}$$

This is Ampère's law. Of course, the experiments show only the proportionality between U and I; the equality is a matter of definition. In this respect our definitions of E and H are different. We cannot define H as the force per unit magnetic charge because magnetic charge does not exist. We can either define H directly in terms of electric currents, as we have done, or develop a concept of "magnetic pole" which could play the part of magnetic charge for the purpose of defining H. In the end, however, we would have to relate H to the current producing it, irrespective of the particular definition we may wish to choose. For a thorough experimental discussion of electromagnetic concepts the reader is referred to R. W. Pohl, *Physical Principles of Electricity and Magnetism*, Blackie and Son, Ltd., 1930.

In the case of an infinitely long straight current filament the magnetic lines of force are circles coaxial with the filament (Fig. 2.3). From symmetry we conclude that H depends only on the distance ρ from the axis of the filament. Hence, from equation 7 we have

$$2\pi\rho H = I, \qquad H = \frac{I}{2\pi\rho}\cdot \tag{8}$$

Figure 2.3 exhibits the relationship between the positive direction of the magnetomotive force and the positive direction of the current.

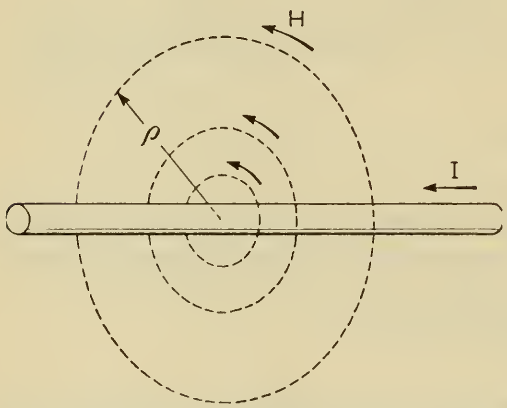


FIG. 2.3 A straight current filament and magnetic lines around it.

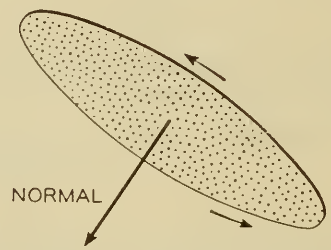


FIG. 2.4 Illustrating the convention regarding the relation between the positive direction around a closed curve and the positive direction of the normal to the plane of the curve.

There will be other occasions when we shall have to relate quantities representing current or flux across an area and line integrals round the edge of the area involving quantities in the nature of force. We need therefore a convention that relates the positive direction of the normal to the area and the positive direction of integration round the edge. This convention is shown in Fig. 2.4.

Ampère's law (equation 7) refers to the magnetomotive force round a closed curve " enclosing " electric current. This requires that the path of integration and the path taken by the current be linked as two closed links of a chain. Otherwise the equation is meaningless. Now steady currents *must* flow in closed conducting paths, and there is no difficulty with equation 7. Varying currents, on the other hand, can exist in any conductor. A charge may fluctuate between the ends of a short wire. Such currents generate magnetic fields, but in its present form equation 7 cannot be applied to them. Maxwell put forward the following hypothesis: *Electric displacement currents generate magnetic fields and equation*

7 *remains true if I includes displacement currents.* The difference between the original Ampère's law for steady currents and the Ampère-Maxwell law for varying currents is illustrated in Fig. 2.5. In the steady case the right-hand side of equation 7 represents the current linked with the path of integration. In Fig. 2.5*a* this current equals nI. In the general case we have to imagine a surface whose edge is the path of integration (Fig. 2.5*b*). This surface may be partly inside a conductor

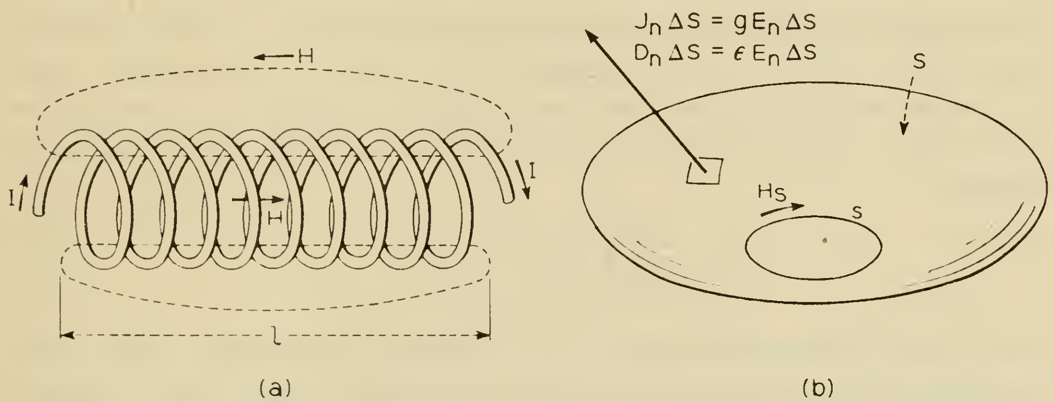


FIG. 2.5 Illustrating the application of the Ampère-Maxwell equations: (*a*) a steady current linked with the path of integration, (*b*) the general case.

and partly outside. In the conductor the current across an element of area ΔS is $J_n\,\Delta S = gE_n\,\Delta S$, where n denotes the normal to the element. In the dielectric there is a displacement $D_n\,\Delta S = \varepsilon E_n\,\Delta S$ across the element. By integration we obtain the total displacement through the surface and the corresponding displacement current. This we add to the conductor current to obtain the total current,

$$I = \iint gE_n\,dS + \frac{\partial}{\partial t}\iint \varepsilon E_n\,dS, \tag{9}$$

to be used in equation 7; thus the Ampère-Maxwell equation is

$$\oint H_s\,ds = \iint gE_n\,dS + \frac{\partial}{\partial t}\iint \varepsilon E_n\,dS. \tag{10}$$

More generally there may be free charged particles in the dielectric medium (electron streams in vacuum tubes, for instance) and we should include the charge carried by them across the surface S; but in antenna design we are not concerned with this case.

Equation 10 obviously implies an assertion that the total currents crossing all surfaces having a common edge are equal. In particular the *total current leaving* (or entering) *a closed surface is zero,*

$$\iint gE_n\,dS + \frac{\partial}{\partial t}\iint \varepsilon E_n\,dS = 0. \tag{11}$$

Let us transfer the first term to the right side of the equation and integrate with respect to t from $t = t_0$ to $t = t_1$,

$$\iint \varepsilon E_n(t_1)\, dS - \iint \varepsilon E_n(t_0)\, dS = -\int_{t_0}^{t_1} dt \iint g E_n\, dS. \tag{12}$$

If our surface integrals are taken in the direction of the normal pointing *out* of the volume enclosed by S, then the left-hand side of equation 12 represents the change in the outward electric displacement across S whereas the right-hand side represents the quantity of electric charge that has entered the volume enclosed by S. If at $t = t_0$ the field is zero (macroscopically) because equal and opposite charges are well mixed, and if at $t = t_1$ the movement of charge has stopped, then equation 12 becomes

$$\iint \varepsilon E_n\, dS = q, \tag{13}$$

where q is the charge enclosed by S. While this equation follows from the Ampère-Maxwell equation 10, it should be noted that it was first

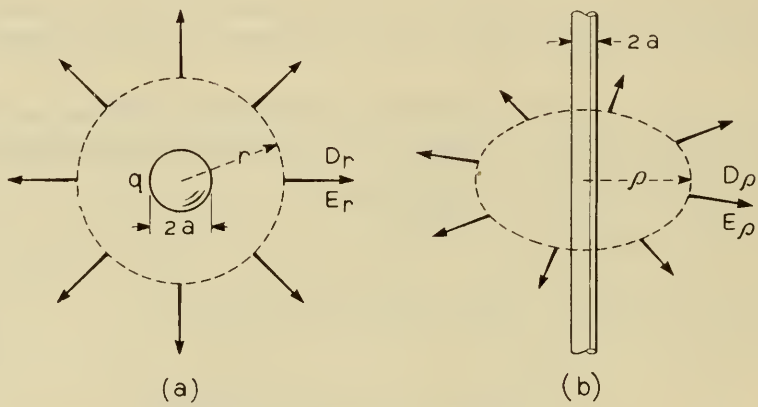


Fig. 2.6 Illustrating the field (a) around a charged sphere and (b) around a uniformly charged cylinder.

established experimentally by Faraday, and, in fact, it suggested to Maxwell the close relationship between the movement of electric charge and the time rate of change of electric displacement in the surrounding medium. This led him to amend Ampère's law and make it general.

If the charge q is distributed uniformly on the surface of a sphere of radius a, the electric intensity is radial (Fig. 2.6a). Applying equation 13 to a concentric sphere of radius $r \geq a$, we have

$$E_r = \frac{q}{4\pi\varepsilon r^2}, \qquad D_r = \varepsilon E_r = \frac{q}{4\pi r^2}, \qquad r \geq a. \tag{14}$$

Inside the charged sphere the field is zero. Similarly for a charge

distributed uniformly on the surface of a cylinder of radius a (Fig. 2.6b), we have

$$E_\rho = \frac{q}{2\pi\varepsilon\rho}, \qquad D_\rho = \frac{q}{2\pi\rho}, \qquad \rho \geq a, \tag{15}$$

where q is the charge per unit length of the cylinder.

The Ampère-Maxwell equation 10 connects the magnetic intensity of a given field with the electric current in conductors (conduction current) and the displacement current in the nonconducting part of the field. There is a corresponding equation which connects the electric intensity with the time rate of change of the magnetic intensity. To formulate this equation we require magnetic concepts analogous to electric displacement, electric displacement current, and their densities. First let us consider a static magnetic field generated by permanent magnets or by steady currents. Let a small single-turn search loop, connected to a voltmeter, be brought into the field. The voltmeter will record a voltage while the loop is moving. The voltage disappears when the loop stops. The voltage depends on the magnetic intensity and the speed of motion; but there is no voltage if the loop is moving through a uniform field provided the angle between H and the plane of the loop is kept constant. The voltage depends on the time rate of change of the component of H normal to the plane of the loop. The same phenomenon is observed if the search loop is stationary while the source of the magnetic field is moving, or if both are stationary but the current generating the field is varying. For sufficiently small loops the voltage is substantially proportional to their areas, assuming, of course, that the time rate of change of H is kept constant. These phenomena are consistent with the idea that the voltage in the search loop is due to some kind of flow through the area of the loop. This flow is called the *magnetic displacement current.* The voltage is taken as its measure. Thus the electric displacement current is measured by the electronic current across a metal search plate, whereas the magnetic displacement current is measured by the voltage acting round a search loop. The time integral of the magnetic displacement current is the *magnetic displacement* (or magnetic flux). Its unit is the volt-second, called the weber. Note the correspondence between this unit and the ampere-second or the coulomb, the unit of electric charge and electric displacement.

Let us now return to the original static field and assume that the search loop is connected to a long period ballistic galvanometer which measures the time integral of the voltage. If the loop is withdrawn from a given point P of the field into a field-free region, the time integral of

the voltage does not depend on the path taken by the loop nor on the speed of its motion. It depends only on the original orientation of the plane of the loop. For some particular plane the time integral of the voltage is maximum. The vector B, perpendicular to this plane, whose magnitude equals the maximum time integral of the voltage per unit area of the search loop is called the *magnetic displacement density* at P. If B is varying with time, dB/dt is called the density of *magnetic displacement current* at P. Since magnetic current in the sense of moving "magnetic charge" does not exist, we can omit the qualifying word "displacement."

In many media B is proportional to H; thus,

$$B = \mu H, \tag{16}$$

where μ is called the permeability of the medium. In iron and other ferromagnetic substances this equation is approximately true for sufficiently weak fields.

We are now in a position to formulate an equation analogous to the Ampère-Maxwell equation. Consider a surface with a hole (Fig. 2.5*b*). The magnetic displacement through an element of this surface is $B_n \, \Delta S$, where B_n is the component of B normal to the element of area ΔS. Integrating over the surface, we obtain the total magnetic displacement through the surface. Differentiating with respect to t, we obtain the magnetic current through the surface. Then we take the voltage along an element Δs of the edge of the surface. This voltage is $E_s \, \Delta s$, where E_s is the component of E tangential to the edge. Integrating round the edge, we obtain the total voltage round the edge. If the positive directions of the normal to the surface and of integration round the edge are related as shown in Fig. 2.4, then *the voltage round the edge of the surface equals the negative of the magnetic current through the surface,*

$$\oint E_s \, ds = -\frac{\partial}{\partial t}\iint B_n \, dS = -\frac{\partial}{\partial t}\iint \mu H_n \, dS. \tag{17}$$

This is the Faraday-Maxwell law.

2.2. Maxwell's equations — general and steady state

The Ampère-Maxwell equation,

$$\oint H_s \, ds = \iint gE_n \, dS + \frac{\partial}{\partial t}\iint \varepsilon E_n \, dS, \tag{18}$$

and the Faraday-Maxwell equation,

$$\oint E_s \, ds = -\frac{\partial}{\partial t}\iint \mu H_n \, dS, \tag{19}$$

express the laws of interaction between electric and magnetic fields. They should be considered as hypotheses suggested and confirmed by certain basic experiments. No set of experiments can prove them completely, for they are supposed to apply to any closed curve and any surface for which this curve is the edge. We can choose this curve $ABCD$ completely on the surface of a closed conducting loop (Fig. 2.7*a*); partly in free space and partly on the surface of a conductor (Figs. 2.7*b*, *d*); or completely in free space (Fig. 2.7*c*). Our confidence in

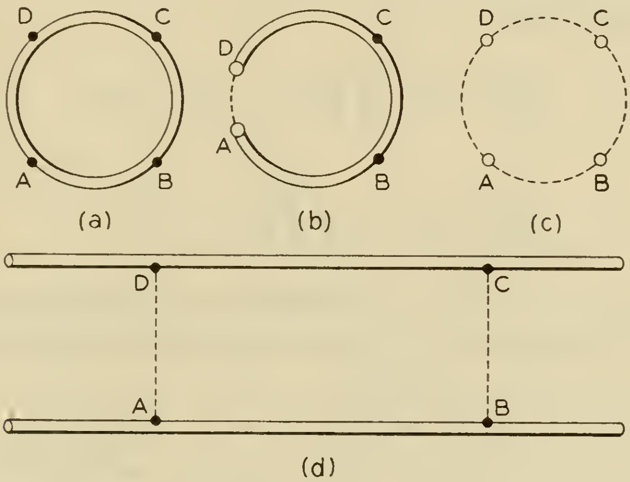


FIG. 2.7 Maxwell's equations are applicable to any closed curve $ABCDA$, which may be located either completely on the surface of a conductor, or partly on such a surface and partly in the dielectric medium, or completely in the dielectric medium.

these equations is great because numerous conclusions obtained from them have later been confirmed by observation, and no experiment has so far contradicted them.

It should be recalled that the current represented by streams of charged particles, not confined to conducting media, was not included in equation 18. Hence, in its present form equation 18 is not valid inside vacuum tubes. This restriction is not important in antenna theory which is concerned with electromagnetic phenomena *outside* electric generators. Such generators may enter antenna theory only as "boundary conditions." A given generator is surrounded by a closed surface, and some information is given or assumed about the field on this surface. Maxwell's equations enable us to extend this field from this surface into the surrounding region. Physically these equations govern propagation of the electromagnetic field from a generator into the surrounding medium.

The electric and magnetic intensities at point $P(x, y, z)$ are functions of the coordinates of the point and of the time t. These functions,

$E(x, y, z, t)$ and $H(x, y, z, t)$, are restricted only by equations 18 and 19 and by the conditions at the boundary of the generator producing the field. If the forces producing the field vary harmonically with time, the field will eventually settle into a steady state in which E and H will also vary harmonically with time. In the steady state we express $E(x, y, z, t)$ and $H(x, y, z, t)$ as the real parts of the exponential time functions, $E(x, y, z) \exp(j\omega t)$ and $H(x, y, z) \exp(j\omega t)$, where $\omega = 2\pi f$ and f is the frequency. Substituting these functions in equations 18 and 19 and canceling the exponential time factor, we obtain the steady-state form of Maxwell's equations,

$$\oint H_s\, ds = \iint (g + j\omega\varepsilon) E_n\, dS, \tag{20}$$

$$\oint E_s\, ds = -\iint j\omega\mu H_n\, dS. \tag{21}$$

In these equations E and H are functions of spatial coordinates only. To obtain the instantaneous values we should multiply these functions by $\exp(j\omega t)$ and take the real part.

Note that in the steady state the electric conduction current density is gE, the electric displacement current density $j\omega\varepsilon E$, and the total electric current density $(g + j\omega\varepsilon)E$. The magnetic (displacement) current density is $j\omega\mu H$. In metals the conductivity g is very large, and $j\omega\varepsilon$ is negligible at all radio frequencies. In good copper $g = 5.8 \times 10^7$ so that only small electric intensities are needed to drive large currents. In vacuum, $\varepsilon \simeq (1/36\pi)10^{-9}$, $\mu = 4\pi 10^{-7}$, hence, local variations in E and H due to displacement currents are very small until the frequencies become very high.

2.3 Differential equations and boundary conditions

Frequently in solving problems we shall apply equations 20 and 21 to infinitesimal circuits to obtain differential equations. No difficulty arises if the electromagnetic parameters g, μ, ε of the medium are continuous functions of position or constants. If, however, these parameters change abruptly across some surface (Fig. 2.8), we obtain two sets of differential equations, one set for each medium. Neither set applies to any region, however small, that includes the interface between the media.

To obtain the boundary equations we apply equations 20 and 21 to a narrow rectangular circuit whose long sides are parallel to the interface and are on the opposite sides of it (Fig. 2.8). Then we let the narrow sides, AD and BC, approach zero. The electric and magnetic

currents through the rectangular area will vanish in the limit. Hence, the magnetomotive and electromotive forces round the rectangle will also vanish. Since AD and BC are vanishingly small, they contribute nothing to either the magnetomotive or the electromotive force. Hence, the electromotive force from A to B must be equal and opposite to the electromotive force from C to D. The electromotive force from A to B is thus equal to the electromotive force from D to C for an arbitrary length AB. As AB approaches zero, these electromotive forces approach $E_t'(AB)$ and $E_t''(AB)$. Hence $E_t' = E_t''$; that is, the *component of E tangential to the interface between two media is continuous.* Similarly, the *tangential component of H is continuous.*

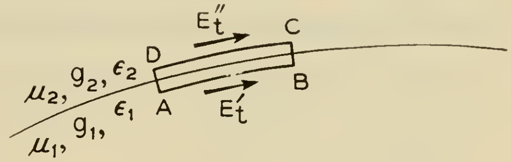


FIG. 2.8 Illustrating the proof of the continuity of the tangential components of E and H across the boundary between two media with different electromagnetic properties.

These boundary conditions enable us to connect the solutions of the differential equations for the two media.

2.4 Electric circuits

An electric circuit, in the conventional sense of the term, is a physical structure in which electric and magnetic fields are well separated from each other and largely concentrated in relatively small regions. The field separation is obtained by using substances whose conductivities, dielectric constants, and permeabilities differ greatly from those of the surrounding medium and by properly distributing these substances in space. The most important of these substances are metals which have very high conductivities. The conductivity of copper, for instance, is about 5.8×10^7 mhos per meter. Hence, small electric intensities will cause strong currents in thin wires. According to equation 8, the magnetic field in the vicinity of such wires will be strong. By winding the wires into coils the magnetic fields of the various turns are superimposed, and the resultant fields are made still stronger. If the coils are wound on toroidal cores, the magnetic fields are confined almost entirely to their interior. Such structures are called inductors, and the electric fields associated with them are concentrated in the vicinity of their terminals.

A strong electric field in a small region may be created by stacking thin metal plates and alternate thin layers of dielectric and then connecting the alternate metal plates together by means of good conductors. If V is the voltage between the two groups of plates and l the

distance between the adjacent plates, the electric intensity between the plates is V/l and can be made large by making l sufficiently small. Outside such a " capacitor " the electric field is small because the effects of the opposite charges on the adjacent plates tend to cancel each other. The displacement density inside the capacitor is $\varepsilon V/l$. The dielectric constant of air is 8.854×10^{-12} farad per meter. The dielectric constant of mica is six to seven times as large. In spite of the strong electric intensity, the density $\omega\varepsilon V/l$ of the electric displacement current is very small even at high frequencies; hence, the magnetic field inside the capacitor is weak.

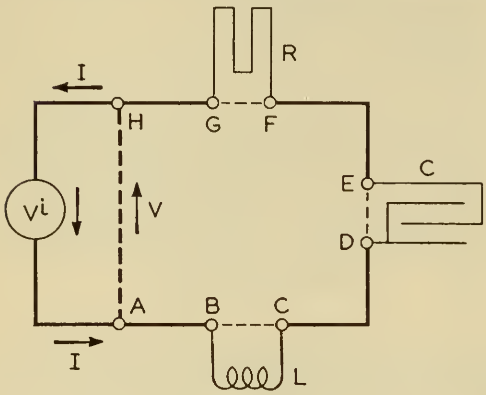


Fig. 2.9 A simplified diagram of an electric circuit consisting of a resistor R, a capacitor C, and an inductor L.

When electric and magnetic fields are widely distributed *and penetrate each other,* the problem of their calculation requires application of Maxwell's equations at all points of an unlimited region; but in an electric circuit (Fig. 2.9) we are not interested in the weak field distribution outside the circuit elements, and we need to apply Maxwell's equations only to the circuit itself. For example, applying the Faraday-Maxwell equation to the closed curve *ABCDEFGHA* which is partly on the surfaces of the connecting leads and partly between the terminals of the circuit elements, we obtain

$$V_{AB} + V_{BC} + V_{CD} + V_{DE} + V_{EF} + V_{FG} + V_{GH} + V_{HA} = -\frac{d\Phi}{dt}, \qquad (22)$$

where Φ is the magnetic displacement through the area *ABCDEFGHA*. Since the leads are good conductors, the voltages V_{AB}, V_{CD}, V_{EF}, V_{GH} are small and for this reason may usually be neglected. The magnetic displacement Φ is also small so that $d\Phi/dt$ may be neglected except when the time rate of change of I is exceptionally large. Hence, considering that $V_{HA} = -V$, equation 22 becomes

$$V_{BC} + V_{DE} + V_{FG} = V, \qquad (23)$$

where V is the voltage across the terminals of the generator. If the generator had no internal resistance, inductance, or capacitance, this voltage would be equal and opposite to the internal electromotive force V^i developed by the generator. In actual generators $V^i \neq V$, the difference depending on the generator.

If the medium surrounding the circuit is not a perfect dielectric, there will be some conduction current from one lead AB to another HG in response to the voltage between them. Such leakage currents are deliberately kept small by the choice of good dielectrics. In any case, however, there will be displacement currents between the leads. At low frequencies these currents are extremely small, and very nearly the same current I flows through each circuit element; then equation 23 becomes

$$L\frac{dI}{dt} + \frac{\int I\,dt}{C} + RI = V. \tag{24}$$

This simplified circuit equation may be made more exact without essentially complicating it. In equation 24 it is assumed that the resistance R_L of the coil is negligible, but its inclusion involves only adding R_LI to the left of equation 24. Similarly, the resistances of the leads affect only the total resistance in the circuit. The principal effect of $d\Phi/dt$ in equation 22 is to add a "stray inductance" to L in equation 24. We can also include "stray capacitances" between the leads, but then the one-mesh circuit in Fig. 2.9 becomes the more complicated network shown in Fig. 2.10. This network however will represent the behavior of our physical circuit accurately at much higher frequencies than the original simple equation 24. This idea of representing the small effects of the medium surrounding a physical network by extra circuit elements in the corresponding electric network diagram is used frequently, and is very advantageous as long as stray fields in the medium are largely in the vicinity of the physical network. When properly understood this device may be used even in the microwave range.

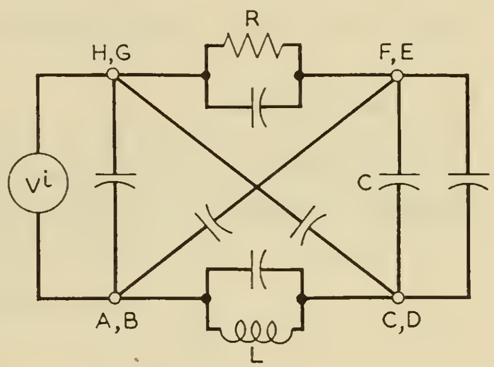


FIG. 2.10 A diagram of the circuit in Fig. 9 showing the stray capacitances between the leads.

The stray capacitance between the terminals of a generator, a resistor, an inductor, or an antenna is often negligible, even at high frequencies, because the distance between the terminals is usually large compared with the diameters of the leads; but in the microwave range these terminals may sometimes be so close that the stray capacitance will tend to short-circuit the terminals. This is important to remember, for the practice of neglecting stray capacitances has created an impression that the size of the gap between the terminals of a physical structure

is unimportant and, in theoretical investigations, may be made equal to zero. This mathematical simplification is permissible only when the terminals are assumed to be tapered to mere points.

The device of representing the electric behavior of physical networks by idealized networks, which include the effects of the stray fields in the surrounding medium, may be extended to represent the structure and the medium *exactly* by an equivalent network with infinitesimal meshes.* On a limited scale this is usually done in the study of "electric transmission lines," although in this case some approximations are still retained. In the general three-dimensional case this network representation of the medium *does not simplify the mathematical solution of electromagnetic problems, but it makes it easy to understand why certain network theorems, such as the reciprocity theorem and the Helmholtz-Thévenin theorem, are applicable to electromagnetic fields,* and it affords insight into electromagnetic phenomena to those who are familiar with electric networks. Furthermore, it provides a method for the experimental solution of electromagnetic problems.†

2.5 Energy flow

Let us consider a circuit (Fig. 2.11) in which a conducting medium is localized between two parallel, perfectly conducting plates, C and D.

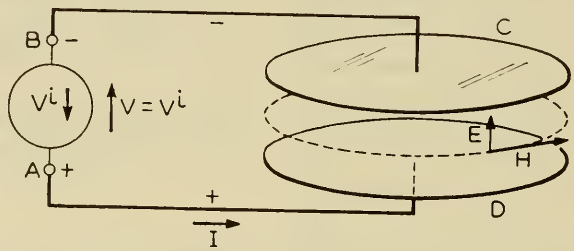


FIG. 2.11 Illustrating the flow of energy in space.

We assume that the generator and the connecting leads, AD and BC, have no resistance. By definition, the electromotive force between two points is the work done by the electric force on a unit charge displaced between these points; hence, in the time interval $(0, t)$ the work done by the impressed emf V^i *against* the opposing emf V produced by the

* Gabriel Kron, Equivalent circuit of the field equations of Maxwell, *IRE Proc.*, **32**, May 1944, pp. 289–299.

† J. R. Whinnery and Simon Ramo, A new approach to the solution of high-frequency field problems, *IRE Proc.*, **32**, May 1944, pp. 284–288.

J. R. Whinnery, C. Concordia, W. Ridgway, and G. Kron, Network analyser studies of electromagnetic cavity resonators, *IRE Proc.*, **32**, June 1944, pp. 360–367.

displaced charge is

$$\mathcal{E} = \int_0^t V^i \, dq, \tag{25}$$

where dq is an element of charge displaced from B to A in the generator. By definition, $dq = I\,dt$; hence,

$$\mathcal{E} = \int_0^t V^i I \, dt. \tag{26}$$

The power P is the time rate of doing this work,

$$P = \frac{d\mathcal{E}}{dt} = V^i I. \tag{27}$$

Now in the resistor it is the emf V of the field that is doing work in moving the charge between the plates. This work and its rate are

$$\mathcal{E} = \int_0^t VI \, dt, \qquad P = VI. \tag{28}$$

If the generator has no internal resistance, capacitance, or inductance, $V^i = V$, and the work done by the generator is equal to that done by the field between the plates; thus the energy has been transferred without loss from the generator to the resistor. Experience shows that in the resistor this energy is transformed into heat.

Expressing the voltage and current in terms of the field quantities, we obtain

$$P = lSEJ = EJ\tau, \tag{29}$$

where τ is the volume of the resistor and l is the distance between the electrodes (Fig. 2.11). Hence the power dissipated in heat per unit volume is

$$\frac{P}{\tau} = EJ = gE^2. \tag{30}$$

Since $V = lE$ and $I = sH$, where s is the circumference of the resistor, the expression 28 for the power may also be written as follows:

$$P = lsEH. \tag{31}$$

The product ls is the lateral surface of the resistor. Since there is no question that the energy leaves the generator and enters the resistor, equation 31 strongly suggests that the flow of power takes place perpendicularly to E and H and at the rate $W = EH$ per unit area. In the present example E and H are perpendicular to each other. In the

general case, the power flow per unit area may be expressed as the vector product,

$$\vec{W} = \vec{E} \times \vec{H}, \tag{32}$$

of the electric and magnetic intensities. The absolute value of this product is $EH \sin \psi$, where ψ is the angle between E and H. The vector $\vec{W}$ is called the Poynting vector. It accounts for energy transfer anywhere in space.

In the steady-state case E and H are sinusoidal quantities. The average value of the product EH is $\frac{1}{2}E_aH_a \cos \vartheta$, where E_a and H_a are the amplitudes of E and H and ϑ is the phase difference. If now E and H are represented by complex exponentials, we have

$$E = E_a e^{j\omega t + j\vartheta}, \qquad H = H_a e^{j\omega t}, \tag{33}$$

where ϑ is the phase difference between E and H. Taking the product of E and the conjugate H^* of H, we have

$$EH^* = E_a H_a e^{j\omega t + j\vartheta} e^{-j\omega t} = E_a H_a e^{j\vartheta}. \tag{34}$$

The real part of this product is

$$\mathrm{re}(EH^*) = E_a H_a \cos \vartheta. \tag{35}$$

Therefore, the *time average of the power flow per unit area W* is the real part of the *complex Poynting vector*,

$$\Psi = \tfrac{1}{2}E \times H^*; \tag{36}$$

that is,

$$W = \tfrac{1}{2}\,\mathrm{re}(E \times H^*). \tag{37}$$

PROBLEMS

2.1–1 Obtain the resistance of an annular disk of thickness h. Let the inner radius be a and the outer b.

Ans. $$R = (2\pi g h)^{-1} \log \frac{b}{a} \cdot$$

2.1–2 Obtain the resistance between two circles of parallel on a thin spherical sheet of radius a and thickness h. Let the distances to these circular electrodes from one of the poles be s_1 and $s_2 > s_1$.

Ans. $$R = (2\pi g h)^{-1} \log\left(\cot \frac{s_1}{2a} \tan \frac{s_2}{2a}\right) \cdot$$

2.1–3 A uniform field in vacuum, 100 volts per centimeter, disappears at a uniform rate in one microsecond. What is the displacement current through a square 10 cm on a side? (See Appendix IX for necessary constants.) *Ans.* 0.8854 milliampere.

2.1–4 Assuming that a current I is distributed uniformly in a wire of radius a, find the magnetic intensity at distance $\rho < a$ from the axis of the wire.

Ans.
$$H_\varphi = \frac{I\rho}{2\pi a^2}\,.$$

2.1–5 Find the magnetic intensity between the closely spaced circular plates of a capacitor (Fig. 2.11). Assume that the frequency is low and that the charging current is I.

Ans.
$$H_\varphi = \frac{I\rho}{2\pi a^2}\,.$$

2.2–1 Consider a uniform magnetic field varying with the frequency $f = 1$ Mc/sec. Let the amplitude of its intensity be one ampere per centimeter. What is the amplitude of the voltage round a square, 10 cm along a diagonal, inclined to the field at 30°? *Ans.* $\pi^2/5$ volts.

2.2–2 Consider a uniform electric field varying with the frequency $f = 1$ Mc/sec. Let the amplitude of E be 100 volts per meter. What is the amplitude of the magnetomotive force round a square, of side 10 cm, normal to E: (*a*) in vacuum, (*b*) in sea water? *Ans.* (*a*) $1/18$ milliampere, (*b*) 5 amperes.

2.2–3 Consider a square, of side 10 cm, normal to a uniform magnetic field varying with the frequency $f = 10$ Mc/sec. Let $H = 10 \cos 2\pi ft$ amperes per meter. What is the difference between the voltages between adjacent vertices directly along the connecting side and round the square? *Ans.* $(8\pi^2/10) \sin 2\pi ft$ volts.

2.3–1 A charged sphere of radius a is surrounded by a concentric spherical shell, of outer radius b, whose dielectric constant is ε_1. The dielectric constant of the medium outside the shell is ε_2. Find E and D.

Ans.
$$D_r = \frac{q}{4\pi r^2}\,; \qquad E_r = \frac{q}{4\pi\varepsilon_1 r^2}\,, \qquad a \le r < b,$$
$$E_r = \frac{q}{4\pi\varepsilon_2 r^2}\,, \qquad r > b.$$

2.3–2 A charged sphere is surrounded by two dielectrics whose interface is a conical surface with its apex at the center of the sphere. Let Ω be the solid angle of the region whose dielectric constant is ε_1. The dielectric constant of the other region is ε_2. Find E and D.

Ans. $E_r = q[\Omega\varepsilon_1 + (4\pi - \Omega)\varepsilon_2]^{-1}r^{-2}$; in region 1, $D_r = \varepsilon_1 E_r$, and, in region 2, $D_r = \varepsilon_2 E_r$.

2.3–3 Solve the preceding problem on the assumption that the voltage between the sphere and a point at infinity (the potential of the sphere) is V_0. What is the charge on the sphere?

Ans. $E_r = V_0 a/r^2$; in region 1, $D_r = \varepsilon_1 V_0 a/r^2$, and, in region 2, $D_r = \varepsilon_2 V_0 a/r^2$; $q = [\Omega\varepsilon_1 + (4\pi - \Omega)\varepsilon_2]V_0 a$.

2.3–4 An infinite cylinder of radius a and carrying current I is surrounded by a coaxial cylindrical layer whose permeability is μ_1. The outer radius of the layer is b, and the permeability of the medium outside the layer is μ_2. Find H and B.

Ans.
$$H_\varphi = \frac{I}{2\pi\rho}\,; \qquad B_\varphi = \frac{\mu_1 I}{2\pi\rho} \quad \text{if} \quad a \le \rho < b,$$

and

$$B_\varphi = \frac{\mu_2 I}{2\pi\rho} \quad \text{if} \quad \rho > b.$$

2.3–5 An infinite cylinder of radius a is carrying current I and is surrounded by two media whose permeabilities are μ_1 and μ_2. The boundaries between the media are half-planes which, if extended to the axis of the cylinder, would form a wedge. Let ψ be the angle of the wedge containing the medium with the permeability μ_1. Find H and B.

Ans. $B_\varphi = \mu_1\mu_2 I[\psi\mu_2 + (2\pi - \psi)\mu_1]^{-1}\rho^{-1}$; in medium 1, $H_\varphi = B_\varphi/\mu_1$, and, in medium 2, $H_\varphi = B_\varphi/\mu_2$.

3

PLANE WAVES

3.1 Classification of waves

Let us consider an electric charge in a small region of a homogeneous medium. If this charge is forced to oscillate about its original position, an oscillating magnetic field is created around it in accordance with the Ampère-Maxwell law. In accordance with the Faraday-Maxwell law, this field in its turn creates an electric field which modifies the original electric field. After a certain lapse of time the field settles into a self-consistent steady state, self-consistent in the sense that the time variations in the electric intensity E produce the magnetic intensity H of just the right magnitude for its time variation to produce E. If the oscillating charge generates a wave, this wave must spread in all directions since there can be no preferred direction in a homogeneous medium. Such waves are called *spherical waves*. Suppose now that an infinitely long uniformly charged cylinder is oscillating in the direction of its axis (or perpendicularly to it). From symmetry considerations we expect the wave to spread in all directions perpendicular to the axis. We expect no wave motion parallel to the axis. Such waves are called *cylindrical waves*. A uniform plane distribution of sources will produce a *plane wave* traveling away from the plane. More precise definitions can be given only after a more detailed consideration of the various types of waves.

In the case of some electromagnetic waves, we shall find that the magnetic vector is perpendicular to the direction of propagation; such waves are called *transverse magnetic waves* (TM waves). If the electric vector is perpendicular to the direction of propagation, the wave is said to be *transverse electric* (TE). If both E and H are perpendicular to the direction of propagation, the wave is *transverse electromagnetic* (TEM).

3.2 Uniform plane waves

Let us imagine an infinite plane sheet of uniformly distributed current parallel to the xy plane (Fig. 3.1). If the current is parallel to the x

axis, the magnetic intensity is parallel to the y axis. As far as the electric field is concerned, there are two cases. The entire sheet may be uniformly charged. In this case there will be two components of E, one perpendicular to the sheet and the other parallel to its motion. If, however, the sheet contains equal charges of opposite sign moving with respect to each other, there can be no component perpendicular to the sheet. In this case the only nonvanishing component of E is E_x. Both

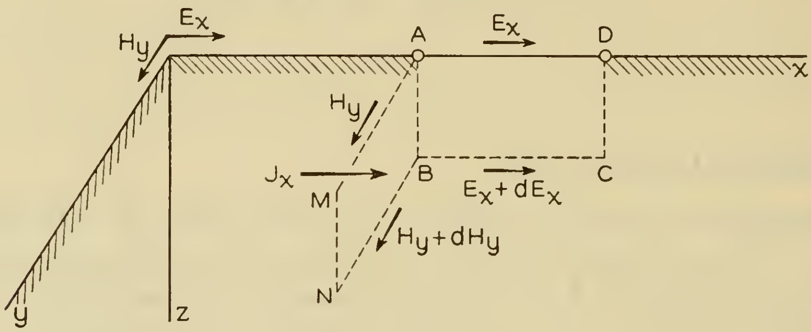


FIG. 3.1 Illustrating the derivation of transmission equations for uniform plane waves.

E_x and H_y are independent of x and y. These conditions are highly idealized, but they approximate actual conditions in front of a large current sheet. Also such uniform plane waves approximate spherical waves in limited regions at large distances from the centers of disturbance. Furthermore, uniform plane waves exhibit most of the essential characteristics of all waves without unnecessary mathematical complications.

Thus we shall consider an electromagnetic field in which the only nonvanishing field components are E_x and H_y, both independent of x and y. The equations of propagation are obtained by applying Maxwell's equations to rectangular circuits, one parallel to the yz plane and the other parallel to the xz plane. There is no need to consider a circuit parallel to the xy plane; from such a circuit we could obtain information only about the field distribution in the xy plane, and this we already have. Since the field is independent of x and y, we can simplify our calculations by assuming $AM = AD = 1$. The vertical sides are infinitesimal, of length dz. The magnetomotive force round the circuit $AMNBA$ is $H_y - (H_y + dH_y) = -dH_y$. According to the Ampère-Maxwell equation 2–20 this magnetomotive force must equal the electric current crossing the rectangular area in the direction shown by the current density J_x. Since $J_x = (g + j\omega\varepsilon)E_x$ and the area is dz, this cur-

rent is $(g + j\omega\varepsilon)E_x\, dz$, and one equation of propagation is

$$\frac{dH_y}{dz} = -(g + j\omega\varepsilon)E_x. \tag{1}$$

The electromotive force round the circuit $ABCDA$ is $(E_x + dE_x) - E_x = dE_x$. According to the Faraday-Maxwell equation 2–21, this must equal the negative of the magnetic current crossing the rectangular area in the y direction. The magnetic current density is $j\omega\mu H_y$, and the area is dz; hence, the total current is $j\omega\mu H_y\, dz$, and the second equation of propagation is

$$\frac{dE_x}{dz} = -j\omega\mu H_y. \tag{2}$$

Eliminating E_x, we obtain

$$\frac{d^2H_y}{dz^2} = \sigma^2 H_y, \tag{3}$$

where

$$\sigma = [j\omega\mu(g + j\omega\varepsilon)]^{1/2}. \tag{4}$$

Solving equation 3, we have

$$H = Ae^{-\sigma z} + Be^{\sigma z}, \tag{5}$$

where A and B are arbitrary constants of integration. Substituting in equation 1, we find

$$E_x = \eta Ae^{-\sigma z} - \eta Be^{\sigma z}, \tag{6}$$

where

$$\eta = \frac{\sigma}{g + j\omega\varepsilon} = \frac{j\omega\mu}{\sigma} = \left(\frac{j\omega\mu}{g + j\omega\varepsilon}\right)^{1/2}. \tag{7}$$

Since g, ε, and μ are normally positive, $j\omega\mu$ is on the positive imaginary axis and $g + j\omega\varepsilon$ is either in the first quadrant or on its boundaries (Fig. 3.2). Hence, the product $j\omega\mu(g + j\omega\varepsilon)$ is in the second quadrant or on its boundaries. There are two square roots, differing only in the algebraic sign. Since the phase of the product is between 90° and 180°, the phase of one square root is between 45° and 90°. This square root will be denoted by the letter σ; the other square root is $-\sigma$. By this definition, the real and imaginary parts of the complex number

$$\sigma = \alpha + j\beta \tag{8}$$

are, in general, positive. In nonconducting media $g = 0$, and, therefore, $\alpha = 0$. The phase of the second parameter η is 90° minus the phase of σ; since $45° \leq \text{ph}(\sigma) \leq 90°$, we have $0 \leq \text{ph}(\eta) \leq 45°$.

Suppose that the source of the wave is at $z = 0$. As $z \to +\infty$, the last terms in equations 5 and 6 approach infinity exponentially (assuming that $g \neq 0$) unless $B = 0$. Infinite current densities imply infinite dissipation of power. On the other hand, at $z = 0$ both E and H are finite, so that the power flow from the source per unit area is finite. The law of conservation of energy will be violated unless $B = 0$ below the plane where z is positive. Hence,

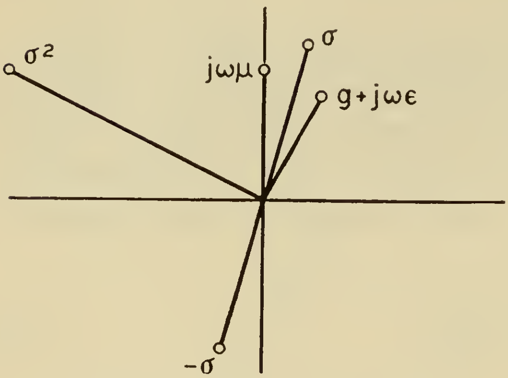


FIG. 3.2 The intrinsic propagation constant σ of a medium lies in the first quadrant of the complex plane or on its boundaries.

$$H_y^+ = Ae^{-\sigma z}, \qquad E_x^+ = \eta H_y^+, \qquad 0 \leq z < \infty. \tag{9}$$

Of course, if the medium does not extend to infinity, our argument does not apply and both terms in equations 5 and 6 should be retained. Similarly,

$$H_y^- = Be^{\sigma z}, \qquad E_x^- = -\eta Be^{\sigma z}, \qquad -\infty < z \leq 0. \tag{10}$$

Let us now reintroduce the time factor in equation 9,

$$H_y^+ e^{j\omega t} = Ae^{-\alpha z}e^{j(\omega t - \beta z)}. \tag{11}$$

The first exponential factor on the right side of the equation affects only the amplitude of the wave, and the second only the phase. The quantity α is called the *attenuation constant* and β the *phase constant.* The entire complex quantity $\sigma = \alpha + j\beta$ is the *propagation constant.*

The phase of the wave is

$$\varphi = \omega t - \beta z, \tag{12}$$

except for a constant phase associated with A. The *period* T of oscillations is the time required for the phase to change by 2π radians at a *given point;* hence,

$$\omega T = 2\pi, \qquad T = \frac{2\pi}{\omega} = \frac{1}{f}, \qquad \omega = \frac{2\pi}{T}. \tag{13}$$

The *wavelength* λ is the distance in which the phase changes by 2π radians at a given instant; hence,

$$\beta\lambda = 2\pi, \qquad \lambda = \frac{2\pi}{\beta}, \qquad \beta = \frac{2\pi}{\lambda}. \tag{14}$$

Suppose that t and z vary in such a way that the phase remains constant. Differentiating equation 12, we obtain

$$\omega\, dt - \beta\, dz = 0, \qquad \frac{dz}{dt} = \frac{\omega}{\beta}\cdot \tag{15}$$

Therefore, to an observer traveling with the velocity

$$v = \frac{\omega}{\beta} = f\lambda, \tag{16}$$

the phase of the wave will have a constant value. This velocity is called the *phase velocity*. In nondissipative media,

$$\beta = \omega(\mu\varepsilon)^{1/2}, \qquad v = (\mu\varepsilon)^{-1/2}. \tag{17}$$

The ratio

$$\frac{E_x^+}{H_y^+} = -\frac{E_x^-}{H_y^-} = \eta \tag{18}$$

is called the *intrinsic impedance* of the medium. In other types of waves the corresponding ratio of E to H depends on η and on the geometry of the wave; then the ratio is called the *wave impedance*. From equation 2–36 we conclude that in the present case the complex Poynting vector has only the z component

$$\Psi_z = \tfrac{1}{2}E_x H_y^*. \tag{19}$$

For the waves on the two sides of the current sheet, we find

$$\begin{aligned} \Psi_z &= \tfrac{1}{2}\eta H_y^+(H_y^+)^* = \tfrac{1}{2}\eta|H_y^+|^2, && 0 \le z < \infty, \\ \Psi_z &= -\tfrac{1}{2}\eta H_y^-(H_y^-)^* = -\tfrac{1}{2}\eta|H_y^-|^2, && -\infty \le z < 0. \end{aligned} \tag{20}$$

Since the real part of η is positive, the average power flow per unit area represented by the real part of Ψ_z is in each case away from the current sheet.

In vacuum,

$$\begin{aligned} \mu_v &= 4\pi \times 10^{-7} = 1.257 \times 10^{-6} \text{ henry per meter}, \\ \varepsilon_v &= 8.854 \times 10^{-12} \simeq (1/36\pi) \times 10^{-9} \text{ farad per meter}, \\ \eta_v &= \left(\frac{\mu_v}{\varepsilon_v}\right)^{1/2} = 376.7 \simeq 120\pi \text{ ohms}, \\ v_v &= (\mu_v\varepsilon_v)^{-1/2} = 2.998 \times 10^8 \simeq 3 \times 10^8 \text{ meters}. \end{aligned} \tag{21}$$

The electromagnetic parameters of air are substantially the same. From equation 16 we obtain the wavelength corresponding to a given frequency,

$$\lambda_v = 3 \times \frac{10^8}{f}\cdot \tag{22}$$

Hence, the phase of uniform plane waves in air changes slowly except at very high frequencies.

In copper $g = 5.8 \times 10^7$, and in comparison $\omega\varepsilon$ is negligible at all radio frequencies. The permeability of copper is the same as that of vacuum. Hence,

$$\sigma_c = (j2\pi\mu f g)^{1/2} = (j46.4f)^{1/2}\pi, \qquad \alpha_c = \beta_c = 15.1f^{1/2},$$
$$\eta_c = \left(\frac{j\omega\mu}{g}\right)^{1/2} = 2\pi\left(\frac{jf}{2.9}\right)^{1/2} \times 10^{-7} = 2.61 \times 10^{-7} f^{1/2}(1 + j). \quad (23)$$

Even at fairly low frequencies, the wave in copper is attenuated very rapidly. Electromagnetic waves do not penetrate deeply into copper or other metals. The reciprocal of the attenuation constant is called the *depth of penetration*,

$$h = \frac{1}{\alpha}. \quad (24)$$

At this depth from the surface of the metal, the field intensity decreases to $1/e = 0.368$ fraction of its value at the surface.

The conductivity of sea water is 5, and the dielectric constant relative to that of vacuum is 80. The depth of penetration is considerably greater than in metals but is still relatively small. The conductivities of soils vary considerably; they may be as low as 0.002 or as high as 0.02. The dielectric constants depend on moisture content. In dry soils the dielectric constant may be 10 times that of vacuum, and in wet soils it may be 30 times that of vacuum. The depth of penetration is thus greater than in sea water.

If g approaches infinity, the attenuation constant approaches infinity and the intrinsic impedance zero. Hence, electromagnetic waves cannot penetrate perfect conductors, and the electric intensity tangential to them vanishes. The normal component of the Poynting vector also vanishes, and the energy flow must necessarily be parallel to the surface of a perfect conductor and almost parallel to the surface of a good conductor. A perfectly conducting closed surface or an infinite perfectly conducting plane divides the medium into electromagnetically independent regions. A field may be excited in one region while the other region remains field-free. Good conductors are effective shields. Any field excited in one region is attenuated so rapidly in passing through a conducting wall that its intensity on the other side of the wall is extremely weak. At first it may appear that low-frequency waves will go through an imperfect conductor since the attenuation constant decreases with frequency. However, the intrinsic impedance also decreases, and, in consequence, low-frequency waves are more strongly "reflected" from conducting surfaces.

3.3 Waves between parallel strips

In the uniform plane wave considered in the preceding section, the electric lines of force are parallel to the x axis and the magnetic lines parallel to the y axis. Two perfectly conducting planes may be introduced into this field perpendicularly to the electric lines without disturbing the field. Since such planes divide the medium into electrically

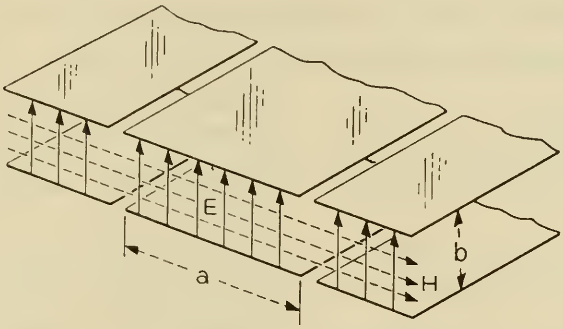


FIG. 3.3 A strip transmission line with guards to keep its field uniform.

independent regions, we can remove the portions of the current sheet outside the region enclosed by the planes without disturbing the field between them. Next we can cut the planes in the direction of wave propagation, as shown in Fig. 3.3. Electric and magnetic lines still remain substantially undisturbed. If the side portions of the planes are removed the field will spread out from the region between the parallel strips. Those magnetic lines that are nearer to the top strip than to the bottom will encircle the top strip. The remaining magnetic lines will encircle the bottom strip. If the width a of the strips is large compared with the distance b between them, the densities of electric and magnetic lines outside the strips are small compared with the densities between the strips. Most of the energy flow is confined to the region between the strips. The field between parallel strips with the "guard strips" on either side is a good approximation to the field between wide and closely spaced strips without guards and is much simpler.

The voltage V between the strips and the current I in the bottom strip (considered positive in the positive z direction) are

$$V = bE_x, \qquad I = aH_y. \tag{25}$$

Substituting in equations 1 and 2, we have

$$\frac{dI}{dz} = -YV, \qquad \frac{dV}{dz} = -ZI, \tag{26}$$

where the "series impedance per unit length" Z and the "shunt

admittance per unit length" Y are

$$Z = \frac{j\omega\mu b}{a}, \qquad Y = (g + j\omega\varepsilon)\frac{a}{b}. \tag{27}$$

3.4 Waves guided by parallel wires

Consider a pair of parallel wires (Fig. 3.4). In the case of the two generators impressing the same voltage V_1 in "push-push," the currents at the same distance from the generators must be equal and similarly directed. The charge densities must also be equal. If equal voltages are impressed in "push-pull," the currents in the wires will be equal and opposite. Likewise the charge densities must be equal and opposite. In the first case the magnetic intensities generated by the currents in the wires oppose each other between the wires and strengthen each other outside. In the second case they add between the wires and tend to cancel outside. The same is true of the electric intensities. Thus, in the second case the field will be concentrated in the vicinity of the parallel pair whereas in the first case it will spread to greater distances. The lines of electric force in the push-pull case run directly from one wire to the other since these wires are equally and oppositely charged. In the push-push case the charges on both wires are of the same sign at equal distances from the generators *on the same side of the generators.* At the same distance on the opposite side of the generators the wires are oppositely charged. Hence, in the push-push case the electric lines run outward from both wires on one side of the generators and gradually bend to meet the wires on the other side. In this case the waves are substantially spherical and will be considered in the next chapter. In the push-pull case the waves are substantially plane and will be considered in this section. In the general case we have a combination of these two types of waves. The total voltages impressed on the wires may always be decomposed into push-push and push-pull combinations. Thus, V_1 is half the sum and V_2 half the difference of the actual voltages impressed on the wires. This decomposition of the total wave into symmetric and antisymmetric components is useful at distances from the generators substantially greater than the distance between the wires. Close to the generators we have two spherical waves, centered at each generator.

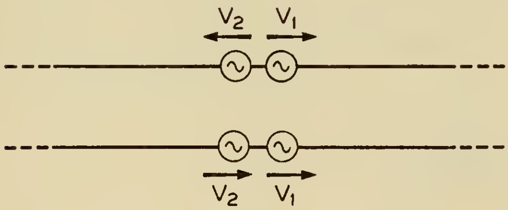


FIG. 3.4 Two modes of propagation on parallel wires of equal radius in free space.

It will be found that at low frequencies the spherical wave generated by the push-push distribution of voltage is very weak. Hence, any distribution of voltage will generate a wave of substantially push-pull type. Since in this wave the field is concentrated near the wires, we conclude that pairs of parallel wires are effective transmission lines for conveying energy from one place to another.

Let the distance between the axes of the parallel wires (Fig. 3.5) be s and the radius of each wire be a. Let us assume that the wires are thin so that any small nonuniformity which may exist in the distribution of current round the periphery of each wire has a negligible effect on the distribution of the magnetic field. Equation 2–8 gives the magnetic intensity as a function of the distance from the axis of the wire. It was derived on the assumption that the current in the wire was steady so

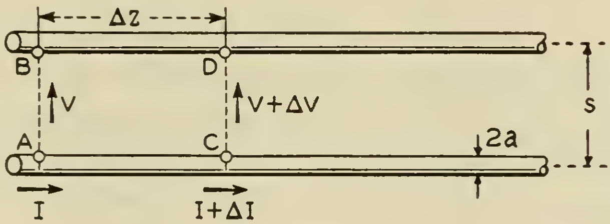


FIG. 3.5 Parallel wires.

that there could be no longitudinal displacement current. However, the longitudinal electric intensity at the surface of the wires is small (zero if the wires are perfect conductors). In the push-pull case it must reverse its direction (unless it is identically zero) as we pass from one wire to the other because the currents in the wires are opposite. To obtain the displacement current density, we have to multiply this small electric intensity by a very small quantity $\omega\varepsilon$. Hence, the effect of the longitudinal displacement current (if any) on the distribution of magnetic intensity in the vicinity of the wires is negligible. As a matter of fact, in the push-pull case there is no reason to expect any longitudinal electric intensity except near the ends of the parallel pair, since the wires are equally and oppositely charged and the electric lines are expected to run from one wire to the other. Hence, we may use equation 2–8 to calculate the magnetic displacement through the rectangular circuit $ACDBA$. The density of the displacement due to the current in the lower wire is $\mu I/2\pi\rho$ where ρ is the distance from its axis. We integrate it from $\rho = a$ to $\rho = s$ to obtain the displacement per unit length along the wires,

$$\Phi = \int_a^s \frac{\mu I}{2\pi\rho}\, d\rho = \frac{\mu I}{2\pi} \log \frac{s}{a} \cdot \tag{28}$$

The magnetic displacement due to the current in the upper wire is aiding; and the total displacement is 2Φ. Hence, the magnetic current passing through the area of circuit $ACDBA$ is $2j\omega\Phi\,\Delta z$, and its direction is toward the reader. According to the Faraday-Maxwell law, the negative of this current must equal the voltage round the circuit in the counterclockwise direction. Hence,

$$V_{AC} + V_{CD} + V_{DB} + V_{BA} = -\frac{j\omega\mu I}{\pi}\,\Delta z \log\frac{s}{a}\cdot \tag{29}$$

If the conductors are perfect, $V_{AC} = V_{DB} = 0$. If V is the transverse voltage from the lower wire to the upper $V_{CD} + V_{BA} = \Delta V$. Substituting in equation 29 and passing to the limit, we obtain

$$\frac{dV}{dz} = -ZI, \qquad Z = j\frac{\omega\mu}{\pi}\log\frac{s}{a}\cdot \tag{30}$$

Thus, the wires force the field to vary in the direction parallel to them by annihilating the longitudinal component of E. If the wires are not perfect conductors, the voltages V_{AC} and V_{DB} depend on the internal impedance of the wires (resistance at low frequencies and some reactance at high frequencies). If Z_i is the internal impedance of each per unit length,

$$Z = 2Z_i + \frac{j\omega\mu}{\pi}\log\frac{s}{a}\cdot \tag{31}$$

To obtain the second transmission equation we note that, if q is the charge per unit length of the lower wire, then the charge on the segment AC is $q\,\Delta z$. The time rate of increase of this charge is $j\omega q\,\Delta z$. If the dielectric is perfect, this must equal the difference $-\Delta I$ between the current entering the segment at A and the current leaving it at C. Hence, as Δz approaches zero,

$$\frac{dI}{dz} = -j\omega q. \tag{32}$$

Now, equation 2–15 gives the electric intensity due to a charged filament. Half the voltage from A to B is due to the charge on one wire. Hence,

$$\tfrac{1}{2}V = \int_a^s \frac{q}{2\pi\varepsilon\rho}\,d\rho = \frac{q}{2\pi\varepsilon}\log\frac{s}{a}\cdot \tag{33}$$

We express q in terms of V and substitute in equation 32,

$$\frac{dI}{dz} = -YV, \qquad Y = \frac{j\omega\pi\varepsilon}{\log(s/a)}\cdot \tag{34}$$

If the dielectric medium between the wires is not perfect, there will

be transverse leakage current which will account for a part of the decrease in the longitudinal current between A and C. In this case,

$$Y = \frac{(g + j\omega\varepsilon)\pi}{\log(s/a)}. \tag{35}$$

3.5 One-dimensional transmission equations

Maxwell's equations are the most general equations governing propagation of electromagnetic waves in three dimensions. Under various special conditions they reduce to the one-dimensional form 26, in which both wave functions depend on some one coordinate and in which the coefficients Y and Z are, in general, functions of this coordinate. If Y and Z are constants, the general solution may be expressed in terms of exponential functions. Comparing equation 26 with equations 1 and 2 and replacing $(g + j\omega\varepsilon)$ and $j\omega\mu$ by Y and Z, we obtain this solution from equations 4, 5, 6, and 7. Thus,

$$I = Ae^{-\Gamma z} + Be^{\Gamma z}, \qquad V = KAe^{-\Gamma z} - KBe^{\Gamma z}, \tag{36}$$

where the propagation constant Γ and the characteristic impedance K are

$$\Gamma = (ZY)^{1/2}, \qquad K = \left(\frac{Z}{Y}\right)^{1/2}. \tag{37}$$

If Z and Y are functions of z, there is no simple general solution. Each equation has to be considered individually. It is only in the special case when Z and Y are slowly varying functions of z that there exist general approximate solutions.*

3.6 Reflection

If infinitely long parallel wires are energized in push-pull at $z = 0$, either A or B in equations 36 must vanish, depending on whether $z < 0$ or $z > 0$. This is required by the law of conservation of energy, when there is dissipation of energy either in the wires or in the medium between them. This, of course, is always the case in practice. In theory it is often convenient to consider idealized nondissipative transmission lines for which the propagation constant is a pure imaginary and both terms in equation 36 remain finite at infinity. This case can be treated as the limit of the general case in which dissipative parameters approach zero. Then it is obvious that the terms that had to be omitted in the dissipative case must also be omitted in the idealized nondissipative case. We have seen that $B = 0$ when $z > 0$. In this case the

* S. A. Schelkunoff, *Applied Mathematics*, D. Van Nostrand, New York, 1948, pp. 218–220. In subsequent references only the title of this book will be given.

amplitude of the wave is attenuated and the phase retarded in the positive z direction. Similarly, we find that on the other side of the generator the amplitude is attenuated and the phase retarded in the negative z direction. Hence, on both sides of the generator the amplitude is attenuated and the phase retarded with increasing distance from the generator. In the nondissipative case the amplitude remains constant but the phase is retarded.

Suppose now that the wires extend only from $z = 0$ to $z = l$ and that the generator is at $z = 0$. In this case we must keep both A and B in equation 36. If the wires are simply terminated at $z = l$, the current must vanish there, and we obtain a relation between A and B. If the voltage across the generator, that is, $V(0)$, is given, we have another relation between A and B. Hence, the constants of integration are completely determined. More generally the wires may be terminated at $z = l$ into some "load impedance." If this impedance is known, we know the ratio $V(l)/I(l)$; hence, we have a relation between A and B. Comparing the waves in this finite section of the transmission line with those in the infinite line, we conclude that the first terms in the wave functions (equation 36) represent a wave generated at $z = 0$ and *incident* on the load, or more generally on some discontinuity, at $z = l$. The second terms represent the wave originating at the discontinuity and traveling back to the generator. This wave is called the *reflected wave*.

The ratio of the reflected and incident wave functions at the discontinuity is called the *reflection coefficient*; we denote it by q with a subscript indicating the particular wave function. Thus,

$$q_I(l) = \frac{B}{A} e^{2\Gamma l}, \qquad q_V(l) = -\frac{B}{A} e^{2\Gamma l}. \tag{38}$$

If Z is the impedance at $z = l$,

$$Z = \frac{V(l)}{I(l)} = K \frac{Ae^{-\Gamma l} - Be^{\Gamma l}}{Ae^{-\Gamma l} + Be^{\Gamma l}} = K \frac{1 + q_V}{1 - q_V}. \tag{39}$$

The ratio

$$k = \frac{Z}{K} \tag{40}$$

is called the *normalized impedance* ("normalized" with respect to the characteristic impedance) at $z = l$. Solving equation 39 for q_V, we obtain

$$q_V = \frac{Z - K}{Z + K} = \frac{k - 1}{k + 1}. \tag{41}$$

Equation 39 is often used in impedance measurements, particularly in microwave transmission lines and wave guides in which it is usually impossible to measure the necessary V and I (or E and H) at the discontinuity. For this purpose we define the *apparent reflection coefficient*

$$q_V(z) = -\frac{B}{A}e^{2\Gamma z} \tag{42}$$

at a typical point $z < l$. Taking the ratio of equations 38 and 42, we have

$$q_V(l) = q_V(z)e^{2\Gamma(l-z)}. \tag{43}$$

In good transmission lines the attenuation constant is small, and its effect for small distances may be neglected. Hence,

$$q_V(l) = q_V(z)e^{2j\beta(l-z)}. \tag{44}$$

Therefore the magnitude of the apparent reflection coefficient is constant along the line but the phase changes. At some points the phase is zero, the incident and reflected waves are in phase, and the voltage is maximum. A quarter wavelength from these points the incident and reflected waves are 180° out of phase, and the voltage is minimum. The *standing wave ratio* n defined by

$$n = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |q|}{1 - |q|} \tag{45}$$

can be determined experimentally at some distance from the discontinuity. From equation 45 we obtain the magnitude of the reflection coefficient

$$|q| = \frac{n-1}{n+1}\cdot \tag{46}$$

From equation 44 we find that, if the voltage is maximum at $z = z$, the phase of the reflection coefficient at $z = l$ is

$$\text{ph}[q_V(l)] = 2\beta(l - z), \tag{47}$$

since $\text{ph}[q_V(z)] = 0$ at the maximum point.

It is evident from equation 41 that there is no reflection if the load impedance equals the characteristic impedance.

It should be noted that all the foregoing equations have been obtained on the assumption that Z and Y are independent of z. In particular, the expression for the reflection coefficient is somewhat more complicated in the general case.* The foregoing theory of reflection has been

* S. A. Schelkunoff, *Electromagnetic Waves*, D. Van Nostrand, New York, 1943, p. 226. In subsequent references only the title of this book will be given.

developed for waves that are one-dimensional in the sense that only one coordinate is required in describing their propagation. In the next chapter we shall consider spherical waves generated by an infinitesimal current element, and find that the field distribution as a function of the angular coordinates is independent of the electromagnetic parameters of the medium. Hence, if the element is at the center of a sphere (Fig. 3.6) which is the boundary between two different homogeneous media, we may confine our attention to a typical radius. The tangential components of E and H, and, hence, their ratio which is called the wave impedance, must be continuous across the boundary. Once this condition is satisfied at one point of the boundary, it is satisfied automatically over the entire boundary because, as we have noted, the relative dependence of the field on the angular coordinates is the same for both media.

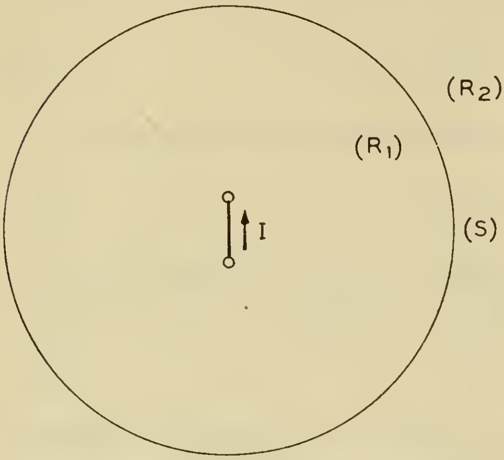


Fig. 3.6 An electric current element concentric with the spherical boundary S between two homogeneous media, R_1 and R_2.

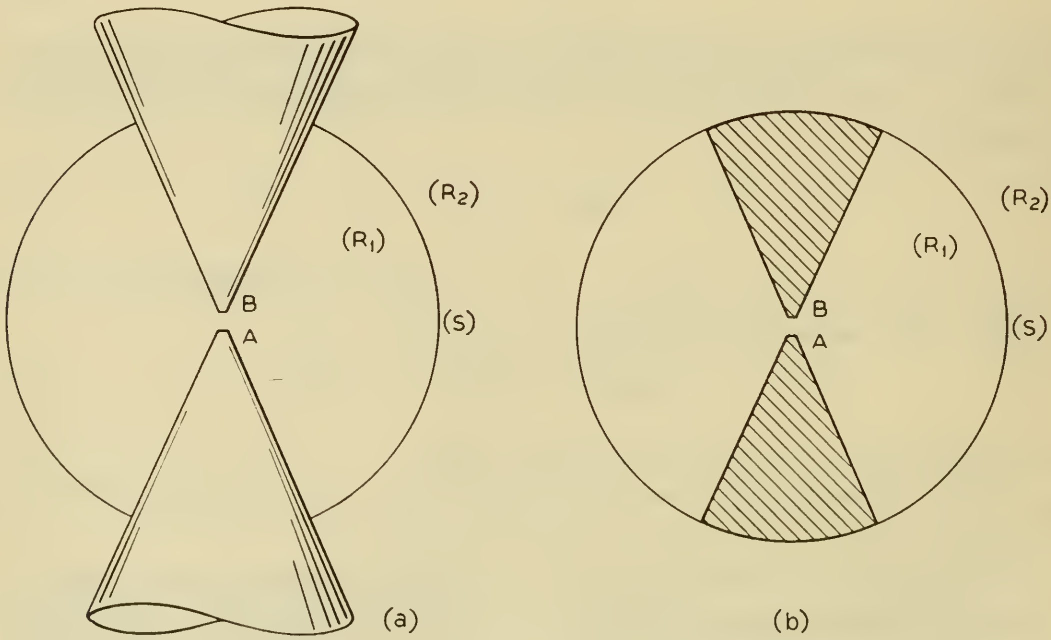


Fig. 3.7 Coaxial cones: (a) of infinite length, (b) of finite length.

If, however, the source of waves is not at the center of the sphere separating the media, then the nature of reflection becomes more com-

plicated. Reflection is no longer uniform. Another situation in which reflection is nonuniform is illustrated in Fig. 3.7. Waves may be generated by a voltage impressed between the apices of the cones. If the cones are infinitely long, the field distribution as a function of the angular coordinates is independent of the electromagnetic parameters of the medium between the cones. Hence, if these parameters change abruptly across some sphere concentric with the apices of the cones, we have an essentially one-dimensional case of reflection. But, if the cones are of finite length, reflection at the sphere of discontinuity S is nonuniform, even if the electromagnetic parameters of the regions R_1 and R_2 are the same. The boundary conditions become essentially three-dimensional because the dependence of various fields on the angular coordinates is different in the free-space region from that in the cone region R_1. The solution of the problem of reflection in this case depends on the decomposition of the fields in both regions into "modes of propagation."

PROBLEMS

3.2–1 Show that the field given by

$$H_\varphi = \frac{Ae^{-j\beta z}}{2\pi\rho}, \qquad E_\rho = \eta H_\varphi, \qquad E_z = E_\varphi = H_\rho = H_z = 0,$$

is consistent with Maxwell's equations and the boundary conditions at the surface of perfectly conducting cylinders coaxial with the z axis.

3.2–2 Using the result of the preceding problem, calculate the characteristic impedance of a coaxial pair of perfectly conducting cylinders whose radii are a and $b > a$.

Ans.
$$K = \frac{\eta}{2\pi} \log \frac{b}{a} \cdot$$

3.2–3 Using the result of the preceding problem show that

$$K = \frac{\eta s}{2\pi c}\left[1 + \frac{1}{3}\left(\frac{s}{2c}\right)^2 + \frac{1}{5}\left(\frac{s}{2c}\right)^4 + \frac{1}{7}\left(\frac{s}{2c}\right)^6 + \cdots\right],$$

where $s = b - a$ and $c = \frac{1}{2}(a + b)$.

3.2–4 Calculate the power carried by the wave described in Problem 3.2–1 outside a *single* cylinder of radius a. What is the important practical conclusion from the result?

3.5–1 Express the voltage and current in a transmission line in terms of their values at $z = l$.

Ans.
$$V(z) = V(l) \cos \beta(l - z) + jK\, I(l) \sin \beta(l - z),$$
$$I(z) = I(l) \cos \beta(l - z) + jK^{-1}\, V(l) \sin \beta(l - z).$$

3.5–2 Show that the input impedance of a nondissipative transmission line of length l, terminated into an impedance Z_t, is

$$Z_i = K \frac{Z_t \cos \beta l + jK \sin \beta l}{K \cos \beta l + jZ_t \sin \beta l} = K \frac{1 + qe^{-2j\beta l}}{1 - qe^{-2j\beta l}},$$

where q is the voltage reflection coefficient.

4

SPHERICAL WAVES

4.1 Introduction

In the preceding chapter we have considered plane waves which may be generated under certain conditions. No matter what precautions are taken, it is rarely possible to avoid generating spherical waves as well, although the intensity of these waves may be small. An a-c generator

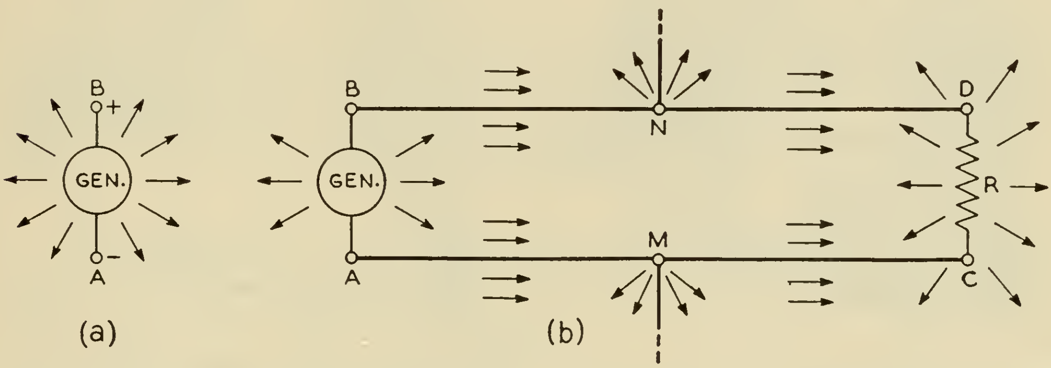


Fig. 4.1 Spherical and plane waves: (*a*) an electric generator with floating terminals generates a spherical wave — a feeble wave if the distance between the terminals is a small fraction of a quarter wavelength; (*b*) when the generator is connected to a pair of closely spaced wires, it generates a strong plane wave guided by the wires and a feeble spherical wave. The other end of the transmission line is also a source of a spherical wave as well as a sink of the plane wave; and, as far as the generator is concerned, the energy carried away by the spherical wave will appear as a change in the resistance R.

with its terminals floating (Fig. 4.1*a*), generates a spherical wave, although at low frequencies its intensity is exceedingly small. If the generator is connected to a pair of wires, the wave is guided — as we have found in the preceding chapter — to the load at the far end (Fig. 4.1*b*). Weak spherical waves are still generated at both ends; they are also generated at intermediate points where some device may be

connected to the transmission line. In such cases spherical waves are of practical interest only as parasitic end effects.*

Radio communication, on the other hand, actually depends on spherical waves. In this chapter we consider spherical waves under two extreme conditions. First we consider waves on infinitely long diverging conductors; then, we consider the case in which the conductor is very short and the wave is entirely in free space.

4.2 Maxwell's equations in spherical coordinates

Spherical waves may be expressed most conveniently in terms of spherical coordinates, in which the position of a point P (Fig. 4.2) is given by the distance r from a fixed point O, called the origin of the coordinate

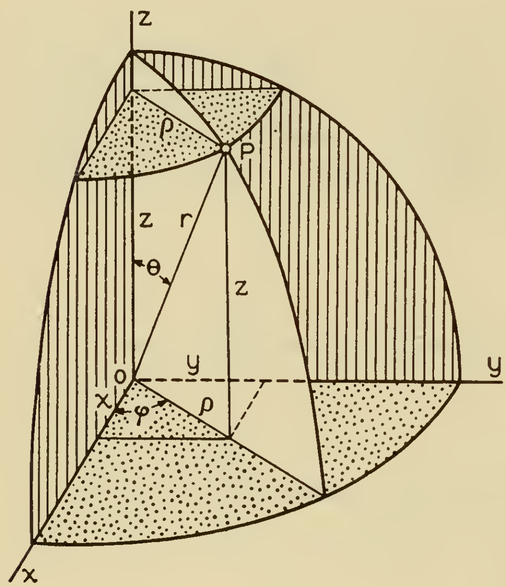


FIG. 4.2 Cartesian (x, y, z), cylindrical (ρ, φ, z), and spherical (r, θ, φ) coordinates.

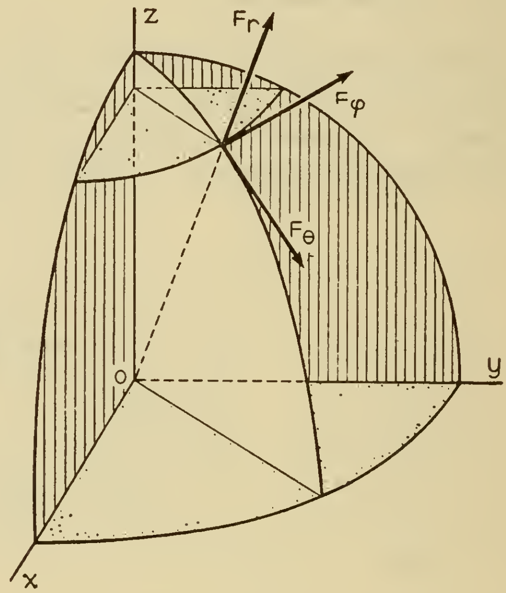


FIG. 4.3 Spherical components of a vector.

system, by the polar angle θ between OP and the axis OZ of the coordinate system, and by the azimuth φ, that is, by the angle between the meridian passing through P and the principal meridian in the plane XOZ. Any vector F may be resolved into three mutually perpendicular components F_r, F_θ, F_φ, as shown in Fig. 4.3. The r component is in the direction of the radius through P, the θ component is tangential to the meridian, and the φ component is tangential to the circle of latitude. The positive directions of these components are chosen to coincide with the directions of increasing coordinates.

* C. Manneback, Radiation from transmission lines, *AIEE Jour.*, **42**, February 1923, pp. 95–105.

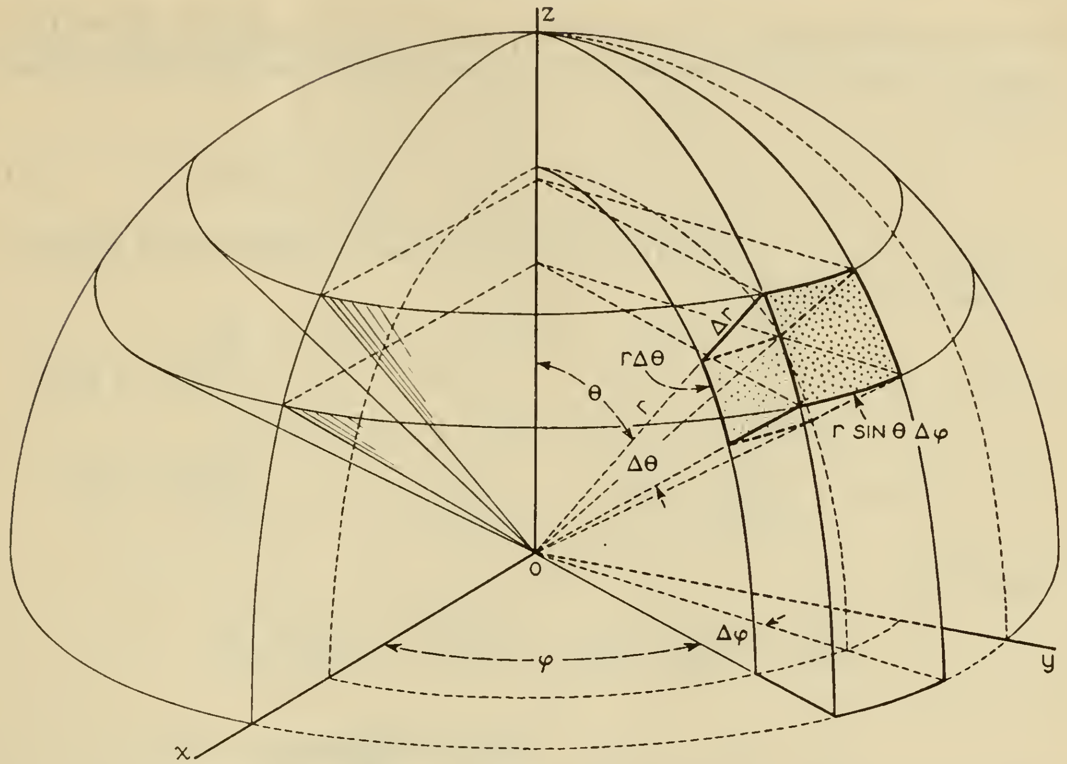


FIG. 4.4 An elementary cell in spherical coordinates is bounded by two nearly equal spheres concentric with the origin O, two nearly equal cones coaxial with OZ, and two half-planes issuing from OZ and forming a small angle. The dimensions of the cell for assigned coordinate differences, Δr, $\Delta\theta$, $\Delta\varphi$, depend on the position of the cell.

To express the field equations in spherical coordinates, we shall consider an elementary cell (Fig. 4.4) formed by two concentric spheres whose radii are r and $r + \Delta r$, by two coaxial cones whose angles with the axis are θ and $\theta + \Delta\theta$, and by two half-planes whose angles with the XOZ plane are φ and $\varphi + \Delta\varphi$. To obtain the complete set of equations, we must apply Maxwell's equations to each of a set of three mutually perpendicular faces of the cell. Take the face $ABCD$ bounded by the r lines and θ lines (Fig. 4.5); it lies in a half-plane given by a constant φ coordinate. By the Faraday-Maxwell law the anticlockwise emf around $ABCD$ must equal the magnetic displacement current through

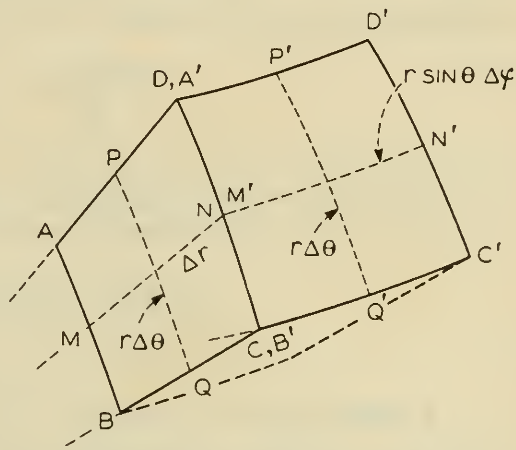


FIG. 4.5 Maxwell's differential equations in any coordinate system are obtained by applying the laws of electromagnetic induction to the faces of a typical elementary cell.

the area of the circuit in the direction away from the reader. Since this current is equal to the product of its density $j\omega\mu H_\varphi$ and the area $r\,\Delta r\,\Delta\theta$, we have

$$V_{AB} + V_{BC} + V_{CD} + V_{DA} = j\omega\mu H_\varphi r\,\Delta r\,\Delta\theta. \tag{1}$$

The four voltages on the left may be expressed in terms of the voltages V_{MN} and V_{PQ} as follows:

$$V_{AB} = V_{PQ} - \frac{\partial V_{PQ}}{\partial r}\left(\tfrac{1}{2}\,\Delta r\right), \qquad V_{DC} = V_{PQ} + \frac{\partial V_{PQ}}{\partial r}\left(\tfrac{1}{2}\,\Delta r\right);$$

$$V_{BC} = V_{MN} + \frac{\partial V_{MN}}{\partial \theta}\left(\tfrac{1}{2}\,\Delta\theta\right), \qquad V_{AD} = V_{MN} - \frac{\partial V_{MN}}{\partial \theta}\left(\tfrac{1}{2}\,\Delta\theta\right).$$

Therefore,

$$V_{AB} + V_{CD} = V_{AB} - V_{DC} = -\frac{\partial V_{PQ}}{\partial r}\,\Delta r,$$

$$V_{BC} + V_{DA} = V_{BC} - V_{AD} = \frac{\partial V_{MN}}{\partial \theta}\,\Delta\theta.$$

Substituting in equation 1, we obtain

$$-\frac{\partial V_{PQ}}{\partial r}\,\Delta r + \frac{\partial V_{MN}}{\partial \theta}\,\Delta\theta = j\omega\mu H_\varphi r\,\Delta r\,\Delta\theta. \tag{2}$$

The voltage V_{PQ} is the product of the electric intensity E_θ and the length $r\,\Delta\theta$ of PQ; similarly, V_{MN} is the product of E_r and Δr. Substituting in equation 2 and canceling $\Delta r\,\Delta\theta$, we have

$$-\frac{\partial(rE_\theta)}{\partial r} + \frac{\partial E_r}{\partial \theta} = j\omega\mu r H_\varphi. \tag{3}$$

Similarly, computing the mmf around the circuit and equating it to the electric current through the area of the circuit, we obtain

$$-\frac{\partial(rH_\theta)}{\partial r} + \frac{\partial H_r}{\partial \theta} = -(g + j\omega\varepsilon) r E_\varphi. \tag{4}$$

A similar pair of equations will be obtained for the circuit $A'B'C'D'$ formed by circles of latitude and meridians on a sphere of radius r (Fig. 4.5). And, finally, the third pair of equations is obtained from a circuit drawn on a cone of angle θ. Thus, we obtain the following complete set of Maxwell's equations in spherical coordinates for a steady-

state electromagnetic field:

$$
\begin{aligned}
\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta} &= -j\omega\mu r H_\varphi, \\
\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta} &= (g + j\omega\varepsilon) r E_\varphi, \\
\frac{\partial}{\partial \theta}(\sin\theta\, E_\varphi) - \frac{\partial E_\theta}{\partial \varphi} &= -j\omega\mu r \sin\theta\, H_r, \\
\frac{\partial}{\partial \theta}(\sin\theta\, H_\varphi) - \frac{\partial H_\theta}{\partial \varphi} &= (g + j\omega\varepsilon) r \sin\theta\, E_r, \\
\frac{\partial E_r}{\partial \varphi} - \sin\theta \frac{\partial (rE_\varphi)}{\partial r} &= -j\omega\mu r \sin\theta\, H_\theta, \\
\frac{\partial H_r}{\partial \varphi} - \sin\theta \frac{\partial (rH_\varphi)}{\partial r} &= (g + j\omega\varepsilon) r \sin\theta\, E_\theta.
\end{aligned} \tag{5}
$$

4.3 Circularly symmetric fields

If the field is independent of φ, the partial derivatives with respect to φ vanish, and equations 5 reduce to two independent sets of equations. One set contains E_r, E_θ, H_φ; thus,

$$\frac{\partial}{\partial \theta}(\sin\theta\, H_\varphi) = (g + j\omega\varepsilon) r \sin\theta\, E_r, \tag{6}$$

$$\frac{\partial}{\partial r}(rH_\varphi) = -(g + j\omega\varepsilon) r E_\theta, \tag{7}$$

$$\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta} = -j\omega\mu r H_\varphi. \tag{8}$$

The magnetic lines are circles coaxial with OZ, and the electric lines lie in planes passing through OZ. These waves are called *circular magnetic waves.*

The other set contains H_r, H_θ, E_φ; thus,

$$\frac{\partial}{\partial \theta}(\sin\theta\, E_\varphi) = -j\omega\mu r \sin\theta\, H_r, \tag{9}$$

$$\frac{\partial}{\partial r}(rE_\varphi) = j\omega\mu r H_\theta, \tag{10}$$

$$\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta} = (g + j\omega\varepsilon) r E_\varphi. \tag{11}$$

In this case the electric lines are coaxial circles, and the magnetic lines are in axial planes. These waves are called *circular electric waves.*

4.4 Fields depending only on the distance from the origin

If the field depends only on r, equation 6 reduces to

$$H_\varphi \cos\theta = (g + j\omega\varepsilon) r E_r \sin\theta,$$

$$\frac{H_\varphi}{rE_r} = (g + j\omega\varepsilon)\tan\theta.$$

This result contradicts our initial assumption that the field depends only on r; hence, there are no uniform spherical electromagnetic waves analogous to uniform sound waves generated by a pulsating sphere.*

4.5 Spherical waves at great distances from their origin

Substituting E_r from equation 6 in equation 8, we obtain

$$\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{(g + j\omega\varepsilon)r^2}\frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[(rH_\varphi)\sin\theta]\right\} = -j\omega\mu r H_\varphi. \quad (12)$$

Thus, we have two equations, 7 and 12, for rE_θ and rH_φ. As r increases, the second term in equation 12 diminishes in comparison with the first, and the two equations approach with increasing accuracy the following set:

$$\frac{\partial}{\partial r}(rE_\theta) = -j\omega\mu(rH_\varphi), \qquad \frac{\partial}{\partial r}(rH_\varphi) = -(g + j\omega\varepsilon)(rE_\theta). \quad (13)$$

Eliminating rE_θ, we obtain

$$\frac{\partial^2}{\partial r^2}(rH_\varphi) = \sigma^2(rH_\varphi), \quad (14)$$

where

$$\sigma^2 = j\omega\mu(g + j\omega\varepsilon), \qquad \sigma = [j\omega\mu(g + j\omega\varepsilon)]^{1/2}. \quad (15)$$

The propagation constant is seen to be the same as for uniform plane waves.

The general solution of equation 14 is

$$rH_\varphi = Ae^{-\sigma r} + Be^{\sigma r}, \quad (16)$$

where A and B are arbitrary constants as far as r is concerned *but are functions of* θ. The first term in equation 16 decreases exponentially

* Electromagnetic waves radiated by a uniformly heated sphere are uniform but only on the average. They consist of a large number of waves in arbitrary phase relationships with arbitrary orientations of the principal planes of radiation.

with increasing distance from the origin (as long as the medium is dissipative, $g \neq 0$). The second term increases exponentially with increasing r; that is, it decreases with decreasing r. Hence, the first term represents a wave coming from the origin and the second a wave going to the origin from infinity.

From equations 13 and 16, we find

$$rE_\theta = \eta A e^{-\sigma r} - \eta B e^{\sigma r}, \tag{17}$$

where η is the intrinsic impedance, already defined in the preceding chapter,

$$\eta = \frac{j\omega\mu}{\sigma} = \frac{\sigma}{g + j\omega\varepsilon} = \left(\frac{j\omega\mu}{g + j\omega\varepsilon}\right)^{1/2}. \tag{18}$$

Similarly from equations 9, 10, and 11 we obtain, for large r,

$$rH_\theta = Ae^{-\sigma r} + Be^{\sigma r}, \qquad rE_\varphi = -\eta A e^{-\sigma r} + \eta B e^{\sigma r}. \tag{19}$$

The last terms in equations 16, 17, and 19 become infinitely large at infinity. Hence, for waves generated inside a sphere of finite radius,

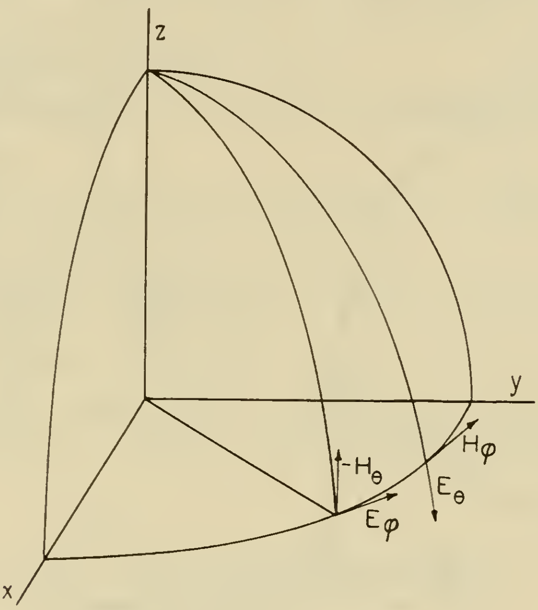


FIG. 4.6 Relative directions of E and H in outgoing traveling waves at large distances from their centers.

B must be zero. Thus, from equations 16 and 17 we find

$$rH_\varphi = Ae^{-\sigma r}, \qquad E_\theta = \eta H_\varphi. \tag{20}$$

Similarly, from equations 19 we have

$$rH_\theta = Ae^{-\sigma r}, \qquad E_\varphi = -\eta H_\theta = \eta(-H_\theta). \tag{21}$$

Figure 4.6 shows that in both cases the direction of propagation is that

in which a right-handed screw advances when its handle is turned through 90° from E to H.

4.6 TEM spherical waves

By definition, the radial components of E and H in TEM spherical waves vanish,

$$E_r = 0, \qquad H_r = 0. \tag{22}$$

For fields with circular magnetic lines the second equation is satisfied automatically; but the first reduces equations 6, 7, and 8 to a simpler set. Thus, equation 6 becomes

$$\frac{\partial}{\partial\theta}(\sin\theta H_\varphi) = 0, \qquad \sin\theta H_\varphi = \hat{H}_\varphi(r), \tag{23}$$

where, as indicated by the notation, $\hat{H}_\varphi(r)$ is a function of r only.

By the Ampère-Maxwell law, the mmf round a typical magnetic line must equal the radial electric current. In the present case there is

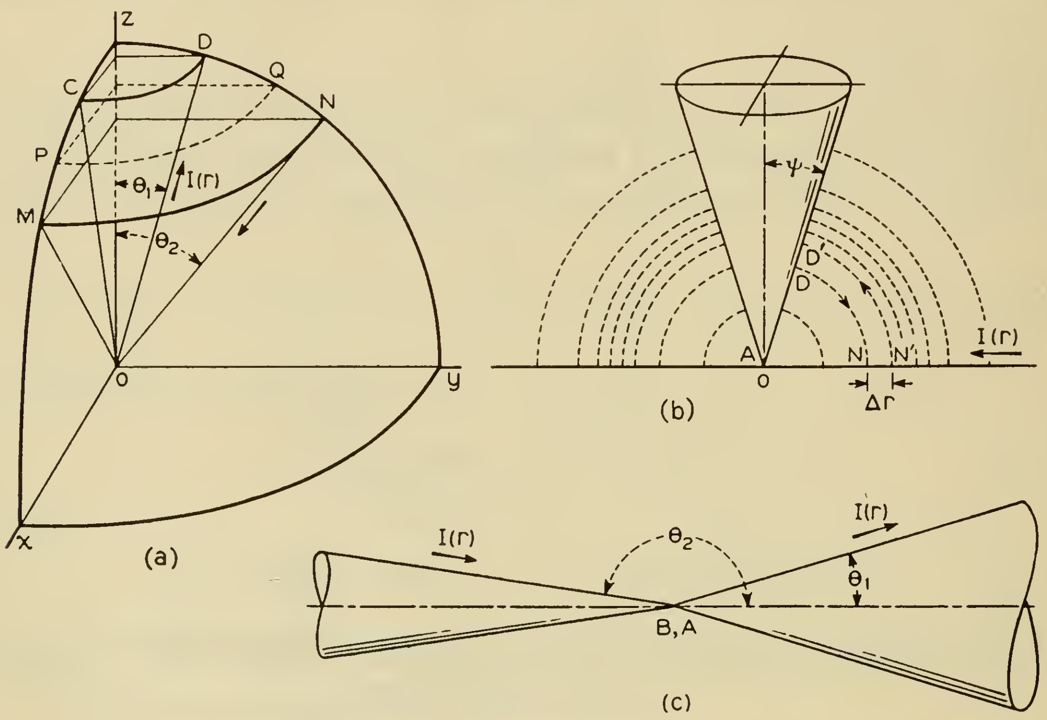


FIG. 4.7 Coaxial conical conductors (a plane b is a special case of a conical surface).

no radial displacement current, and, hence, there must be a conduction current; otherwise, H_φ would have to vanish, and with it the entire field. The assumed symmetry of the field requires that this current be distributed on coaxial cones (Fig. 4.7a) and that its density be inde-

pendent of φ. Since $E_r = 0$, the cones must be perfect conductors. The electric lines run along the meridians from one cone to the other.

Let $I(r)$ be the current at distance r in the cone $\theta = \theta_1$ (Fig. 4.7a); then, by the Ampère-Maxwell law, we have

$$2\pi r \sin\theta\, H_\varphi = I, \qquad H_\varphi = \frac{I}{2\pi r \sin\theta}, \qquad \theta_1 < \theta < \theta_2. \tag{24}$$

The same current flows in the cone $\theta = \theta_2$ but in the opposite direction.

Equations 7 and 8 are identical with 13; hence, their solutions are of the form given by equations 16 and 17 with the constants of integration inversely proportional to $\sin\theta$. Thus,

$$\begin{aligned} H_\varphi &= \frac{I_0{}^+ e^{-\sigma r} + I_0{}^- e^{\sigma r}}{2\pi r \sin\theta}, \\ E_\theta &= \frac{\eta I_0{}^+ e^{-\sigma r} - \eta I_0{}^- e^{\sigma r}}{2\pi r \sin\theta}, \end{aligned} \tag{25}$$

where the constants of integration have been expressed in terms of the currents ($I_0{}^+$, the outgoing current, and $I_0{}^-$, the incoming current) associated with the two traveling waves.

The *transverse voltage* $V(r)$ along a typical meridian from $\theta = \theta_1$ to $\theta = \theta_2$ is

$$V(r) = \int_{\theta_1}^{\theta_2} rE_\theta\, d\theta = KI_0{}^+ e^{-\sigma r} - KI_0{}^- e^{\sigma r}, \tag{26}$$

where the characteristic impedance K of the "biconical transmission line" is

$$K = \eta \int_{\theta_1}^{\theta_2} \frac{d\theta}{2\pi \sin\theta} = \frac{\eta}{2\pi}\int_{\theta_1}^{\theta_2} \frac{d\theta}{2 \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta} = \frac{\eta}{2\pi}\int_{\theta_1}^{\theta_2} \frac{d\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\theta};$$

that is,

$$K = \frac{\eta}{2\pi} \log\left(\tan\tfrac{1}{2}\theta_2 \cot\tfrac{1}{2}\theta_1\right). \tag{27}$$

In particular, for a cone above a conducting plane (Fig. 4.7b), $\theta_2 = \frac{1}{2}\pi$, $\tan\frac{1}{2}\theta_2 = 1$, and

$$K = \frac{\eta}{2\pi} \log\cot\tfrac{1}{2}\psi = 60 \log\cot\tfrac{1}{2}\psi. \tag{28}$$

The numerical value is for cones in free space.

For two cones with equal angles (Fig. 4.8), we have

$$K = \frac{\eta}{\pi} \log\cot\tfrac{1}{2}\psi = 120 \log\cot\tfrac{1}{2}\psi. \tag{29}$$

If θ_1 and θ_2 are both small, then $\cot \frac{1}{2}\theta_1 \simeq 2/\theta_1$, and $\tan \frac{1}{2}\theta_2 \simeq \frac{1}{2}\theta_2$; hence,

$$K = \frac{\eta}{2\pi} \log \frac{\theta_2}{\theta_1} \cdot \tag{30}$$

Let a and b be the radii of the cones at distance r from the apex; then, $\theta_1 = a/r$, $\theta_2 = b/r$, and

$$K = \frac{\eta}{2\pi} \log \frac{b}{a} \cdot \tag{31}$$

As θ_1 and θ_2 approach zero, the cones approach a pair of coaxial cylinders, and equation 31 gives the value of the corresponding characteristic impedance.

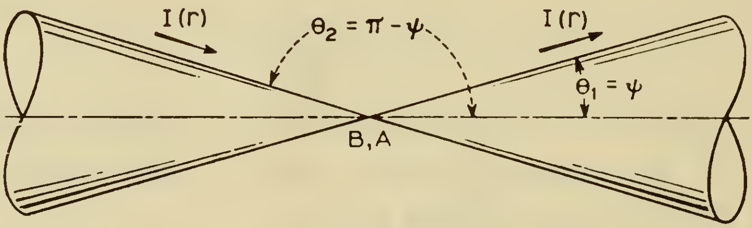


FIG. 4.8 Two equal coaxial cones.

It should be noted that, while the field intensities increase indefinitely as r approaches zero, the voltage and current remain finite. Equations 7 and 8 can be expressed in terms of V and I, since, by equations 24, 25, and 26, we have

$$H_\varphi = \frac{I}{2\pi r \sin \theta}, \qquad E_\theta = \frac{\eta V}{2\pi r K \sin \theta} \cdot \tag{32}$$

Thus, we obtain

$$\frac{dV}{dr} = -j\omega LI, \qquad \frac{dI}{dr} = -(G + j\omega C)V, \tag{33}$$

where the distributed inductance, conductance, and capacitance of the biconical transmission line are

$$\begin{aligned} L &= \frac{\mu}{2\pi} \log(\cot \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2), \\ G &= \frac{2\pi g}{\log(\cot \frac{1}{2}\theta_1 \tan \frac{1}{2}\theta_2)}, \\ C &= \frac{2\pi \varepsilon}{\log(\cot \frac{1}{2}\theta_1 \tan \frac{1}{2}\theta_2)} \cdot \end{aligned} \tag{34}$$

TEM waves with circular electric lines are obtained from equations 9, 10, and 11 by letting $H_r = 0$. In this case the magnetic lines run along the meridians, and the cones must be perfect magnetic conductors. No such conductors have been observed, and this case is impossible to realize in practice.

4.7 Principal waves on nonconical wires

In the case of nonconical wires (Fig. 4.9), strictly transverse electromagnetic waves cannot exist. However, there exist waves which are very nearly transverse and in which electric lines of force still run principally from one wire to the other. For reasons that will become clear

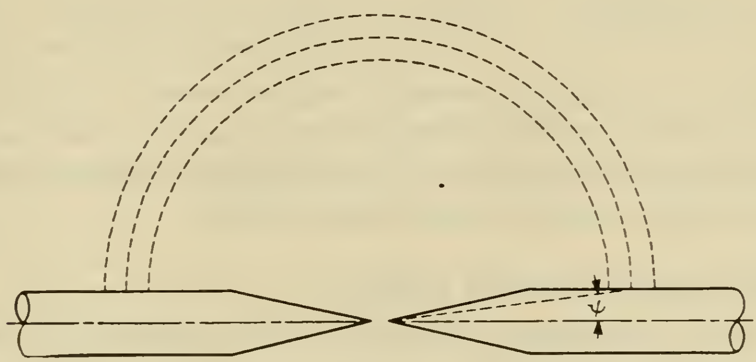


Fig. 4.9 Coaxial cylindrical conductors with tapered ends and electric lines of force in principal waves.

later, these waves, as well as TEM waves, are grouped together under the name "principal waves." When the wires are thin, we shall assume that the field distribution, as a function of θ, is still given by the equations of the preceding section.* This field does not quite satisfy the boundary condition that the tangential electric intensity should vanish on the surface of the wires. Thus, where the wires are cylindrical (Fig. 4.9), this field component is

$$E_z = -E_\theta \sin \psi, \tag{35}$$

where ψ is the "cone angle" at distance r. In the exact expressions there must be another small term which reduces E_z on the wires to zero. This term we shall neglect when we integrate E_θ to obtain V, and H_φ to obtain I in equations† 33. In this way we shall obtain L, G, C as given

* For a mathematical justification of this assumption, see S. A. Schelkunoff, Principal and complementary waves on antennas, *IRE Proc.*, 34, January 1946, pp. 23–32.

† Because of this integration, the errors in our final results will be of the second order of magnitude.

by equations 34, but with $\theta_1 = \psi$ and $\theta_2 = \pi - \psi$ varying with r. Since we have already assumed that ψ is small, we may also use this approximation to simplify equations 34. Thus, noting that $\psi = a/r$ where $a(r)$ is the radius of the wire at distance r, we obtain

$$L = \frac{\mu}{\pi}\log\frac{2r}{a}, \qquad C = \frac{\pi\varepsilon}{\log\dfrac{2r}{a}}, \qquad G = \frac{\pi g}{\log\dfrac{2r}{a}}. \tag{36}$$

When $g = 0$, the characteristic impedance becomes

$$K = \left(\frac{L}{C}\right)^{1/2} = \frac{\eta}{\pi}\log\frac{2r}{a} = 120\log\frac{2r}{a}, \tag{37}$$

the numerical coefficient applying as usual to free space. If the radii a_1, a_2 of the two wires are different, the same equations apply with $a = \sqrt{a_1 a_2}$. To the present order of approximation r is equal to z.

In the above approximations, we must assume that the wires are tapered when r is comparable to or less than the diameter of the wires. This region will be considered in Section 12.10.

4.8 TEM waves on coaxial cylinders

We have already discussed in Section 4.6 the possibility of considering a pair of coaxial cylinders as a limiting case of two coaxial cones whose angles θ_1 and θ_2 approach zero, and thus we obtained the characteristic impedance of the coaxial pair. To obtain the expressions for the field we note that, as θ approaches zero, r approaches z; also,

$$r\sin\theta = \rho, \tag{38}$$

where ρ is the distance from the axis of the cylinders. Hence, equations 24 and 25 become

$$\begin{aligned} H_\varphi &= \frac{I}{2\pi\rho} = \frac{I_0{}^+e^{-\sigma z} + I_0{}^-e^{\sigma z}}{2\pi\rho}, \\ E_\rho &= \frac{\eta I_0{}^+e^{-\sigma z} - \eta I_0{}^-e^{\sigma z}}{2\pi\rho}, \end{aligned} \tag{39}$$

at any point between the cylinders $(a < \rho < b)$.

4.9 TEM waves on parallel wires

Let us assume that the radius of the outer cylinder in the preceding section is very large, so that equations 39 are approximately true, even if the two cylinders are not quite coaxial. Let us insert a second inner cylinder parallel to the first (Fig. 4.10) and assume that the interaxial distance l is fairly large compared with the sum of the radii a_1, a_2 (at

least twice as large). There will be a "proximity effect" consisting of a displacement of the electric charge across one cylinder due to the tangential electric intensity of the charge on the other; but this effect will diminish as a_1, a_2 become smaller or l becomes larger. If the currents in the inner cylinders are equal and opposite, the total current in the distant "coaxial" cylinder will vanish. Since there will be some magnetic intensity tangential to this latter cylinder, some current will flow in it; but it will have to be in one direction on one side and in the opposite direction on the other. As the distances ρ_1, ρ_2 to a point P on the outer cylinder become larger and larger, the fields of the two cylinders tend to cancel more completely, and the currents induced in the outer cylinder will become smaller and smaller. In the limit we have just two parallel wires.

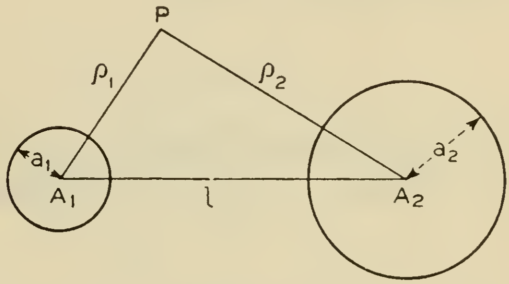


FIG. 4.10 Cross section of two parallel cylinders.

By equation 39 the electric intensity on the line A_1A_2 joining the axes of the wires, between A_1 and A_2 (Fig. 4.10) is

$$E_{\rho_1} = \frac{\eta I_0{}^+ e^{-\sigma z} - \eta I_0{}^- e^{\sigma z}}{2\pi\rho_1} - \frac{-\eta I_0{}^+ e^{-\sigma z} + \eta I_0{}^- e^{\sigma z}}{2\pi\rho_2}, \tag{40}$$

provided E_{ρ_1} is positive in the direction A_1A_2. Integrating from a point on the surface of one wire to a point on the surface of the other, we obtain the transverse voltage between the wires. It is better, however, to let ρ_1 vary from a_1 to l, so that the corresponding voltage is taken to an average position on the second wire. Similarly we let ρ_2 vary from l to a_2. Thus, we obtain

$$V = \int E_{\rho_1}\, d\rho_1 = KI_0{}^+ e^{-\sigma z} - KI_0{}^- e^{\sigma z},$$

where

$$K = \frac{\eta}{2\pi}\left(\log\frac{l}{a_1} + \log\frac{l}{a_2}\right) = \frac{\eta}{\pi}\log\frac{l}{\sqrt{a_1a_2}}. \tag{41}$$

It is not very difficult to obtain the exact formula* for K, but the above approximation is sufficient for our purposes.

4.10 Principal waves on diverging wires

Comparing equation 37 for the characteristic impedance of principal

* *Electromagnetic Waves*, pp. 283–285.

waves on wires diverging at an angle of 180° with equation 41 for parallel wires, we find that the equations are the same. This is not a coincidence, for it can be shown* that both formulas are special cases of a more general equation,

$$K = \frac{\eta}{\pi} \log k, \qquad k = \frac{d(r)}{[a_1(r)a_2(r)]^{1/2}}, \tag{42}$$

for two diverging wires (Fig. 4.11) whose radii are a_1 and a_2 while d is the distance between the corresponding elements of the wires.

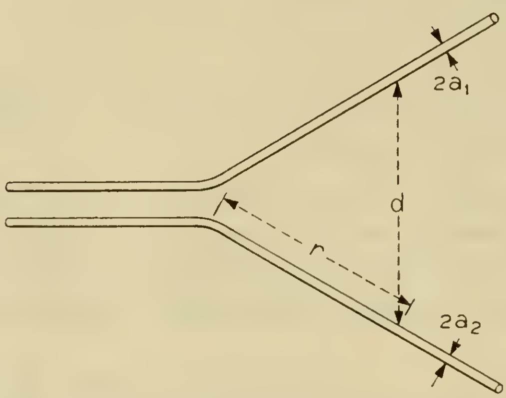


FIG. 4.11 Diverging wires connected to parallel wires.

4.11 Principal waves on cage structures

It can also be shown by the method given elsewhere† that for a "cage" (Fig. 4.12) consisting of $2n$ conical wires equidistributed on the surfaces of two coaxial cones with angles ψ and $\pi - \psi$, the characteristic impedance is given by the same equation as for solid cones with an effective angle determined by

$$\tan(\tfrac{1}{2}\psi_{\text{eff}}) = \tan(\tfrac{1}{2}\psi)\left[\frac{n\tan(\frac{1}{2}\psi_0)}{\tan(\frac{1}{2}\psi)}\right]^{1/n}, \tag{43}$$

where ψ_0 is the angle of each conical wire.

When the wires are thin and ψ is small, the effective radius of the cage may be obtained from equation 43; thus,

$$a_{\text{eff}} = a\left(\frac{na_0}{a}\right)^{1/n}, \tag{44}$$

where a_0 is the radius of each wire and a is the radius of the cage. In this form we may use the equation to obtain the equivalent radius of a

* *Ibid.*, p. 293.
† *Ibid.*, pp. 292–293.

cage formed by wires equispaced on a cylindrical surface. When $n = 2$, the formula gives $a_{\text{eff}} = \sqrt{a_0 s}$, where s is the interaxial distance between the wires.

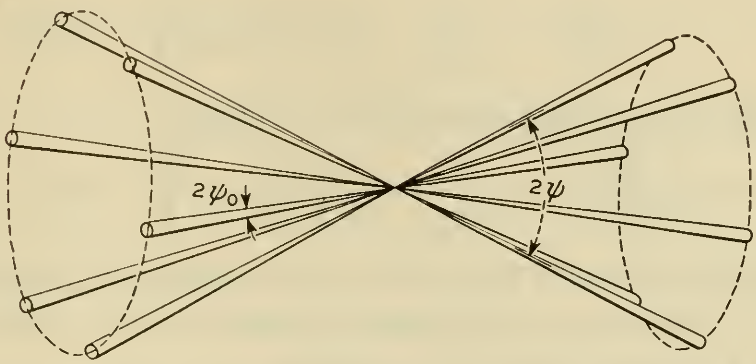


FIG. 4.12 A conical cage formed by thin conical conductors equispaced on a conical surface, $n = 6$.

4.12 Higher modes of propagation

So far we have been considering the principal waves in which the electric lines of force run from one conductor to the other, or from one group of conductors to the other, and in which the electric intensity in the direction of propagation is either exactly equal to or nearly equal to zero. In this section we shall examine more general circularly symmetric waves in which the radial field does not vanish.

First let us conside· circular magnetic waves given by equations 6, 7, 8. From these equations, we have

$$E_r = \frac{1}{(g + j\omega\varepsilon)r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, H_\varphi), \tag{45}$$

$$E_\theta = -\frac{1}{(g + j\omega\varepsilon)r} \frac{\partial}{\partial r} (rH_\varphi). \tag{46}$$

Substituting in equation 8, we obtain

$$\frac{\partial^2}{\partial r^2} (rH_\varphi) + \frac{1}{r^2} \frac{\partial}{\partial\theta} \left\{\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} [(rH_\varphi) \sin\theta]\right\} = \sigma^2(rH_\varphi). \tag{47}$$

Let us now assume that rH_φ is a product of a function of r and a function of θ,

$$rH_\varphi = R(r) \frac{d\Theta(\theta)}{d\theta} \cdot \tag{48}$$

The second factor is represented as the derivative of a function merely in anticipation of mathematical simplifications. Substituting in equa-

tion 47 and integrating with respect to θ, we have

$$\frac{\partial^2}{\partial r^2}(R\Theta) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(R\frac{d\Theta}{d\theta}\sin\theta\right) = \sigma^2 R\Theta,$$

$$\Theta\frac{d^2R}{dr^2} + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\sin\theta\right) = \sigma^2 R\Theta.$$

Multiplying by r^2 and dividing by $R\Theta$, we obtain

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\sin\theta\right) = \sigma^2 r^2. \tag{49}$$

The second term is not a function of r (by our initial assumption), and it cannot be a function of θ because the remaining two terms are independent of θ; hence, it must be a constant,

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\sin\theta\right) = k. \tag{50}$$

Inserting this constant in equation 49, we have

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + k = \sigma^2 r^2. \tag{51}$$

These two equations may be expressed as follows:

$$\sin\theta\frac{d^2\Theta}{d\theta^2} + \cos\theta\frac{d\Theta}{d\theta} - k\sin\theta\,\Theta = 0, \tag{52}$$

$$\frac{d^2R}{dr^2} = \left(\sigma^2 - \frac{k}{r^2}\right)R. \tag{53}$$

To obtain equation 52 we multiply equation 50 by $\Theta\sin\theta$, perform the indicated differentiation, and transpose all terms to the left-hand side of the equation; to obtain equation 53 we multiply equation 51 by R and divide by r^2.

From equations 48 and 45, using equation 50, we find

$$E_r = \frac{kR\Theta}{(g + j\omega\varepsilon)r^2}\cdot \tag{54}$$

Also from equations 46 and 48, we have

$$E_\theta = -\frac{1}{(g + j\omega\varepsilon)r}\frac{dR}{dr}\frac{d\Theta}{d\theta}\cdot \tag{55}$$

Thus, the entire field has been expressed in terms of R and Θ, which must satisfy equations 53 and 52.

Similarly, in the case of circular electric waves, we find

$$rE_\varphi = R(r)\,\frac{d\Theta(\theta)}{d\theta}\,, \qquad rH_\theta = \frac{1}{j\omega\mu}\,\frac{dR}{dr}\,\frac{d\Theta}{d\theta}\,,$$
$$H_r = -\,\frac{kR\Theta}{j\omega\mu r^2}\,. \tag{56}$$

TEM waves are obtained by letting k vanish; higher modes of propagation are obtained when $k \neq 0$. The parameter k is not arbitrary but takes on a set of values depending on the cone angles. Thus, for circular magnetic waves E_r must vanish at the surface of each cone at all distances; therefore Θ must satisfy the boundary conditions,

$$\Theta(\theta_1) = \Theta(\theta_2) = 0, \tag{57}$$

in addition to being a solution of equation 52. It is found that these conditions cannot be satisfied unless k is restricted to a certain set of values.

In free space there are no boundary conditions similar to equation 57, and at first sight it appears that the parameter k is unrestricted. It is found, however, that, unless

$$k = -n(n+1), \qquad n = 1,2,3,\cdots, \tag{58}$$

the solutions of equation 52 become infinite either for $\theta = 0$ or for $\theta = \pi$ or for both values. Since there is no physical reason for the field to become infinite at a point in free space, the "permissible" values of k are those given by equation 58; that is:

$$k = -2, \quad -6, \quad -12, \cdots.$$

However, there is an easier method of calculating the "proper values" of k for free-space modes of transmission. Equation 52 does not involve the frequency; in equation 53 the frequency becomes unimportant as r approaches zero. Therefore, the Θ functions are the same for static and alternating fields, while the R functions are nearly the same sufficiently near the origin. Thus we may start with static fields, which can be examined by elementary methods, and then generalize the results. This is the method we shall use in the following sections.

The significance of the higher-order modes of propagation is easy to understand. In the case of TEM waves between coaxial cones, the electric lines follow the meridians. If the cones are of finite length, such waves alone cannot possibly give the complete field. Some charge will be pushed to the ends of the cones, and the electric lines will bulge out into free space. This will cause a change in the shape of the lines

between the cones. Hence, Maxwell's equations must provide solutions to express the end effect: that is, the change in the field caused by the termination of the cones. The waves expressing this effect are the higher-order waves.

In free space there can be no waves with electric lines strictly along the meridians for there are no conductors on which these lines could terminate. Again Maxwell's equations must provide solutions expressing the waves of the type that *can* travel in free space.

4.13 A point charge, a doublet, and an electric current element

Conductors are not essential to the existence of electromagnetic waves, although, as we shall subsequently see, they play a very important role in the process of generating these waves. A charged particle is surrounded by an electric field. If the particle oscillates back and forth, the field also oscillates. Our next problem is to calculate this field.

To solve this problem we must find an appropriate solution of equations 52 and 53. These equations possess infinitely many solutions since they must be satisfied by all waves whose magnetic lines of force are circles coaxial with the z axis, and such waves may be produced by an arbitrary distribution of oscillating charged particles on the z axis. The particular solution we want is that which corresponds to a single particle oscillating near a fixed particle having an equal charge of opposite sign. This is the solution which may be used to examine the oscillations of electrons in electrically neutral conductors under the influence of impressed electromotive forces.

As indicated at the end of the preceding section we shall obtain the required solution in two main steps:

1. First we shall solve our problem *for the static case* in which $\sigma = 0$; this will represent the limiting form of the general solution of equations 52 and 53 as σr approaches zero.

2. This solution we shall then *generalize to all values* of σr. For simplicity we shall assume at first that the medium is nondissipative ($g = 0$).

When the particle is stationary (Fig. 4.13*a*), its field is radial, and, in a nondissipative medium, equation 2–14 gives

$$E_r = \frac{q}{4\pi\varepsilon r^2}\cdot \tag{59}$$

An *electric doublet* is a pair of equal and oppositely charged particles (Fig. 4.13*b*), an infinitesimal distance s apart. The product qs is called the *moment of the doublet.* The field of the doublet may be obtained by adding vectorially the fields of the two particles. It is simpler, how-

ever, to make use of a certain scalar function which will enable us to replace vectorial addition by ordinary addition. This function is called the *potential* V and is defined as *the work done by the forces of the electric field when a unit charge is transferred from a given point P to a fixed point Q which is usually taken at infinity.* Since the electric intensity E is the

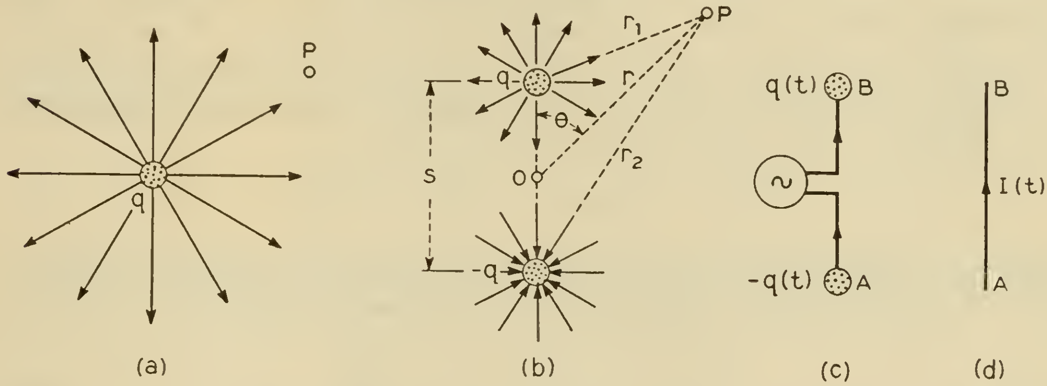


FIG. 4.13 (*a*) The electric field of a point charge; (*b*) a dipole formed by two equal and opposite point charges; (*c*) a dipole with varying end charges; (*d*) an electric current element.

force per unit charge, the total force on a charge q is Eq. The component of this force along the tangent to an element ds of the path of the particle (Fig. 4.14) is E_sq; hence, the work done when the particle travels through distance ds is $E_sq\,ds$, and the total work is the integral of this quantity. By definition $q = 1$ in computing the potential of point P; thus,

$$V = \int_{PQ} E_s\,ds. \qquad (60)$$

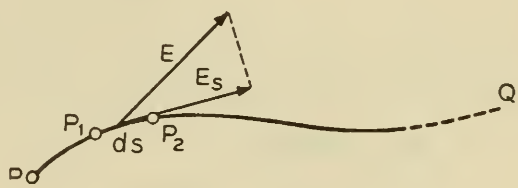


FIG. 4.14 In a *static* electric field the integral of the tangential component E_s of the electric intensity E from a given point P to a point Q at infinity is independent of the path of integration and is called the potential of the field at P.

Obviously, this definition has a meaning only if the integral is independent of the path between P and Q; otherwise, V would not be a property of the point P but rather a property of the path PQ. By the Faraday-Maxwell law the above integral is independent of the path of integration only when the field is static.* In Section 8.4 we shall define a potential function for nonstatic fields, but the definition will be different from

* If we are interested only in small regions, use only short paths of integration, and avoid regions of strong magnetic intensity (interiors of coils), we can define approximate potentials of various points. In this approximate but highly useful sense, the concept of potential is used in a-c network theory.

equation 60 and will reduce to it only in the limiting case of fields that vary infinitely slowly. The present definition, however, is sufficient for the purposes of this section.

The fixed point Q, which we shall take at infinity, is at zero potential by definition; hence, the work done in carrying a unit charge from P to infinity is equal to the *potential drop* from P to infinity. More generally, the work done in carrying a unit charge from P_1 to P_2 is the potential drop from P_1 to P_2. If P_1 and P_2 are infinitely close, this work may also be expressed as $E_s\,\Delta s$, where E_s is the electric intensity in the direction P_1P_2, and we have an equation,

$$E_s\,\Delta s = -\Delta V,$$

where ΔV represents the *increment* of potential from P_1 to P_2. Therefore, E_s is the negative of the *directional derivative* of V,

$$E_s = -\lim \frac{\Delta V}{\Delta s} = -\frac{\partial V}{\partial s}\cdot \tag{61}$$

If the electric intensity is a vector sum of several intensities,

$$E = E_1 + E_2 + E_3 + \cdots,$$

the potential,

$$\begin{aligned} V &= \int E_{1,s}\,ds + \int E_{2,s}\,ds + \int E_{3,s}\,ds + \cdots \\ &= V_1 + V_2 + V_3 + \cdots, \end{aligned}$$

is a sum of scalar functions. The vector sum may thus be obtained by directional differentiation of a scalar sum; this is the simplification we have been seeking.

For the point charge, we have

$$V = \int_r^\infty \frac{q\,dr}{4\pi\varepsilon r^2} = -\frac{q}{4\pi\varepsilon r}\Big|_r^\infty = \frac{q}{4\pi\varepsilon r}\cdot \tag{62}$$

For the doublet,

$$V = \frac{q}{4\pi\varepsilon r_1} - \frac{q}{4\pi\varepsilon r_2} = \frac{q(r_2 - r_1)}{4\pi\varepsilon r_1 r_2}\cdot \tag{63}$$

From the triangles (Fig. 4.13b),

$$\begin{aligned} r_1 &= (r^2 - sr\cos\theta + \tfrac{1}{4}s^2)^{1/2} = r\left(1 - \frac{s}{r}\cos\theta + \frac{s^2}{4r^2}\right)^{1/2}, \\ r_2 &= (r^2 + sr\cos\theta + \tfrac{1}{4}s^2)^{1/2} = r\left(1 + \frac{s}{r}\cos\theta + \frac{s^2}{4r^2}\right)^{1/2}\cdot \end{aligned} \tag{64}$$

By the binomial theorem, the square root of $1 + x$ is $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$;

hence, neglecting squares and higher powers of s/r, we have

$$r_1 = r - \tfrac{1}{2}s \cos \theta, \qquad r_2 = r + \tfrac{1}{2}s \cos \theta. \tag{65}$$

Substituting in equation 63 and again neglecting the square of s, we obtain the potential of the doublet,

$$V = \frac{qs \cos \theta}{4\pi\varepsilon r^2}. \tag{66}$$

Taking the directional derivatives in the r and θ directions, we find

$$E_r = -\frac{\partial V}{\partial r} = \frac{qs \cos \theta}{2\pi\varepsilon r^3}, \quad E_\theta = -\frac{\partial V}{r\, \partial \theta} = \frac{qs \sin \theta}{4\pi\varepsilon r^3}, \quad E_\varphi = 0. \tag{67}$$

Comparing E_r in equations 59 and 67 with equation 54, we find that R and Θ are constants for the point charge, and, for the doublet,

$$\Theta \propto \cos \theta, \qquad R \propto \frac{1}{r}. \tag{68}$$

Substituting a constant for Θ, in the point charge case, in equation 52, we find $k = 0$; substituting this value and a constant for R in equation 53, we find that R must vanish except when σ vanishes. The latter case occurs when $\omega = 0$ or when $j\omega = -g/\varepsilon$; hence, a strictly radial electric field must be either static or transient; in the latter case the time factor is $\exp(j\omega t) = \exp(-gt/\varepsilon)$. In a perfect dielectric, only the static case is possible, as we might have anticipated from the principle of conservation of charge. In a conducting medium, the point charge will gradually disperse radially under the influence of Coulomb forces.

Substituting for Θ from equation 68 into equation 52, we find that, for the doublet case,

$$k = -2. \tag{69}$$

This value belongs to the set defined by equation 58.

Comparing equations 54 and 67, and noting that in the present case $g = 0$, we have

$$\Theta = \cos \theta, \qquad R = -\frac{j\omega\varepsilon r^2 E_r}{2\Theta} = -\frac{j\omega qs}{4\pi r}. \tag{70}$$

In a nonstatic doublet, electric charge must flow between the two points, either freely or under the influence of an impressed voltage (Fig. 4.13c); the corresponding current will be

$$I(t) = \frac{dq(t)}{dt} = j\omega q, \qquad q = \frac{I}{j\omega}. \tag{71}$$

Substituting in equation 70, we find

$$R = -\frac{Is}{4\pi r}, \qquad \Theta = \cos\theta. \tag{72}$$

From equation 48, we now have

$$H_\varphi = \frac{Is \sin\theta}{4\pi r^2}. \tag{73}$$

In the nonstatic case the R function in equations 72 and H_φ in equation 73 are approximate, since the exact value of R must satisfy equation 53 and thus must depend on σ.

With the information in our possession we are now able to solve equation 53 exactly; but first let us see what happens to the above approximate equations when the conductivity of the medium does not vanish. In addition to the current (equation 71) producing electric displacement, there will be another component producing conduction current. The general relation between the total current density and the electric intensity is

$$J = (g + j\omega\varepsilon)E. \tag{74}$$

Conduction and displacement currents are similarly distributed, and, since both contribute to the magnetic intensity, equation 73 remains unaltered, provided I is the total current flowing between the ends of the doublet. Therefore R in equation 72 remains unaltered. Then from equations 54 and 55, we obtain

$$E_r = \frac{Is\cos\theta}{2\pi(g + j\omega\varepsilon)r^3}, \qquad E_\theta = \frac{Is\sin\theta}{4\pi(g + j\omega\varepsilon)r^3}. \tag{75}$$

We have already established that, in the doublet case, $k = -2$; hence the exact differential equation 53 is

$$\frac{d^2R}{dr^2} = \left(\sigma^2 + \frac{2}{r^2}\right)R. \tag{76}$$

When r is large, the second term in parentheses may be neglected, and the solution becomes

$$R = Ae^{-\sigma r} + Be^{\sigma r}. \tag{77}$$

Since the real part of σ is positive, the second term becomes exponentially infinite when r is infinite. For a field which is supposed to be produced in the vicinity of $r = 0$, B must vanish to satisfy the principle of the conservation of energy. As r approaches zero, the first term in equation 77 approaches A, and R must also satisfy equation 72. Hence, the exact solution must contain at least two terms, one giving the near

field and the other the far field. Thus, we write tentatively

$$R = -\frac{Is}{4\pi r} e^{-\sigma r} + Ae^{-\sigma r}. \tag{78}$$

The factor $\exp(-\sigma r)$ is inserted in the first term because without it we cannot possibly satisfy equation 76. The exponential function must

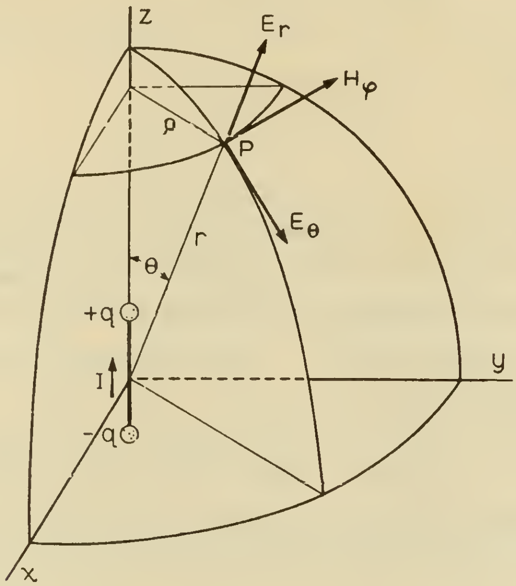


FIG. 4.15 An electric current element at the origin and the components of its field.

occur in all terms so that it can be canceled. Differentiating equation 78 twice and substituting in equation 76, we find

$$A = -\frac{\sigma Is}{4\pi} \cdot \tag{79}$$

Substituting in equation 78, we have

$$R = -\frac{\sigma Is}{4\pi}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r}. \tag{80}$$

Using this value of R in equations 48, 54, 55, we obtain the *exact field of an infinitesimal electric current element of moment* Is, located at the origin (Fig. 4.15) along the z axis:

$$\begin{aligned} E_\theta &= \frac{j\omega\mu Is}{4\pi r}\left(1 + \frac{1}{\sigma r} + \frac{1}{\sigma^2 r^2}\right) e^{-\sigma r} \sin\theta, \\ H_\varphi &= \frac{\sigma Is}{4\pi r}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r} \sin\theta, \\ E_r &= \frac{\eta Is}{2\pi r^2}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r} \cos\theta. \end{aligned} \tag{81}$$

In nondissipative media, these equations become

$$E_\theta = \frac{j\eta Is}{2\lambda r}\left(1 - \frac{\lambda^2}{4\pi^2 r^2} - j\,\frac{\lambda}{2\pi r}\right) e^{-j\beta r} \sin\theta, \qquad \beta = \frac{2\pi}{\lambda},$$

$$H_\varphi = \frac{jIs}{2\lambda r}\left(1 - j\,\frac{\lambda}{2\pi r}\right) e^{-j\beta r} \sin\theta, \tag{82}$$

$$E_r = \frac{\eta Is}{2\pi r^2}\left(1 - j\,\frac{\lambda}{2\pi r}\right) e^{-j\beta r} \cos\theta.$$

This wave generated by an infinitesimal current element is the *dominant free-space wave.* It is dominant in the sense that it expresses accurately the wave generated by a small current element of any shape at points whose distances from the element are large compared with the largest dimension of the element. The differences in size and shape are expressed by the higher-order waves ($k = -6, -12, \cdots$ in equation 58). These waves are important near the element but become imperceptible at greater distances unless the " element " becomes large.

Any given current distribution may be subdivided into infinitesimal current elements, and the wave generated by it may thus be considered as due to the superposition of dominant free-space waves emerging from the various points of the given current distribution. This is the kind of analysis we shall find convenient in our study of radiation patterns of antennas and antenna arrays. In some other antenna problems, however, we shall find it more convenient to surround the entire current distribution by a spherical surface and express the wave outside this sphere as the resultant of certain waves of a relatively simple type which *appear* to originate at the center of the sphere. Only one of these waves is of the type that can be generated by an infinitesimal current element. The other waves correspond to the solutions of Maxwell's equations with $k = -6, -12, \cdots$ in equation 58. The lines of force describing some of these waves will be shown in Section 4.17.

Equations 82 were first derived by Hertz.* They are the most important equations in the theory and practice of antennas.

4.14 The distant field of an electric current element in free space

The intrinsic impedance of free space is very nearly 377 or 120π ohms. Substituting this value for η in equations 82 and assuming $r/\lambda \gg 1/2\pi$,

* *Collected Works*, Vol. II, Third Edition, Johann Ambrosius Barth, Leipzig, 1914, p. 147.

we have

$$E_\theta = j\,\frac{60\pi Is}{\lambda r}\,e^{-j\beta r}\sin\theta, \qquad H_\varphi = j\,\frac{Is}{2\lambda r}\,e^{-j\beta r}\sin\theta,$$
$$E_r = \frac{60Is}{r^2}\,e^{-j\beta r}\cos\theta. \tag{83}$$

The ratio of the amplitudes of the radial and transverse electric intensities is

$$\frac{|E_r|}{|E_\theta|} = \frac{\lambda}{\pi r}\cot\theta. \tag{84}$$

Thus, at great distances the field is substantially transverse, except very near the axis of the current element where $\cot\theta$ is large. At a small distance $\rho = r\sin\theta$ from the axis of the element, we have

$$E_\theta = j\,\frac{60Is}{r^2}\,\frac{\pi\rho}{\lambda}\,e^{-j\beta r}, \qquad E_r = \frac{60Is}{r^2}\,e^{-j\beta r}. \tag{85}$$

Hence, E_θ is smaller than E_r only when $\pi\rho < \lambda$.

4.15 Comparison between free-space waves and principal waves on diverging wires

The simplest example of an oscillating electric doublet is a very small (compared with λ) a-c generator, whose terminals are floating. Under the influence of the internal emf V_i, a small charge will fluctuate back and forth between the terminals, depending on the capacitance between them. In the immediate vicinity of the generator the field voltage between the terminals will be equal and opposite to the internal emf. As seen from equations 82, this field diminishes very rapidly as the distance from the generator increases, until the distance becomes comparable to λ. At greater distances the field varies more slowly — inversely as the distance — but it has already become very weak.

On the other hand, if we connect a pair of diverging wires to the terminals of the generator, the field varies inversely as the distance even for small distances — as in the case given by equations 32. The loss in the intensity of the field within the first wavelength is not as great as in the preceding case. Thus, the wires help to increase the distant field for a given voltage developed by the generator.

4.16 Lines of power flow

Energy flows normally to E and H (Section 2.5). Hence, in any TEM wave the flow of energy is in the direction of wave propagation. In

particular, in a spherical TEM wave the flow is radial. From equations 32 and 2–37, we obtain the average power flowing per unit area in a progressive TEM wave between coaxial cones,

$$W = \tfrac{1}{2}E_\theta H_\varphi^* = \frac{\eta VI^*}{8\pi^2r^2K\sin^2\theta} = \frac{\eta II^*}{8\pi^2r^2\sin^2\theta}. \tag{86}$$

Hence, the power flowing between a conical surface $\theta = \theta_m$ and the equatorial plane $\theta = \pi/2$ is

$$P(\theta_m) = \int_0^{2\pi}\int_{\theta_m}^{\pi/2} Wr^2 \sin\theta \, d\theta \, d\varphi$$

$$= \frac{\eta II^*}{4\pi}\int_{\theta_m}^{\pi/2}\frac{d\theta}{\sin\theta} = \frac{\eta II^*}{4\pi}\log\cot\tfrac{1}{2}\theta_m. \tag{87}$$

If the surface of the upper conical conductor corresponds to $\theta = \psi$, the power flow between the conductor and the equatorial plane is

$$P(\psi) = \frac{\eta II^*}{4\pi}\log\cot\tfrac{1}{2}\psi. \tag{88}$$

Therefore,

$$P(\theta_m) = P(\psi)\frac{\log\cot\frac{1}{2}\theta_m}{\log\cot\frac{1}{2}\psi}. \tag{89}$$

Using this equation, we can subdivide the space between the conductor and the equatorial plane into regions of equal power flow. In Fig. 4.16 we have five such regions for which

$$P(\theta_1) = \tfrac{1}{5}P(\psi), \qquad P(\theta_2) - P(\theta_1) = \tfrac{1}{5}P(\psi), \qquad \text{etc.} \tag{90}$$

The power flow tends to be concentrated near the conductor.

In the case of a free-space wave generated by an electric current element, the instantaneous power flow is not radial except at great distances; there is some flow of power along meridians, normally to E_r and H_φ. However, from equations 82 we find that these field components are in quadrature; hence, *on the average there is no flow of power along meridians.* Thus, the lines of average power flow are radial; but this time the power flow is concentrated near the equatorial plane of the current element (Fig. 4.17). To obtain this result we calculate the complex radial power flow per unit area from equations 82,

$$\Psi = \tfrac{1}{2}E_\theta H_\varphi^* = \frac{\eta II^*s^2}{8\lambda^2r^2}\left(1 - j\frac{\lambda^3}{8\pi^3r^3}\right)\sin^2\theta; \tag{91}$$

and then the average power flow

$$W = \operatorname{re} \Psi = \frac{\eta I I^* s^2}{8\lambda^2 r^2} \sin^2 \theta. \tag{92}$$

The power flow within the cone $\theta = \theta_m$ is

$$\begin{aligned} P(\theta_m) &= \int_0^{2\pi} \int_0^{\theta_m} r^2 W \sin \theta \, d\theta \, d\varphi \\ &= \frac{\pi \eta I I^* s^2}{6\lambda^2} (1 - \tfrac{3}{2} \cos \theta_m + \tfrac{1}{2} \cos^3 \theta_m). \end{aligned} \tag{93}$$

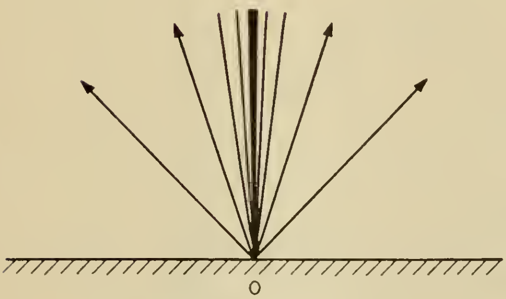


FIG. 4.16 The flow of power in a TEM wave between a conducting plane and an infinitely long conical conductor perpendicular to the plane is radial. If the cone is thin, the greater part of the flow is near the cone.

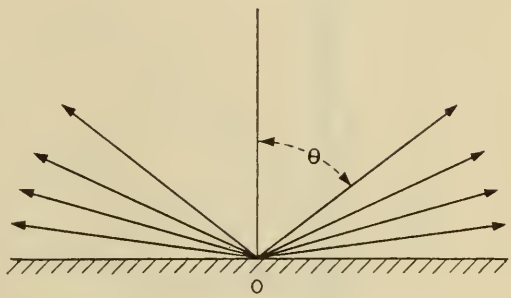


FIG. 4.17 The flow of power from a current element perpendicular to a conducting plane is radial. The greater part of the flow is near the plane.

The power flow in the entire upper hemisphere is

$$P\left(\frac{\pi}{2}\right) = \frac{\pi \eta I I^* s^2}{6\lambda^2}. \tag{94}$$

Hence,

$$P(\theta_m) = P(\tfrac{1}{2}\pi)(1 - \tfrac{3}{2} \cos \theta_m + \tfrac{1}{2} \cos^3 \theta_m). \tag{95}$$

Assigning to the ratio $P(\theta_m)/P(\pi/2)$ the successive values $1/5$, $2/5$, $3/5$, and $4/5$, and evaluating the corresponding angles, we obtain Fig. 4.17.

If the conical conductors are of finite length, the lines of average power flow are more complicated. From inspection of power flow diagrams and of field equations, it is evident that the TEM wave between the cones does not match the wave from a current element. The problem of calculating the field in this case is difficult and will not be considered here. As might be expected, the calculations show that the lines of power flow, beginning at the apex where the source of power is assumed to be located, follow at first the radial pattern for a pair of infinitely long cones; then they veer away from the biconical antenna

and form another radial pattern. If the length of each arm of the antenna does not greatly exceed a quarter wavelength, the final pattern resembles closely the pattern in Fig. 4.17 for a current element. The transition regions are shown in Figs. 4.18 and 4.19 for two antennas whose arms are a quarter wavelength long. In the first case, the ratio

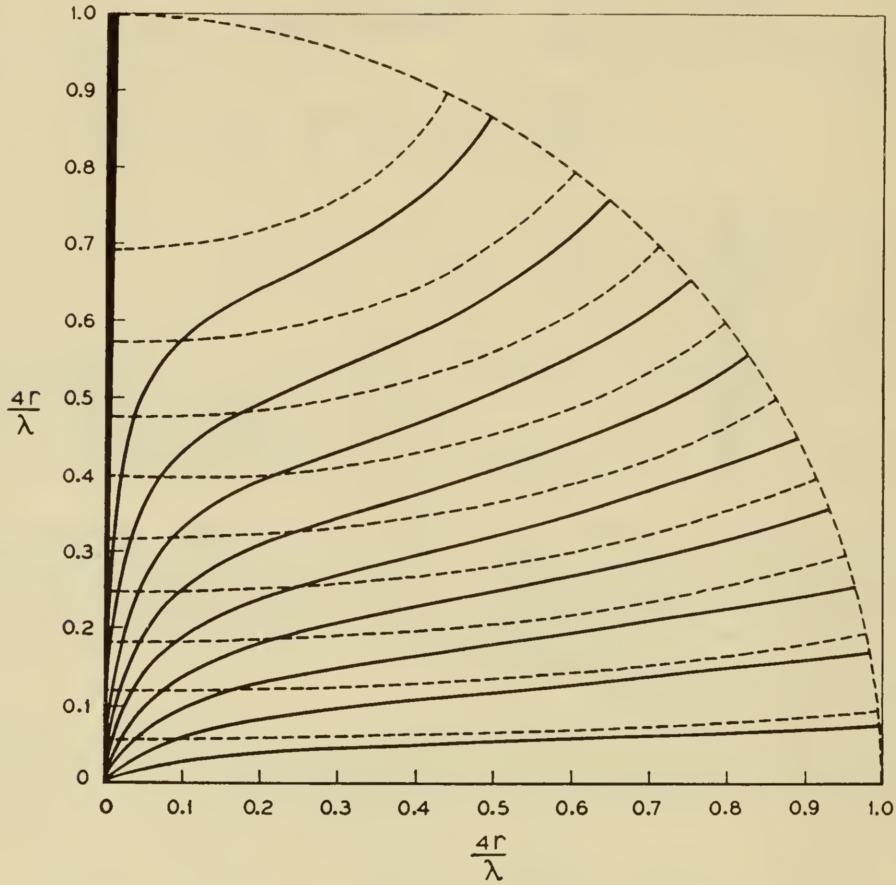


FIG. 4.18 Lines of average power flow from a half-wave antenna in free space or a quarter-wave antenna perpendicular to a conducting plane. The solid lines are for the case in which $l/a = \lambda/4a = 74$; the dotted lines are for $a = 0$.

of the length l of one antenna arm to the maximum radius a is 74; in the second case, $l/a = 11{,}000$. Thus, as the antenna becomes thinner, the lines of power flow tend to hug the antenna to a greater extent before veering off. In each figure the dotted lines represent the power flow lines for an infinitely thin antenna. They *appear* to emerge from the antenna rather than from the source of power at the apex. Actually the lines emerge from the source; but they are all concentrated within a cylinder of infinitesimal radius surrounding the infinitely thin antenna until they are ready to break away, and for this reason they are indistinguishable from the vertical axis. It should also be noted that, in the

case of an infinitely thin antenna supporting a finite current, the stored energy is infinite and is located in an infinitely thin layer surrounding the antenna. From this layer the energy is sprayed at right angles to the antenna. However, it would take an infinite time to "reach" the steady state.

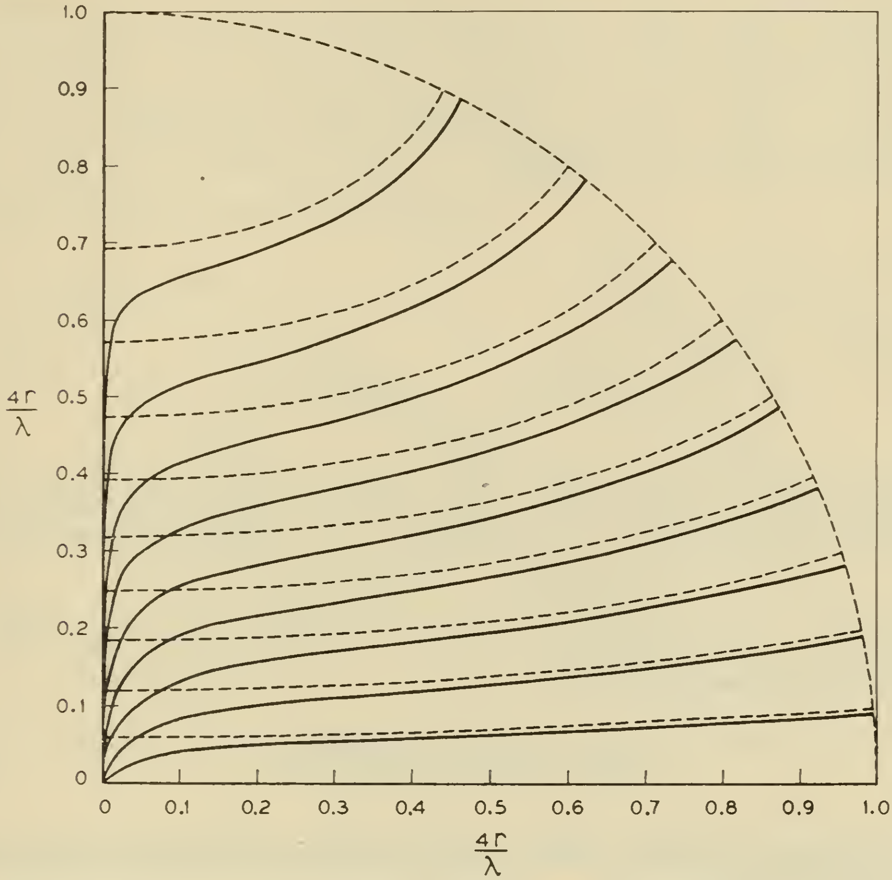


FIG. 4.19 Lines of average power flow from a half-wave antenna in free space or a quarter-wave antenna perpendicular to a conducting plane. The solid lines are for the case in which $l/a = \lambda/4a = 11{,}000$; the dotted lines are for $a = 0$.

For any practical antenna, no matter how thin, the build-up time is only a few periods; the lines of power flow differ considerably from the limiting case in the immediate vicinity of the antenna, but not very much at greater distances.

An imperfectly conducting antenna absorbs power and dissipates it in heat. Some power flow lines, the ones very close to the antenna, will terminate on the antenna and will thus represent the power flow into the antenna. On the scale used in Figs. 4.18 and 4.19 it is not feasible to show these lines; they are too close to the antenna.

4.17 Electric lines of force

Electric lines of force are, by definition, lines tangential to the electric vector. Hence, an element of length ds along a line of force coincides in direction with vector E (Fig. 4.20), and the components of ds along

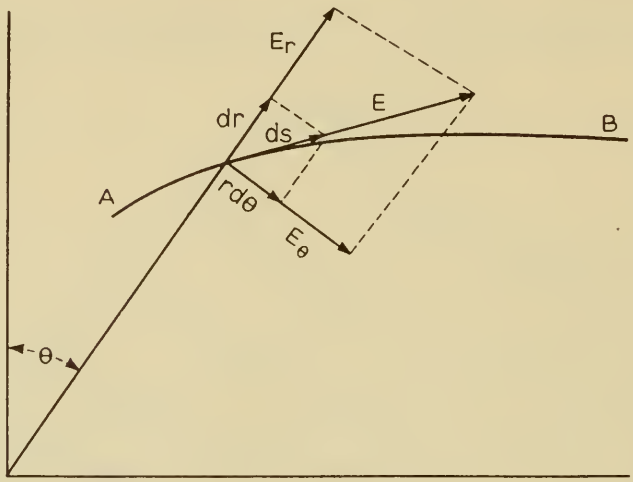


FIG. 4.20 Elementary differentials, dr, $r\,d\theta$, along a tangent to an electric line of force are proportional to the components, E_r, E_θ, of the electric intensity.

the coordinate lines must be proportional to the components of E. In spherical coordinates, we have

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta} = \frac{r \sin\theta\, d\varphi}{E_\varphi}. \tag{96}$$

For circularly symmetric fields, in which electric lines lie in axial planes, equation 96 becomes

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta}. \tag{97}$$

For a static doublet E_r and E_θ are given by equations 67, and the lines of force are given by

$$\frac{dr}{r} = \frac{2\cos\theta\, d\theta}{\sin\theta}. \tag{98}$$

Integrating, we find

$$\log r = 2 \log \sin\theta + C, \tag{99}$$

where C is a constant of integration. For a point in the equatorial plane, $\theta = \pi/2$, and the corresponding value r_0 of r may be found from

$$\log r_0 = C.$$

This value may be used as a new constant of integration. Substituting in equation 99 and taking the antilogarithms, we have

$$\frac{r}{r_0} = \sin^2 \theta, \qquad \sin \theta = \left(\frac{r}{r_0}\right)^{1/2}. \tag{100}$$

Since $\sin \theta$ is never greater than unity, r is never greater than r_0, and the lines of force have the shape shown in Fig. 4.21. All lines of force start from the positive charge and return to the negative; the lines spread throughout the entire space. The electric lines are often so drawn that equal displacements are enclosed by successive pairs; but in some cases this is not practicable.

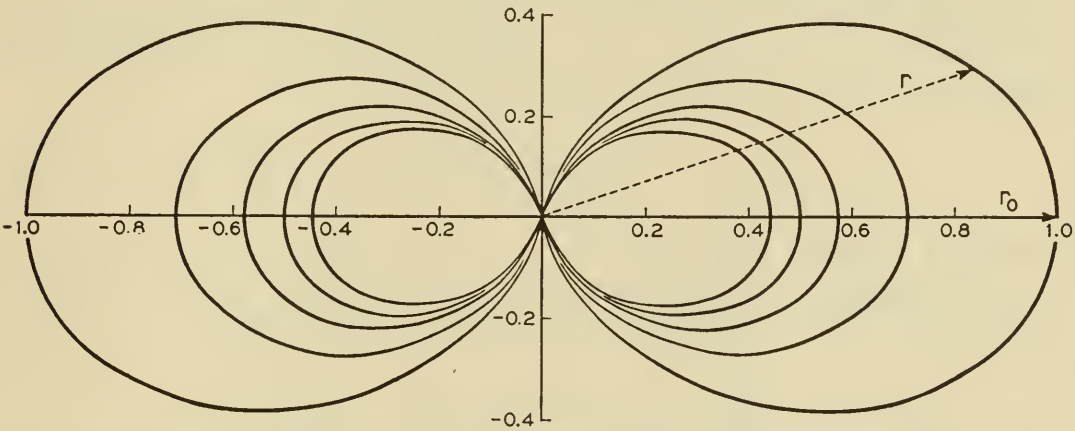


FIG. 4.21 Electric lines of force surrounding a static infinitesimal dipole, situated on the vertical axis at the origin.

For an a-c electric doublet, equation 97 retains its form; but the denominators will be functions of time, and the equation for the lines of force is

$$\frac{dr}{\tilde{E}_r} = \frac{r\,d\theta}{\tilde{E}_\theta}, \tag{101}$$

where $\tilde{E}_r$ and $\tilde{E}_\theta$ represent the field at some particular instant. From equations 54 and 55, we find that, for nondissipative media,

$$\tilde{E}_r = \mathrm{re}\,\frac{kR\Theta \exp(j\omega t)}{j\omega\varepsilon r^2}, \qquad \tilde{E}_\theta = -\mathrm{re}\,\frac{1}{j\omega\varepsilon r}\,\frac{dR}{dr}\,\frac{d\Theta}{d\theta}\,e^{j\omega t}. \tag{102}$$

In cases of particular interest to us, Θ and k are real while R is complex. Letting

$$R = R_r + jR_i, \tag{103}$$

we have, from equations 102,

$$\begin{aligned} \widetilde{E}_r &= \frac{k\Theta}{\omega\varepsilon r^2}\,(R_r \sin \omega t + R_i \cos \omega t), \\ \widetilde{E}_\theta &= -\frac{1}{\omega\varepsilon r}\,\frac{d\Theta}{d\theta}\left(\frac{dR_r}{dr}\sin \omega t + \frac{dR_i}{dr}\cos \omega t\right)\cdot \end{aligned} \tag{104}$$

Substituting in equation 101 and integrating, we obtain the equation for the electric lines of force; thus,

$$R_r \sin \omega t + R_i \cos \omega t = C \exp\left[-k \int \frac{\Theta\, d\theta}{d\Theta/d\theta}\right]\cdot \tag{105}$$

For an electric current element, $\Theta = \cos\theta$ and $k = -2$; the R function is given by equation 80. Assuming that the medium is non-dissipative, we let $\sigma = j\beta$. The constant factor does not affect the shape of the lines of force and may be omitted; thus,

$$R = \left(\cos \beta r - \frac{\sin \beta r}{\beta r}\right) - j\left(\sin \beta r + \frac{\cos \beta r}{\beta r}\right), \tag{106}$$

save for a constant factor.

Substituting in equation 105 and introducing the radial coordinate r_0 of a point in the equatorial plane in place of C, we find

$$\sin^2\theta = \frac{\left(\cos \beta r_0 - \dfrac{\sin \beta r_0}{\beta r_0}\right)\sin \omega t - \left(\sin \beta r_0 + \dfrac{\cos \beta r_0}{\beta r_0}\right)\cos \omega t}{\left(\cos \beta r - \dfrac{\sin \beta r}{\beta r}\right)\sin \omega t - \left(\sin \beta r + \dfrac{\cos \beta r}{\beta r}\right)\cos \omega t}\cdot \tag{107}$$

Assigning different values to t, we are able to plot electric lines in the various phases of oscillation. Figure 4.22 shows these lines at the instant when the doublet is completely discharged, so that all electric lines are free (only the upper right quadrant is shown). A quarter period later the lines assume the shape shown in Fig. 4.23; a new set of lines emerges from the doublet and pushes the free lines outward. Figure 4.24 shows what happens shortly before the doublet becomes neutral; it shows one line on the verge of splitting into two: a large oval which will become free and a small oval which will contract. The free line will carry off some energy; energy associated with the contracting line will return to the doublet.

In the case of two oppositely directed current elements, electric lines will look, at the instant of complete discharge, as shown in Fig. 4.25. For three properly proportioned current elements, the lines will look as shown in Fig. 4.26.

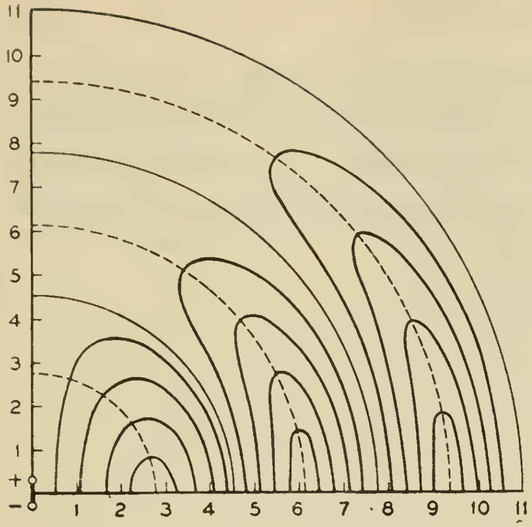


FIG. 4.22 Electric lines of force surrounding an electric current element or doublet at the instant of complete discharge.

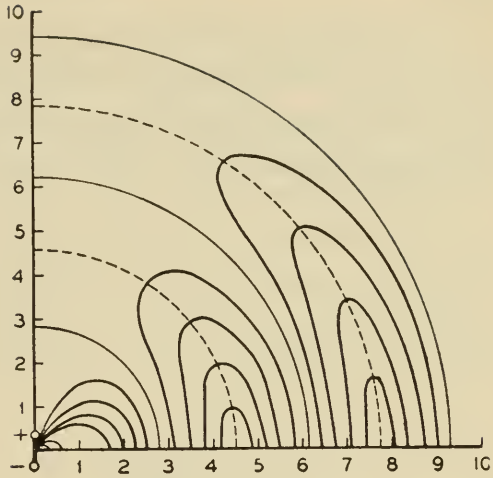


FIG. 4.23 Electric lines of force a quarter period later.

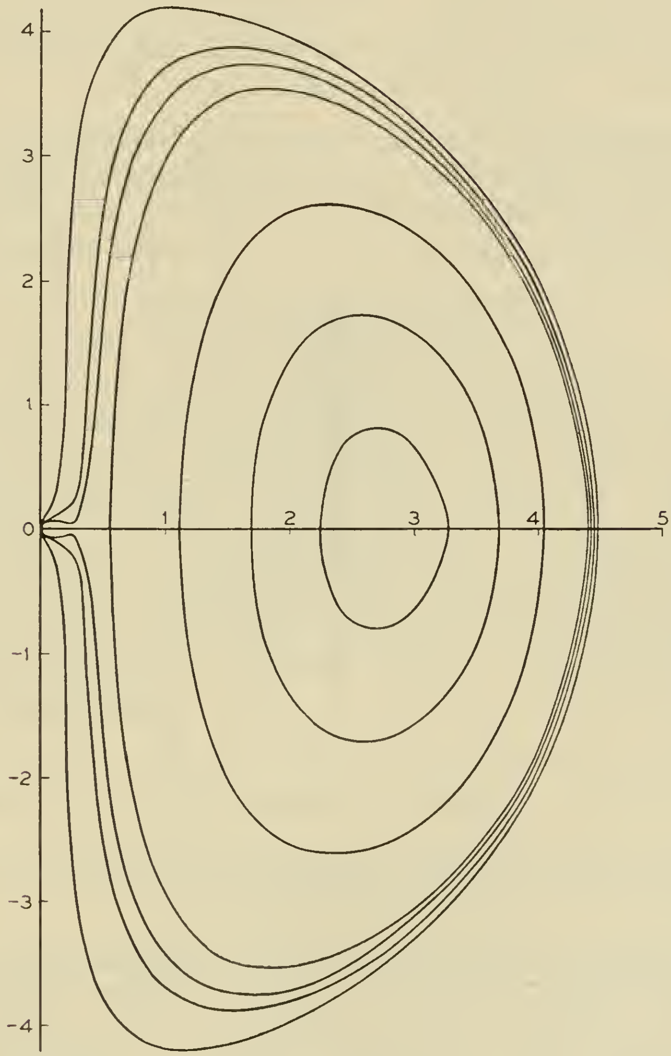


FIG. 4.24 Electric lines of force just before the complete discharge.

In the case of two coaxial cones, there is one mode of propagation which has no counterpart in free space: the TEM mode in which electric lines follow meridians from one cone to the other. In addition to this principal mode, there are other modes corresponding to free-space

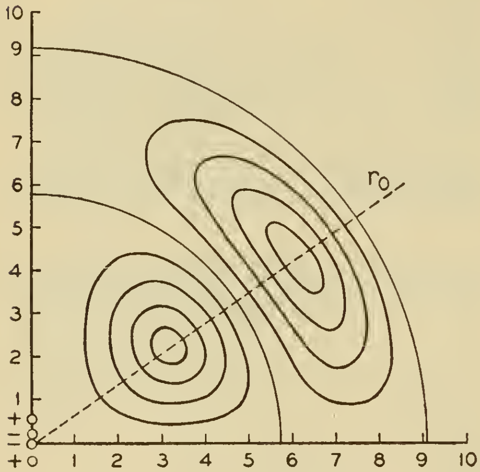


Fig. 4.25 Electric lines of force in the wave produced by a tripole (+1, −2, +1) at the instant of complete discharge.

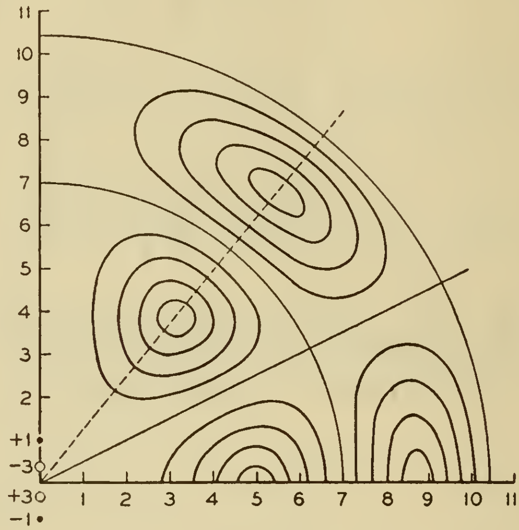


Fig. 4.26 Electric lines of force in the wave produced by a quadrupole (+1, −3, +3, −1) at the instant of complete discharge.

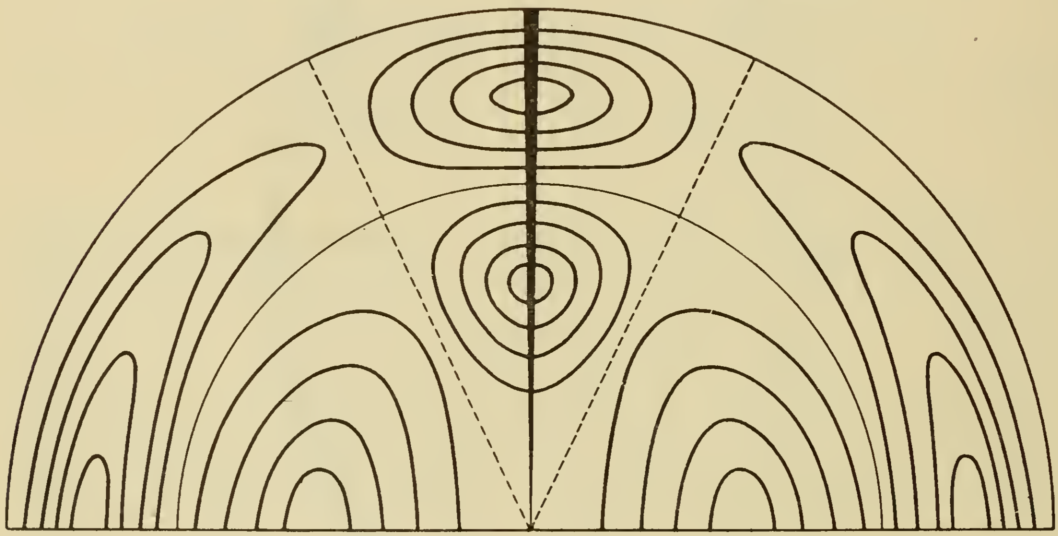


Fig. 4.27 A sketch illustrating the modification of the field shown in Fig. 4.22 caused by a conducting cone.

modes as modified by the presence of the conducting cones. The electric lines must be normal to the conductors and will terminate on their charged surface. Figure 4.27 illustrates the mode which corresponds to that generated by an electric current element. These higher modes become very important in advanced antenna theory.

There are many mathematical forms in which we can express electromagnetic fields; the most convenient form depends on the particular type of problem. If the current distribution is given, we can subdivide it into current elements and obtain the total field by integrating the field of a typical element. In actual practice, however, the current distribution is never given; it is the voltage developed by the generator that is usually given. This complicates the problem tremendously, except in certain cases in which approximate solutions are relatively simple. Even in those cases, there are some questions that can be reliably answered only by advanced analysis.

4.18 Theory of images

In the equatorial plane of an electric current element the electric intensity normal to the element vanishes (see equations 82); hence, a perfectly conducting sheet in this plane will not disturb the field. The current element will be cut in half; the voltage impressed on the element will be cut in half. If the conducting sheet is infinite, it will separate the space into two electrically independent regions. To prove this we recall that the Poynting vector is normal to the electric vector and, therefore, tangential to the surface of a perfect conductor. Hence, no energy can ever pass through a perfectly conducting surface. Thus, in the above case we can remove the lower half of the current element without affecting the field in the upper half and obtain the field of a current element just above a perfectly conducting plane.

More generally, in order to calculate the electromagnetic field due to a given system of sources in the presence of a perfectly conducting plane, we superimpose, on the original free-space field of the given system of sources, another field due to a proper "image system." From symmetry considerations we can always choose the image sources in such a way that the electric intensity tangential to the plane vanishes; then we can repeat the preceding argument and obtain the required field. For example, the components of E perpendicular to and parallel to a current element are, respectively,

$$\begin{aligned} E_\rho &= E_r \sin\theta + E_\theta \cos\theta, \\ E_z &= E_r \cos\theta - E_\theta \sin\theta. \end{aligned} \tag{108}$$

From these equations and from equations 82 we find that at two points, (r, θ) and $(r, \pi - \theta)$, on the same perpendicular to the equatorial plane and at equal distances above and below the plane, we have

$$E_\rho(r, \theta) = -E_\rho(r, \pi - \theta), \qquad E_z(r, \theta) = E_z(r, \pi - \theta). \tag{109}$$

Therefore, the image of a vertical current element (Fig. 4.28) has the same moment as the given element, but the image of the horizontal element is 180° out of phase.

These two rules are sufficient for all cases, since any element may be resolved into vertical and horizontal components. In all cases the sum of the field due to the source and that due to the image represents the field of the source above the perfectly conducting plane; below it, the field vanishes identically (assuming of course that there are no real sources below the plane).

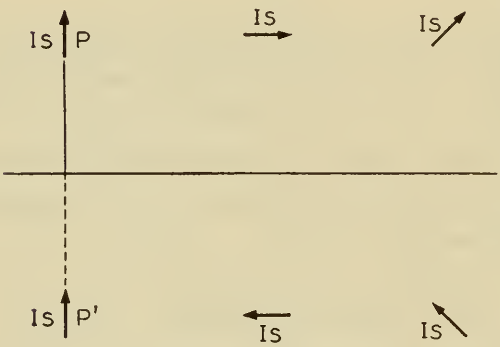


FIG. 4.28 Current elements above a conducting plane and their images.

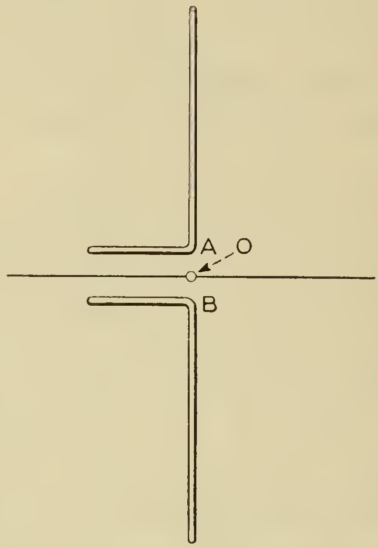


FIG. 4.29 A vertical antenna above perfect ground and its image.

From the preceding considerations, we conclude that the field of a symmetrically fed antenna will not be disturbed if we introduce a perfectly conducting plane halfway between the terminals and perpendicular to the antenna (Fig. 4.29). Hence, the voltage between the center O and the upper terminal equals one half of the voltage between A and B. From this we conclude that the *impedance of the vertical antenna above a perfect ground equals one half of the impedance of the corresponding antenna in free space formed by the actual antenna and its image.*

PROBLEMS

4.2–1 Express Maxwell's equations in Cartesian coordinates.

Ans.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x, \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (g + j\omega\varepsilon)E_x,$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y, \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = (g + j\omega\varepsilon)E_y,$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (g + j\omega\varepsilon)E_z.$$

4.2–2 Express Maxwell's equations in cylindrical coordinates (ρ, φ, z).

Ans.
$$\frac{\partial E_z}{\partial \varphi} - \rho \frac{\partial E_\varphi}{\partial z} = -j\omega\mu\rho H_\rho, \qquad \frac{\partial H_z}{\partial \varphi} - \rho \frac{\partial H_\varphi}{\partial z} = (g + j\omega\varepsilon)\rho E_\rho,$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\varphi, \qquad \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = (g + j\omega\varepsilon)E_\varphi,$$

$$\frac{\partial}{\partial \rho}(\rho E_\varphi) - \frac{\partial E_\rho}{\partial \varphi} = -j\omega\mu\rho H_z, \qquad \frac{\partial}{\partial \rho}(\rho H_\varphi) - \frac{\partial H_\rho}{\partial \varphi} = (g + j\omega\varepsilon)\rho E_z.$$

4.2–3 From the equations in Problem 4.2–1, show that, if g, ε, μ are independent of x, y, z, then,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0.$$

4.2–4 Maxwell's equations for fields varying arbitrarily with time are obtained if we replace $j\omega$ by $\partial/\partial t$. Hence, show that, in a transient state,

$$\frac{\partial \mu H_x}{\partial x} + \frac{\partial \mu H_y}{\partial y} + \frac{\partial \mu H_z}{\partial z} = f(x, y, z),$$

where $f(x, y, z)$ is an arbitrary function of integration. This function is proportional to the density of magnetic charge and must vanish for actual magnetic fields; but this conclusion cannot be drawn from Maxwell's equations.

4.2–5 Show that, in a transient state,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = f(x, y, z)e^{-(g/\varepsilon)t},$$

provided g and ε are independent of x, y, z. Note that, if $g = 0$, the right side is independent of time.

4.2–6 Using Maxwell's equations and assuming that g, ε, μ are constants, obtain

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\rho) + \frac{1}{\rho}\frac{\partial H_\varphi}{\partial \varphi} + \frac{\partial H_z}{\partial z} = f_1(\rho, \varphi, z),$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\varphi}{\partial \varphi} + \frac{\partial E_z}{\partial z} = f_2(\rho, \varphi, z)e^{-(g/\varepsilon)t}.$$

Note that f_1 and f_2 are independent of time and must vanish for simple harmonic fields.

4.2–7 Using Maxwell's equations and assuming that g, ε, μ are constants, obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 H_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, H_\theta) + \frac{1}{r\sin\theta}\frac{\partial H_\varphi}{\partial \varphi} = f_1(r, \theta, \varphi),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial \varphi} = f_2(r, \theta, \varphi)e^{-(g/\varepsilon)t}.$$

(See the comment in the preceding problem.)

4.4–1 Show that the conclusion in Section 4.4 is based on a tacit assumption that $g + j\omega\varepsilon$ does not vanish and that, consequently, it does not apply to static fields in perfect dielectrics.

4.4–2 What conclusions may be drawn from equations 6, 7, 8 when $g = \omega = 0$?

$$\textit{Ans.} \qquad H_\varphi = \frac{A}{r}\sin\theta, \qquad E_r = \frac{\partial U}{\partial r}, \qquad E_\theta = \frac{\partial U}{r\,\partial\theta},$$

where A is a constant and U is a differentiable function of r and θ.

4.4–3 Using the answers to Problems 4.2–7 and 4.4–2, we obtain the following equation for U:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) = f_2(r, \theta).$$

Assume that $f_2(r, \theta) = 0$, and find that particular solution which is independent of θ. What is the corresponding electric intensity?

$$\textit{Ans.} \qquad U = \frac{A}{r} + B; \qquad E_r = \frac{-A}{r^2}, \qquad E_\theta = 0.$$

4.5–1 Obtain E_r for large values of r.

$$\textit{Ans.} \quad E_r = \frac{1}{(g + j\omega\varepsilon)r^2}\left[\left(A\cot\theta + \frac{\partial A}{\partial\theta}\right)e^{-\sigma r} + \left(B\cot\theta + \frac{\partial B}{\partial\theta}\right)e^{\sigma r}\right].$$

4.5–2 From equations 5, obtain approximate equations on the assumption that r is very large.

4.9–1 Explain why a request for a 300-ohm coaxial line would be considered unreasonable whereas a request for a 300-ohm parallel pair would be reasonable.

4.11–1 Consider a cage formed by four wires of diameter d equispaced on a cylinder of diameter $20d$. What is the effective diameter of the cage? *Ans.* $13.4d$.

4.12–1 If $g = 0$, the expression on the right of equation 51 is $-(2\pi r/\lambda)^2$. This expression approaches zero either when r approaches zero or when λ increases indefinitely. Hence, the R function governing the variation of any field in radial directions is approximately independent of the frequency at distances for which $(2\pi r/\lambda)^2 \ll k$. Find these approximate solutions.

Ans. $R = Ar^{\nu_1} + Br^{\nu_2}$, where ν_1, ν_2 are the roots of $\nu^2 - \nu + k = 0$.

4.12–2 In accordance with equation 54, the R function is proportional to r^2E_r. The electrostatic field of a point charge in free space is proportional to $1/r^2$; that of a doublet varies as $1/r^3$; and, in general, electric charges of opposite signs may be so distributed that the field will vary as $1/r^m$, where m is an *integer* greater than unity. The corresponding R function will then be r^{-n}, where n is a positive integer, including zero. Show that these considerations restrict the values of k to those given by $k = -n(n + 1)$, $n = 0, 1, 2, \cdots$, and that the general approximate solution for R is $Ar^{-n} + Br^{n+1}$. The above is another way of deriving the condition 58.

4.13–1 Let the origin of time be so chosen that the current in a given element is $I = I_a\exp(j\omega t)$. Show that the amplitudes and phases of the various field intensities produced by the element are

$$\mathrm{am}(E_\theta) = \frac{\eta I_a\,ds}{2\lambda r}\left(1 - \frac{1}{\beta^2r^2} + \frac{1}{\beta^4r^4}\right)^{1/2}\sin\theta,$$

$$\mathrm{ph}(E_\theta) = \omega t - \beta r + \tan^{-1}\left(\beta r - \frac{1}{\beta r}\right),$$

$$\mathrm{am}(H_\varphi) = \frac{I_a\,ds}{2\lambda r}\left(1 + \frac{1}{\beta^2 r^2}\right)^{1/2} \sin\theta, \qquad \mathrm{ph}(H_\varphi) = \omega t - \beta r + \tan^{-1}\beta r,$$

$$\mathrm{am}(E_r) = \frac{\eta I_a\,ds}{2\pi r^2}\left(1 + \frac{1}{\beta^2 r^2}\right)^{1/2} \cos\theta, \qquad \mathrm{ph}(E_r) = \omega t - \beta r - \tan^{-1}\frac{1}{\beta r},$$

where the inverse tangents have values between $-\pi/2$ and $\pi/2$.

4.13–2 Assuming that the instantaneous value of the current is $\tilde{I} = I_a \cos \omega t$, show that

$$\tilde{E}_r = \frac{60 I_a\,ds}{r^2}\left[\left(\cos\beta r - \frac{\sin\beta r}{\beta r}\right)\cos\omega t + \left(\frac{\cos\beta r}{\beta r} + \sin\beta r\right)\sin\omega t\right]\cos\theta,$$

$$\tilde{E}_\theta = \frac{60\pi I_a\,ds}{\lambda r}\left[\left(\sin\beta r + \frac{\cos\beta r}{\beta r} - \frac{\sin\beta r}{\beta^2 r^2}\right)\cos\omega t + \right.$$
$$\left.\left(-\cos\beta r + \frac{\sin\beta r}{\beta r} + \frac{\cos\beta r}{\beta^2 r^2}\right)\sin\omega t\right]\sin\theta,$$

$$\tilde{H}_\varphi = \frac{I_a\,ds}{2\lambda r}\left[\left(\frac{\cos\beta r}{\beta r} + \sin\beta r\right)\cos\omega t + \left(\frac{\sin\beta r}{\beta r} - \cos\beta r\right)\sin\omega t\right]\sin\theta.$$

4.13–3 Show that, if r is substantially greater than $\lambda/2\pi$, then,

$$\tilde{E}_\theta \backsimeq \frac{60\pi I_a\,ds}{\lambda r}\sin(\beta r - \omega t)\sin\theta, \qquad \tilde{H}_\varphi = \frac{\tilde{E}_\theta}{120\pi};$$

$$\tilde{E}_r \backsimeq \frac{60 I_a\,ds}{r^2}\cos(\beta r - \omega t)\cos\theta.$$

4.13–4 Noting that $\beta = \omega/v$, where $v = 1/\sqrt{\mu\varepsilon}$, and that $j\omega = \partial/\partial t$, rewrite equations 82 for the general case in which $I(t)$ is an arbitrary function of time. There is no loss in generality if we assume that $I(t) = 0$, $t < 0$. Let

$$\int_0^t I(t)\,dt = q(t).$$

Ans.
$$E_\theta = \left[\frac{s\,q[t-(r/v)]}{4\pi\varepsilon r^3} + \frac{\eta s\,I[t-(r/v)]}{4\pi r^2} + \frac{\eta s\,I'[t-(r/v)]}{4\pi v r}\right]\sin\theta,$$

$$H_\varphi = \left[\frac{s\,I[t-(r/v)]}{4\pi r^2} + \frac{s\,I'[t-(r/v)]}{4\pi v r}\right]\sin\theta,$$

$$E_r = \left[\frac{s\,q[t-(r/v)]}{2\pi\varepsilon r^3} + \frac{\eta s\,I[t-(r/v)]}{2\pi r^2}\right]\cos\theta.$$

4.13–5 Show that, if $q(t)$ is differentiable any number of times when $t > 0$,

the electric intensity along the axis $\theta = 0$ for $t > r/v$ may be expressed as follows:

$$E_r = \frac{s\, q(t)}{2\pi\varepsilon r^3} - \frac{\mu s\, I'(t)}{4\pi r} + \frac{\eta s}{6\pi} v^{-2}\, I''(t) + \cdots.$$

4.13–6 Consider a current element and let $q(t) = a \sin(2\pi t/T)$ when $0 \leq t \leq T$. At other times $q(t) = 0$. Calculate the field.

Ans. If $t > T$, the field exists only between two concentric spheres whose radii are vt and $v(t - T)$. The field intensities are

$$E_\theta = \frac{as}{4\pi r}\left[\frac{1}{\varepsilon r^2} \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) + \frac{2\pi\eta}{Tr} \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) - \frac{4\pi^2\eta}{T^2 v} \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)\right] \sin\theta,$$

$$H_\varphi = \frac{as}{4\pi r}\left[\frac{2\pi}{Tr} \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) - \frac{4\pi^2}{T^2 v} \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)\right] \sin\theta,$$

$$E_r = \frac{as}{2\pi r^2}\left[\frac{1}{\varepsilon r} \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) + \frac{2\pi\eta}{T} \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)\right] \cos\theta.$$

4.13–7 Show that the energy passing through the sphere of radius r, centered at the current element, in the time interval $(r/v,\, t)$ is

$$\mathcal{E} = \frac{1}{6\pi} s^2 \left\{\frac{1}{2\varepsilon r^3}\left[q\left(t - \frac{r}{v}\right)\right]^2 + \frac{\eta}{r^2} q\left(t - \frac{r}{v}\right) I\left(t - \frac{r}{v}\right) + \frac{\mu}{r}\left[I\left(t - \frac{r}{v}\right)\right]^2 + \frac{\eta}{v^2}\int_0^{t-r/v} [\dot{I}(t)]^2\, dt\right\} - \frac{1}{6\pi} s^2 \left\{\frac{1}{2\varepsilon r^3}[q(0)]^2 + \frac{\eta}{r^2} q(0)\, I(0) + \frac{\mu}{r}[I(0)]^2\right\}.$$

Note that if $q(t - r/v) = q(0)$, $I(t - r/v) = I(0)$, then the energy which has passed through the sphere is

$$\mathcal{E} = \frac{\eta}{6\pi v^2} s^2 \int_0^{t-r/v} [\dot{I}(t)]^2\, dt;$$

that is, this amount of energy is lost to the source. The rate of loss of energy by radiation is, therefore,

$$P_{\text{rad}} = \frac{d\mathcal{E}}{dt} = \frac{\eta s^2}{6\pi v^2}\left[\frac{dI(t)}{dt}\right]^2.$$

4.13–8 In a nondissipative medium, Maxwell's equations remain unaltered if j is replaced by $-j$ and E by $-E$. Use this invariance to obtain the solution for a spherical wave similar to that given by equations 82 but converging to the origin.

Ans. $$E_\theta = \frac{j\eta A}{2\lambda r}\left(1 - \frac{\lambda^2}{4\pi^2 r^2} + j\frac{\lambda}{2\pi r}\right) e^{j\beta r} \sin\theta,$$

$$H_\varphi = -\frac{jA}{2\lambda r}\left(1 + j\frac{\lambda}{2\pi r}\right)e^{j\beta r}\sin\theta,$$

$$E_r = -\frac{\eta A}{2\pi r^2}\left(1 + j\frac{\lambda}{2\pi r}\right)e^{j\beta r}\cos\theta.$$

4.13–9 Let a current element of moment Is be surrounded by a perfectly conducting sphere centered at the element. Assume that the radius a of the sphere is large compared with λ. Calculate the field.

Ans. $$E_\theta = \frac{j\eta Is}{2\lambda r\sin\beta a}\left[\left(1 - \frac{1}{\beta^2 r^2}\right)\sin\beta(a-r) - \frac{\cos\beta(a-r)}{\beta r}\right]\sin\theta,$$

$$H_\varphi = \frac{Is}{2\lambda r\sin\beta a}\left[\cos\beta(a-r) + \frac{\sin\beta(a-r)}{\beta r}\right]\sin\theta,$$

$$E_r = -\frac{j\eta Is}{2\pi r^2\sin\beta a}\left[\cos\beta(a-r) + \frac{\sin\beta(a-r)}{\beta r}\right]\cos\theta.$$

4.13–10 Treating Is in equations 82 as an arbitrary constant of integration, show that, when $Is = -A$, where A is the constant in the answer to Problem 4.13–8, the sum of the fields is finite at $r = 0$ and is

$$E_\theta = -\frac{\eta A}{\lambda r}\left[\frac{\cos\beta r}{\beta r} + \left(1 - \frac{1}{\beta^2 r^2}\right)\sin\beta r\right]\sin\theta,$$

$$H_\varphi = j\frac{A}{\lambda r}\left(\frac{\sin\beta r}{\beta r} - \cos\beta r\right)\sin\theta,$$

$$E_r = \frac{\eta A}{\pi r^2}\left(\frac{\sin\beta r}{\beta r} - \cos\beta r\right)\cos\theta.$$

Note that, as r approaches zero, H_φ also approaches zero. Consequently, there is no current element at $r = 0$. This type of field expresses natural oscillations inside a perfectly conducting sphere of radius a, the frequency of which is found from

$$\frac{\sin\beta a}{\beta a} + \left(1 - \frac{1}{\beta^2 a^2}\right)\cos\beta a = 0, \qquad \beta a = \omega(\mu\varepsilon)^{1/2}a,$$

so that $\omega = (\beta a)(\mu\varepsilon)^{-1/2}/a$. The corresponding wavelength is $\lambda = 2\pi a/\beta a$.

4.13–11 Consider two large concentric spheres, one of radius a and the other of radius $a - \frac{1}{4}\lambda$. Assume that the first sphere is a perfect conductor, and the second a very thin resistive sheet with its *surface resistance* (ratio of E to the linear current density) equal to R. Find the steady-state field of the current element of moment Is situated at the center of the spheres. Solve first for the case $R = \eta$.

Ans. If $R = \eta$, the field inside the inner sphere is given by equations 82. Between the resistive and perfectly conducting spheres we have

$$rE_\theta = j\eta B\sin\beta(a-r)\sin\theta, \qquad rH_\varphi = B\cos\beta(a-r)\sin\theta,$$

where $B = (jIs/2\lambda)\,e^{-j\beta a}$. This is the case of perfect absorption of the wave by

the resistive sheet. The power absorbed per unit area is $\frac{1}{2}\eta H_\varphi H_\varphi^*$, where H_φ is the magnetic intensity just inside the resistive sheet and is equal to the linear current density J_θ in the sheet.

If $R \neq \eta$, then the field between the spheres is given by the above expression where

$$B = \frac{Is}{2\lambda}\left[\sin\beta a - \frac{j\eta}{R}\cos\beta a\right]^{-1}.$$

Inside the resistive sphere the field is given by the sum of equations 82 and the field in Problem 4.13–10 with

$$A = j\lambda e^{-j\beta a}\left(1 - \frac{\eta}{R}\right)B.$$

4.14–1 Consider a current filament along the z axis extending from $z = -\lambda/4$ to $z = \lambda/4$, and assume that the current at various points is $I(z) = I_0 \cos\beta z$. Find the distant field in the equatorial plane.

Ans.
$$E_\theta = \frac{60 I_0}{r} j e^{-j\beta r}.$$

4.14–2 Expand in power series expressions 82 for the field of the current element.

Ans.
$$E_\theta = \frac{j\eta Is}{2\lambda r}\left(-\frac{1}{\beta^2 r^2} + \tfrac{1}{2} - \tfrac{2}{3}j\beta r - \tfrac{3}{8}\beta^2 r^2 + \tfrac{2}{15}j\beta^3 r^3 + \cdots\right)\sin\theta,$$

$$H_\varphi = \frac{jIs}{2\lambda r}\left(\frac{1}{j\beta r} - \tfrac{1}{2}j\beta r - \tfrac{1}{3}\beta^2 r^2 + \tfrac{1}{8}j\beta^3 r^3 + \tfrac{1}{30}\beta^4 r^4 + \cdots\right)\sin\theta,$$

$$E_r = \frac{\eta Is}{2\pi r^2}\left(\frac{1}{j\beta r} - \tfrac{1}{2}j\beta r - \tfrac{1}{3}\beta^2 r^2 + \tfrac{1}{8}j\beta^3 r^3 + \tfrac{1}{30}\beta^4 r^4 + \cdots\right)\cos\theta.$$

4.14–3 Calculate the first two terms of E_z and E_ρ in phase with I.

Ans.
$$E_z' = -\frac{2\pi\eta Is}{3\lambda^2} + \frac{\pi\eta Is}{10\lambda^2}\beta^2 r^2 - \frac{\pi\eta Is}{30\lambda^2}\beta^2 r^2 \cos 2\theta,$$

$$E_\rho' = 0 + 0 - \frac{\pi\eta Is}{30\lambda^2}\beta^2 r^2 \sin 2\theta.$$

Note that the field of the element opposes the flow of current; hence, it presents a resistance to the driving voltage.

5

DIRECTIVE RADIATION

5.1 Fundamental formula

One of the major problems in antenna design is the calculation of the distant or "radiation" field in terms of an assumed current distribution. In the solution of this problem equations 4–83, which give the distant field of an electric current flowing through an infinitesimal element, are fundamental. At great distances, the radial electric intensity becomes vanishingly small compared with the transverse intensity and may be neglected. If the origin of the spherical coordinate system is chosen at the element and the equatorial plane is taken perpendicularly to the element (Fig. 4.15), then,

$$E_\theta = \left(\frac{60\pi I\, dz}{\lambda r}\sin\theta\right) je^{-j\beta r}, \qquad H_\varphi = \left(\frac{I\, dz}{2\lambda r}\sin\theta\right) je^{-j\beta r}, \tag{1}$$

where I is the current and dz is the length of the element.

In the preceding chapter we obtained this formula from the fundamental laws of electromagnetism; but it would not be too difficult to establish it directly by experiment. The analysis in this and the following chapters could then be carried out without any reference to Maxwell's equations. In fact, radiation patterns and the relative directivity gain of one current distribution over another may be calculated from a simpler formula,

$$E_\theta = A\,\frac{\sin\theta}{r}\, e^{-j\beta r}, \tag{2}$$

where A is a constant of proportionality which does not appear in the final results.

We can even rationalize this formula as follows.* At great distances from the source, the curvature of the wavefront is small, and the wave is substantially plane; hence, the phase retardation along the

* See also Sections 1.7 and 1.12.

radius between two points $r = r_1$ and $r = r$ is proportional to the distance $(r - r_1)$, and the coefficient of proportionality β is the same as for plane waves: that is, $\omega\sqrt{\mu\varepsilon}$. Thus, we have the phase retardation factor $\exp(-j\beta r)$ in the equation; the constant factor $\exp(-j\beta r_1)$ may be absorbed in the constant of proportionality A, which will thus include a possible effect of the curvature of the wavefront on the phase retardation of the wave in the vicinity of the source. Similarly, the power flow per unit area must be given by the same formula as that for a wave between two parallel conducting strips; that is, the power flow per unit area must be proportional to the square of the amplitude of E_θ. The total radiated power can then be obtained by integration over a sphere of large radius. By the principle of conservation of energy, this total power is independent of the radius r; hence, the power per unit area of the sphere is inversely proportional to the square of the radius, and, therefore, the amplitude of E_θ is inversely proportional to the radius. Finally, the meridian component E_θ of the field produced by the equal and opposite charges at the ends of the current element varies as $\sin\theta$. This may be shown by adding vectorially the radial electric intensity pointing one way from the positive charge and an equal radial intensity pointing the opposite way from the negative charge. Thus we have equation 2 except for the constant A.

The constant A may also be partially determined from simple considerations. It should be proportional to the current I and to the length dz of the element; then E_θ will be proportional to $I\,dz/r$. The physical dimensions of E_θ are those of a voltage per unit length whereas the physical dimensions of $I\,dz/r$ are those of electric current. To make the equation dimensionally correct, A must contain a physical quantity with the dimensions of an impedance in the numerator and a length in the denominator. These quantities must be characteristic of the medium and the wave. The intrinsic impedance, $\eta = \sqrt{\mu/\varepsilon}$, of the medium and the wavelength λ are two such quantities. Thus, equation 2 may be expanded into

$$E_\theta = A'\eta \frac{I\,dz}{\lambda r} \sin\theta\, e^{-j\beta r}. \tag{2'}$$

In free space, $\eta = 120\pi$. Comparing equations 2′ and 1, we find that A' is a dimensionless factor $\frac{1}{2}j$. This factor cannot be found from simple considerations such as those above. The factor j indicates that the total phase delay to a distant point corresponds not to the actual distance r but to the distance $r - \frac{1}{4}\lambda$.

However, this simple approach to the problem of radiation is not adequate for solving all the pertinent problems. In order to determine

the current distribution in a given antenna, or to calculate the input impedance, we have to use Maxwell's equations. Without them we can solve only those problems that are common to all waves, regardless of their particular physical characteristics.

5.2 Radiation intensity and radiated power

The *radiation intensity* Φ of a spherical wave in a given direction is the power radiated per unit solid angle. The *solid angle* is the region enclosed by a conical surface which is generated by a radius emerging from a fixed point and sliding round a fixed closed curve; its measure Ω is the ratio S/r^2, where S is the area intercepted by the solid angle on a sphere of radius r centered at the apex of the angle. The total radiated power P may thus be expressed in the following form:

$$P = \iint \Phi \, d\Omega. \tag{3}$$

Since $(r\,d\theta)(r \sin\theta\, d\varphi) = r^2 \sin\theta\, d\theta\, d\varphi$ is the expression in spherical coordinates for the elementary area intercepted by $d\Omega$ (see Fig. 4.4), we have

$$d\Omega = \sin\theta\, d\theta\, d\varphi; \tag{4}$$

hence,

$$P = \int_0^{2\pi} \int_0^{\pi} \Phi(\theta, \varphi) \sin\theta\, d\theta\, d\varphi. \tag{5}$$

In section 2.5 (equation 2–37), the average power flowing per unit area was expressed in the following form:

$$W = \tfrac{1}{2}\,\mathrm{re}(E \times H^*). \tag{6}$$

Since the distant field of a current element is perpendicular to the radius, the distant field of any combination of current elements in a given finite region is also perpendicular to the radius. Hence, the flow of power is strictly radial and is given by

$$W = \tfrac{1}{2}\,\mathrm{re}(E_\theta H_\varphi^* - E_\varphi H_\theta^*). \tag{7}$$

For a wave traveling outward in free space,

$$E_\theta = 120\pi H_\varphi, \qquad E_\varphi = -120\pi H_\theta; \tag{8}$$

and equation 7 becomes

$$W = 60\pi(H_\theta H_\theta^* + H_\varphi H_\varphi^*) = \frac{E_\theta E_\theta^* + E_\varphi E_\varphi^*}{240\pi}. \tag{9}$$

Since the area intercepted by the unit solid angle is r^2, the radiation

intensity is

$$\Phi = 60\pi r^2(H_\theta H_\theta^* + H_\varphi H_\varphi^*) = \frac{r^2(E_\theta E_\theta^* + E_\varphi E_\varphi^*)}{240\pi}. \tag{10}$$

5.3 The radiation intensity of a current element

From equations 1 and 10 we obtain the radiation intensity of *a current element parallel to the z axis,*

$$\Phi = 15\pi\left(\frac{I\,dz}{\lambda}\right)^2 \sin^2\theta. \tag{11}$$

The distance r does not occur in this equation, and the actual position of the element is irrelevant; but its orientation is important. To make the preceding equation general, we should write

$$\Phi = 15\pi\left(\frac{I\,ds}{\lambda}\right)^2 \sin^2\alpha, \tag{12}$$

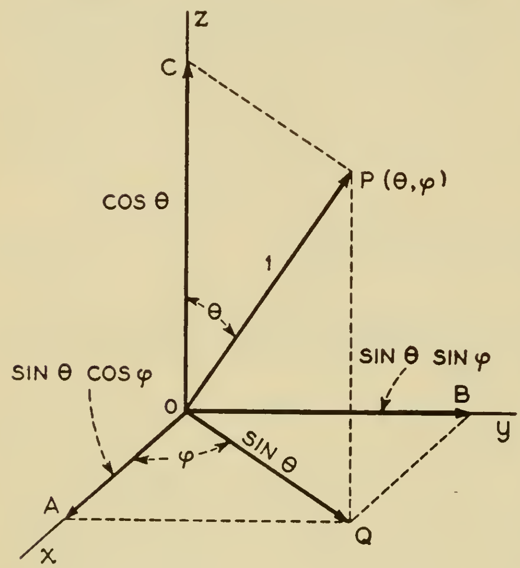


FIG. 5.1 The cosine of the angle between any two directions may be obtained by projecting a unit vector parallel to one of these directions on the other.

where ds is the length of the element and α is the angle between the axis of the element and a typical direction (θ, φ) in space.

To express α in terms of spherical coordinates we take a unit vector in the direction OP (Fig. 5.1). Its projections on the coordinate axes are the cosines of the angles between OP and the axes. The projection of OP on the x axis equals the projection of its projection on the xy plane. Since the projection of OP on the xy plane is $OQ = \sin\theta$ and the projection of OQ on the x axis is $OQ\cos\varphi$, we have the cosine of the angle between the x axis and a typical direction (θ, φ) in space,

$$\cos\alpha = \sin\theta\cos\varphi. \tag{13}$$

From this we obtain $\sin\alpha$ in equation 12 for an element parallel to the x axis. Similarly, for an element parallel to the y axis,

$$\cos\alpha = \sin\theta\sin\varphi. \tag{14}$$

The cosine of the angle between any two directions (θ_1, φ_1) and (θ_2, φ_2) is obtained by resolving a unit vector in one of these directions

along the coordinate axes and projecting the components on the remaining direction; thus, in general,

$$\cos \alpha = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2). \tag{15}$$

5.4 The power radiated by a current element

Substituting from equation 11 in equation 5, we obtain the power radiated by a current element,

$$P_0 = 15\pi \left(\frac{I\,dz}{\lambda}\right)^2 \int_0^{2\pi} \int_0^{\pi} \sin^3 \theta \, d\theta \, d\varphi = 40\pi^2 \left(\frac{I\,dz}{\lambda}\right)^2 . \tag{16}$$

If du is the moment of the current element per wavelength,

$$du = \frac{I\,dz}{\lambda}, \tag{17}$$

then,

$$P_0 = 40\pi^2\, du^2. \tag{18}$$

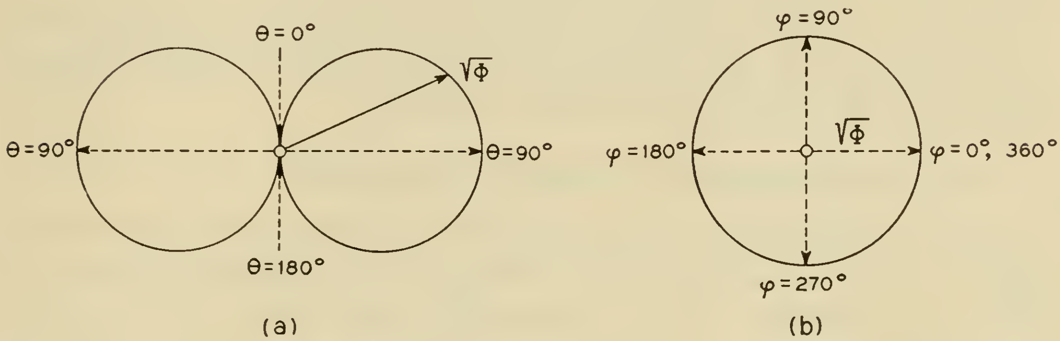


FIG. 5.2 Radiation patterns of a current element in free space: (*a*) the axial or "vertical" pattern; (*b*) the equatorial or "horizontal" pattern.

5.5 Radiation patterns

Radiation patterns are graphical representations of the radiation intensity or its square root. The graphs of $\sqrt{\Phi}$ give a better idea of the relative importance of the radiation in those directions in which its value is small.

"Vertical" and "horizontal" radiation patterns of an electric current element are shown in Fig. 5.2. In any plane through the axis of the element, $\sqrt{\Phi}$ is proportional to $\sin \theta$, and the polar pattern is a circle tangential to the element (Fig. 5.2*a*). In the equatorial plane of the element the radiation is uniform in all directions, and the radiation pattern is a circle concentric with the element (Fig. 5.2*b*). Figure 5.3 shows the vertical pattern in the form of a Cartesian plot.

5.6 Isotropic radiator

An *isotropic radiator* radiates equally in all directions; a uniformly pulsating sphere emitting sound waves is an example. There are no isotropic radiators of *coherent* electromagnetic waves, that is, of waves

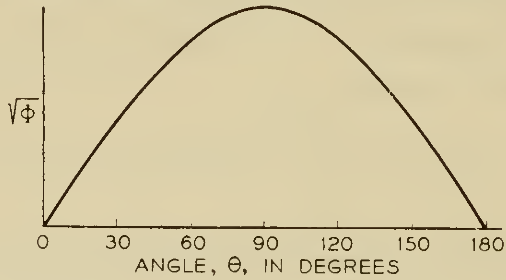


FIG. 5.3 The Cartesian plot of the axial radiation pattern of a current element is half a sine wave.

possessing a well-defined phase. The least directive simple sources of electromagnetic waves are the current element and an elementary current loop; their directive patterns are the same.

In spite of the nonexistence of isotropic radiators of coherent electromagnetic waves, such radiators are very useful as ideal reference standards in the analysis of directive antennas.

5.7 Wave interference and directive radiation

A simple method of directing more radiation in some directions than in others consists in using several radiators and distributing them over several wavelengths. If the sources are of equal intensities, the amplitudes of the waves generated by them tend to become equal as the dis-

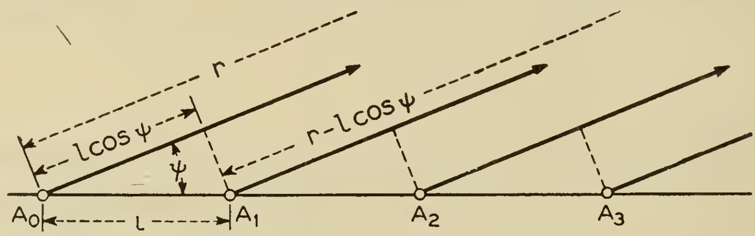


FIG. 5.4 A linear array of equispaced isotropic radiators.

tance from the sources increases. The phases, on the other hand, will depend on the local distances between the sources and on the direction of travel. In some directions the waves from the various sources may arrive in phase and thus reinforce each other; in others they may arrive out of phase and destroy each other. Hence, by a proper arrangement of sources we can obtain constructive superposition of waves in some directions and destructive interference in others.

Consider a linear array of equispaced isotropic radiators (Fig. 5.4)

operating in phase and of equal intensity. At a great distance from the array, the amplitudes of the waves arriving from the various sources will be equal, but the phases will differ. The wave from A_1 will travel a shorter distance than the wave from A_0, and its phase will be ahead of the phase of the wave arriving from A_0 by the amount $\xi = \beta l \cos \psi = (2\pi l/\lambda) \cos \psi$. If the field intensity of the wave from A_0 is represented by $OP_0 = 1$ (Fig. 5.5), the field intensity of the wave from A_1 will be represented by P_0P_1, that of the wave from A_2 by P_1P_2, etc. In the direction perpendicular to the array, $\psi = \pi/2$, all these vectors will be in phase and the field intensity will be n, where n is the number of elements. In other directions the field intensity will be smaller and may even vanish. Thus, in consequence of interference between the waves arriving from the various points, the resultant wave is stronger in some directions than in others.

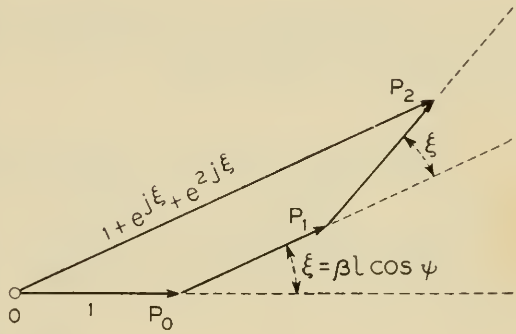


FIG. 5.5 A graphical method of obtaining the field intensity of an array.

The analytic analogue of the graphical diagram in Fig. 5.5 is

$$F = 1 + e^{j\xi} + e^{2j\xi} + e^{3j\xi} + \cdots + e^{(n-1)j\xi},$$
$$\xi = \beta l \cos \psi = \frac{2\pi l}{\lambda} \cos \psi. \tag{19}$$

If the strengths and the phases of the sources of the array relative to A_0 are given by $A_1 \exp(j\vartheta_1)$, $A_2 \exp(j\vartheta_2)$, $A_3 \exp(j\vartheta_3)$, $\cdots$, then at a distant point the field intensity of the array relative to that of the reference element will be

$$F = 1 + A_1 e^{j(\xi+\vartheta_1)} + A_2 e^{j(2\xi+\vartheta_2)} + A_3 e^{j(3\xi+\vartheta_3)} + \cdots + A_{n-1} e^{j[(n-1)\xi+\vartheta_{n-1}]}. \tag{20}$$

5.8 Space factor

Suppose now that the sources A_0, A_1, A_2, etc., are similar and similarly oriented antennas, parallel current elements or parallel half-wave antennas, for example. If E_0 is the electric intensity from the antenna at A_0, then the electric intensity from the antenna at A_1 is $E_0A_1 \exp j(\xi + \vartheta_1)$, where A_1, ξ, and ϑ_1 have the meanings assigned in the preceding section. Consequently, the electric intensity of the wave

from the entire array is

$$E = (1 + A_1 e^{j(\xi+\vartheta_1)} + A_2 e^{j(2\xi+\vartheta_2)} + A_3 e^{j(3\xi+\vartheta_3)} + \cdots + A_{n-1} e^{j(n-1)\xi+j\vartheta_{n-1}})E_0. \quad (21)$$

The absolute value S of the quantity F in parentheses is called the *space factor* of the array,

$$\begin{aligned} S &= |F| \\ &= |1 + A_1 e^{j(\xi+\vartheta_1)} + A_2 e^{j(2\xi+\vartheta_2)} + \cdots + A_{n-1} e^{j(n-1)\xi+j\vartheta_{n-1}}|. \end{aligned} \quad (22)$$

Since the radiation intensity is proportional to the square of the amplitude of the electric intensity, we may write

$$\Phi = S^2 \Phi_0, \quad (23)$$

where Φ_0 is the radiation intensity of the reference element of the array.* In general, the best way to calculate S^2 is to multiply F by its conjugate F^*,

$$S^2 = FF^*, \qquad S = \sqrt{FF^*}. \quad (24)$$

For example, for two elements,

$$\begin{aligned} S^2 &= (1 + Ae^{j(\xi+\vartheta)})(1 + Ae^{-j(\xi+\vartheta)}) \\ &= 1 + Ae^{j(\xi+\vartheta)} + Ae^{-j(\xi+\vartheta)} + A^2 \\ &= 1 + 2A\cos(\xi + \vartheta) + A^2. \end{aligned} \quad (25)$$

However, various short-cuts may be devised in special cases.

5.9 End-fire couplets

An *end-fire couplet* is a pair of equal sources operating in quadrature a quarter wavelength apart (Fig. 5.6*a*). The name " end-fire " is intended to describe a pronounced tendency of the couplet to radiate in the direction of the line of sources (Fig. 5.6*b*).

Assume that the phase lag of the source at A_1 with respect to the source at A_0 is 90°. In the direction A_0A_1 the wave from A_0 has to travel an extra quarter wavelength and will arrive at a distant point in phase with the wave from A_1; the field intensity will thus be doubled. In the opposite direction the wave from A_1 has to travel an extra quarter wavelength and will be 180° out of phase with the wave from A_0; thus, the field intensity will vanish. At right angles to the axis of the couplet, the waves from both sources will travel equal distances, the initial phase

* If this array is used as an element of an array whose space factor is S_1, the radiation intensity of the array of arrays is $\Phi_1 = S_1{}^2\Phi = S_1{}^2S^2\Phi_0$.

relationship is preserved, and the resultant field intensity is $\sqrt{2}$ times that from a single source.

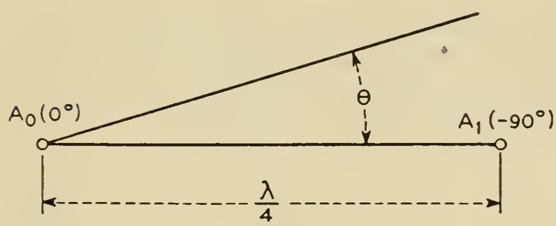
To find the space factor from equations 19, 22, and 24, we substitute

$$A_0 = A_1 = 1, \quad \vartheta_1 = -\frac{\pi}{2}, \quad l = \frac{\lambda}{4}, \quad \xi = \frac{\pi}{2}\cos\theta;$$

thus,

$$\begin{aligned} S^2 &= \{1 + \exp[\tfrac{1}{2}j\pi(1 - \cos\theta)]\}\{1 + \exp[-\tfrac{1}{2}j\pi(1 - \cos\theta)]\} \\ &= 2 + 2\cos[\tfrac{1}{2}\pi(1 - \cos\theta)] = 4\cos^2[\tfrac{1}{4}\pi(1 - \cos\theta)], \qquad (26) \\ S &= 2\cos[\tfrac{1}{4}\pi(1 - \cos\theta)]. \end{aligned}$$

The polar plot of S is shown in Fig. 5.6*b*.



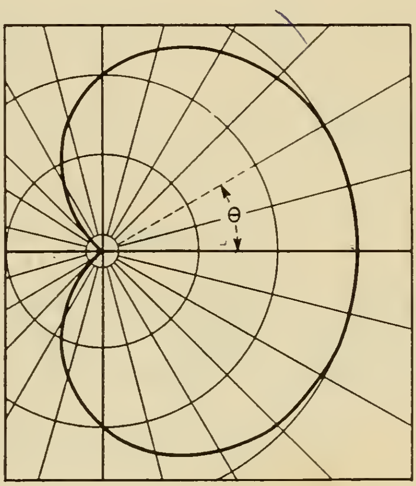


Fig. 5.6 (*a*) An end-fire couplet, and (*b*) its space factor.

From equations 12, 23, and 26 we find the radiation intensity of an end-fire couplet of current elements perpendicular to the line joining them. Assuming that the elements are parallel to the x axis and using equation 13, we have

$$\Phi = 60\pi\left(\frac{I\,dx}{\lambda}\right)^2 \cos^2[\tfrac{1}{4}\pi(1 - \cos\theta)](1 - \sin^2\theta\cos^2\varphi). \qquad (27)$$

In the plane perpendicular to the current elements $\cos\varphi = 0$ and the radiation pattern has still the shape shown in Fig. 5.6*b*; but in other planes the pattern is altered due to the directive properties of the current elements. Two nulls, for instance, appear in the positive and negative x directions.

5.10 Broadside couplets

If two isotropic radiators operate in phase, they reinforce each other most strongly in the plane perpendicular to the line joining them; they form a *broadside couplet.* If the radiators are on the z axis (Fig. 5.7), then,

$$S^2 = |1 + e^{i\beta l \cos\theta}|^2 = 4\cos^2\left(\frac{\pi l}{\lambda}\cos\theta\right). \tag{28}$$

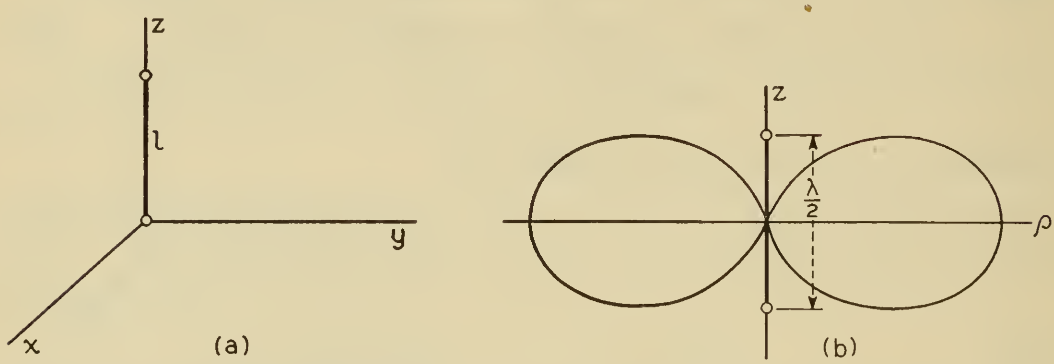


FIG. 5.7 (*a*) A broadside couplet, and (*b*) its space factor, for the case $l = \lambda/2$.

When $l < \lambda/2$, $|(\pi l/\lambda)\cos\theta| < \pi/2$; hence, S does not vanish in any direction. When $l = \lambda/2$, S vanishes in the direction of the line joining the sources. When $l > \lambda/2$, the null directions are given by

$$\begin{aligned} \frac{\pi l}{\lambda}\cos\theta &= (2m+1)\frac{\pi}{2}, \\ \cos\theta &= \frac{(2m+1)\lambda}{2l}, \qquad m = 0, \quad \pm 1, \quad \pm 2, \quad \cdots. \end{aligned} \tag{29}$$

As l/λ increases, the number of separate "radiation lobes" also increases.

5.11 Uniform linear arrays with a progressive phase delay

In a uniform linear array the amplitudes of the sources are equal. If the phase delay is uniformly progressive, then,

$$\vartheta_m = -m\vartheta, \tag{30}$$

where ϑ is the phase delay from one source to the next. Hence from equation 22 the space factor is

$$\begin{aligned} S &= |1 + e^{j(\xi-\vartheta)} + e^{2j(\xi-\vartheta)} + e^{3j(\xi-\vartheta)} + \cdots + e^{j(n-1)(\xi-\vartheta)}| \\ &= \frac{|\exp jn(\xi-\vartheta) - 1|}{|\exp j(\xi-\vartheta) - 1|} = \frac{|\exp[\frac{1}{2}jn(\xi-\vartheta)] - \exp[-\frac{1}{2}jn(\xi-\vartheta)]|}{|\exp[\frac{1}{2}j(\xi-\vartheta)] - \exp[-\frac{1}{2}j(\xi-\vartheta)]|} \\ &= \frac{|\sin[\frac{1}{2}n(\xi-\vartheta)]|}{|\sin[\frac{1}{2}(\xi-\vartheta)]|}, \end{aligned} \tag{31}$$

where $\xi = \beta l \cos\psi$. In the expression for Φ, the space factor S is squared, and the vertical bars indicating the absolute value in the final expression 31 may be omitted.

The space factor is maximum when

$$\xi = \vartheta, \qquad \beta l \cos\psi = \vartheta, \qquad \cos\psi = \frac{\vartheta\lambda}{2\pi l}; \tag{32}$$

then,

$$S_{\max} = n. \tag{33}$$

More generally, the space factor assumes its maximum value n, when

$$\xi - \vartheta = 2k\pi, \qquad k = 0, \quad \pm 1, \quad \pm 2, \quad \cdots;$$
$$\cos\psi = (\vartheta + 2k\pi)\frac{\lambda}{2\pi l} = \frac{\vartheta\lambda}{2\pi l} + \frac{k\lambda}{l}. \tag{34}$$

If l is sufficiently large, there may be several directions in which the fields add in phase and $S_{\max} = n$; if l is sufficiently small, there may be no direction in which the fields add in phase. In the latter case there still exist directions in which S is maximum; but the maximum is smaller than n. In intermediate cases there are one large maximum and several smaller maxima.

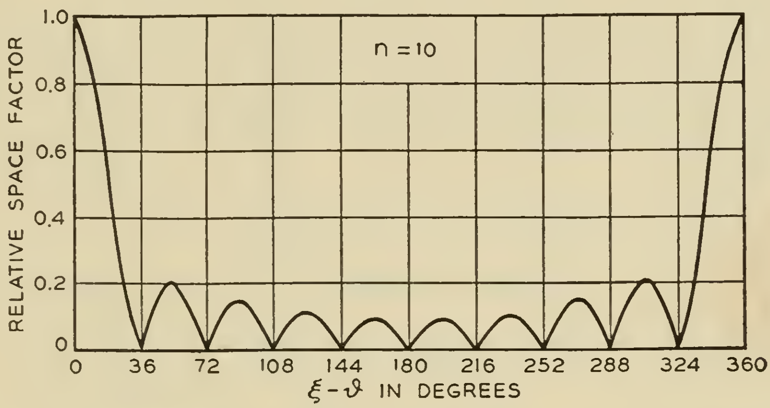


FIG. 5.8 Universal space factor for linear arrays of ten sources with equal amplitudes.

Figure 5.8 shows the Cartesian plot of the space factor as a function of the universal variable $\xi - \vartheta$ for a ten-element array. As the angle ψ (Section 5.7) varies from 0° to 180°, which is the maximum span for ψ, ξ varies from $\beta l - \vartheta$ to $\beta l + \vartheta$. If this range is within a 360° span, then the space factor contains only one major lobe; the minor lobes diminish as the angle made with the axis of the major lobe increases. When the range is greater, the minor lobes will eventually become larger.

The directions of maximum radiation are affected by the directive pattern of the elements of the array, in some instances very profoundly.

5.12 Uniform broadside arrays

In a *broadside array* the elements are operated in phase and are so oriented that they radiate maximum power in any direction perpendicular to the line of the array. For the broadside array $\vartheta = 0$, and the space factor becomes

$$S = \frac{\left|\sin \frac{n\xi}{2}\right|}{\left|\sin \frac{\xi}{2}\right|} = \frac{\left|\sin\left(\frac{n\pi l}{\lambda} \cos \psi\right)\right|}{\left|\sin\left(\frac{\pi l}{\lambda} \cos \psi\right)\right|}. \tag{35}$$

Two types of broadside arrays may be constructed out of current elements: (1) a "broadcast" array (Fig. 5.9*a*), and (2) a "point-to-

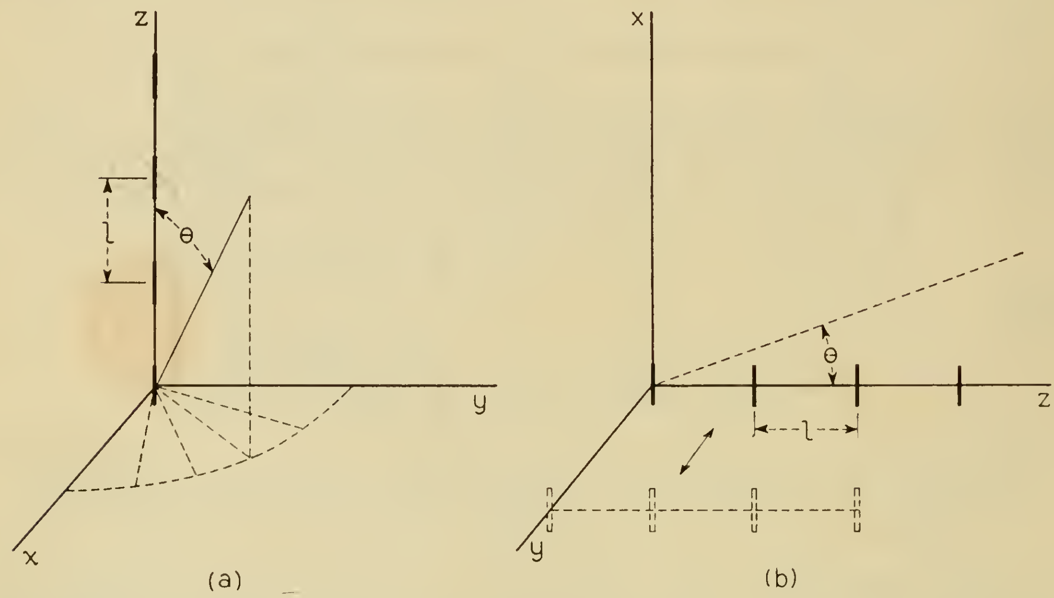


FIG. 5.9 Broadside arrays: (*a*) the "broadcast" type; (*b*) the "point-to-point" type.

point" array (Fig. 5.9*b*). In the first case the axes of the elements coincide with the line of the array, $\psi = \theta$, and the radiation intensity,

$$\Phi = 15\pi \left(\frac{I\,dz}{\lambda}\right)^2 \frac{\sin^2\left(\frac{n\pi l}{\lambda} \cos \theta\right) \sin^2 \theta}{\sin^2\left(\frac{\pi l}{\lambda} \cos \theta\right)}, \tag{36}$$

is the same in all directions at right angles to the elements. In the second case, the axes of the elements are perpendicular to the line of the array, and the radiation intensity is maximum only in those two directions that are perpendicular to the line of the array and to its elements. In this case,

$$\Phi = 15\pi\left(\frac{I\,dx}{\lambda}\right)^2 \frac{\sin^2\left(\dfrac{n\pi l}{\lambda}\cos\theta\right)}{\sin^2\left(\dfrac{\pi l}{\lambda}\cos\theta\right)}(1 - \sin^2\theta\cos^2\varphi). \tag{37}$$

The mathematical expression for Φ depends not only on the physical configuration but also on the choice of the spherical coordinate system. When the radiated power is calculated by integrating Φ, it is essential to choose the coordinates to make the form of Φ as simple as possible. The space factor is usually the more complicated factor in Φ; for this reason, in both arrangements of the elements (Fig. 5.9) we chose the line of the array as the z axis.

In the "point-to-point" broadside array of current elements (Fig. 5.9*b*), the radiation is maximum in two directions parallel to the y axis. By adding another broadside array parallel to the first and at a distance of a quarter wavelength in the direction of the y axis, we can cancel the backward radiation. Since the square of the space factor of the array of arrays is

$$S_1^2 = 4\cos^2[\tfrac{1}{4}\pi(1 - \sin\theta\sin\varphi)],$$

the radiation intensity of the double-row array is

$$\Phi = 60\pi\left(\frac{I\,dx}{\lambda}\right)^2 \frac{\sin^2\left(\dfrac{n\pi l}{\lambda}\cos\theta\right)}{\sin^2\left(\dfrac{\pi l}{\lambda}\cos\theta\right)} \times$$
$$(1 - \sin^2\theta\cos^2\varphi)\cos^2[\tfrac{1}{4}\pi(1 - \sin\theta\sin\varphi)]. \tag{38}$$

5.13 Uniform end-fire arrays

In an *end-fire array* the phase delay from one element to the next equals the phase delay in a plane wave traveling in the same direction, and the maximum radiation from each element is also in this direction. Thus, for the end-fire array, $\vartheta = \beta l$, and

$$S = \frac{\left|\sin\dfrac{n\pi l}{\lambda}(1 - \cos\psi)\right|}{\left|\sin\dfrac{\pi l}{\lambda}(1 - \cos\psi)\right|}. \tag{39}$$

For current elements arranged along the z axis ($\psi = \theta$) as in Fig. 5.9b, with the phase delay in the positive z direction, we have

$$\Phi = 15\pi \left(\frac{I\,dx}{\lambda}\right)^2 \frac{\sin^2 \dfrac{n\pi l}{\lambda}(1 - \cos\theta)}{\sin^2 \dfrac{\pi l}{\lambda}(1 - \cos\theta)} (1 - \sin^2\theta \cos^2\varphi). \qquad (40)$$

Figure 5.10 shows the space factor of an end-fire array of eight elements, spaced one-quarter wavelength apart.

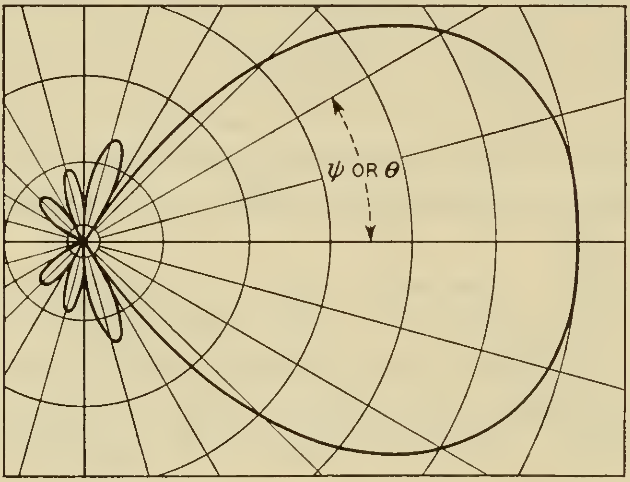


FIG. 5.10 The space factor of an end-fire array of eight elements, spaced one-quarter wavelength apart.

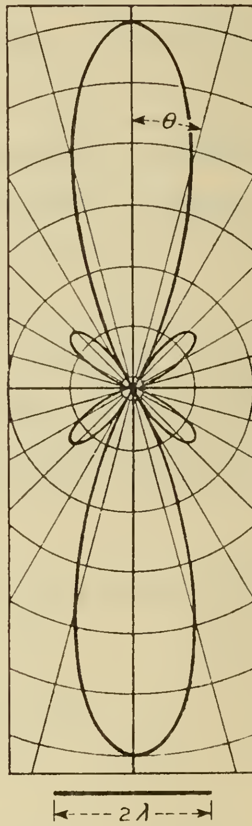


FIG. 5.11 The space factor for a continuous broadside array, two wavelengths long.

5.14 Continuous arrays

The elements of an array may be arranged continuously. In this case the amplitude and phase of an element are continuous functions of the position of the element, and the summation is replaced by integration. For example, for a continuous uniform array of nondirective sources, situated on the z axis from $z = 0$ to $z = l$, with a progressive linear phase delay, $\vartheta = kz$, we have

$$S = \left| \int_0^l e^{j(\beta \cos\theta - k)z}\,dz \right| = \left| \frac{e^{j(\beta\cos\theta - k)l} - 1}{j(\beta\cos\theta - k)} \right|$$

$$= \frac{2|\sin\frac{1}{2}(\beta\cos\theta - k)l|}{|\beta\cos\theta - k|}. \qquad (41)$$

Figure 5.11 shows the space factor for a continuous broadside array ($k = 0$), two wavelengths long.

Since the absolute maximum of $(\sin x)/x$ occurs when $x = 0$, we find that, if $k < \beta$, the major beam makes an angle with the array. The cosine of this angle equals k/β.

5.15 Rectangular arrays

In a rectangular array the elements are arranged in a rectangular pattern (Fig. 5.12). The array may be continuous, as in the case of a rectangular current sheet (Fig. 5.13). If all the elements are operated in phase,

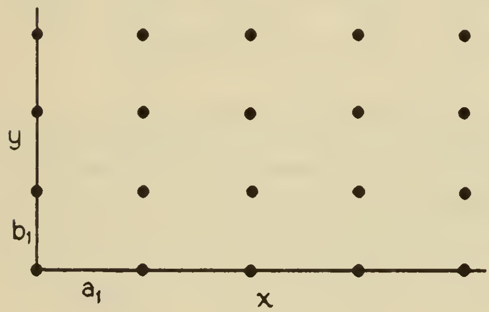


FIG. 5.12 The arrangement of elements in a rectangular array.

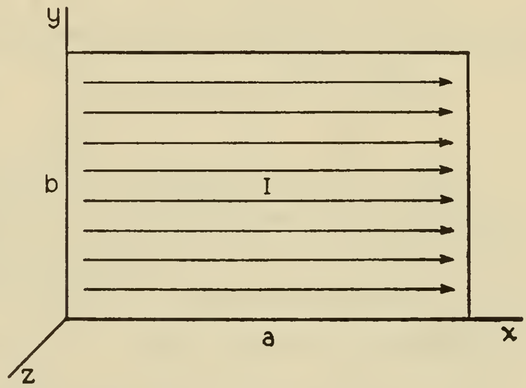


FIG. 5.13 A rectangular current sheet.

the array is broadside, and the maximum radiation is along the line perpendicular to the plane of the array (assuming, of course, that the radiation intensity of each element is also maximum in this direction). If the broadside rectangular array is uniform, it can be considered as a *linear array of linear arrays*, and its space factor is *the product of the space factors of the two arrays*,

$$S = S_1 S_2,$$

$$S_1 = \frac{\left|\sin\left(\dfrac{m\pi a_1}{\lambda}\cos\psi_x\right)\right|}{\left|\sin\left(\dfrac{\pi a_1}{\lambda}\cos\psi_x\right)\right|}, \qquad S_2 = \frac{\left|\sin\left(\dfrac{n\pi b_1}{\lambda}\cos\psi_y\right)\right|}{\left|\sin\left(\dfrac{\pi b_1}{\lambda}\cos\psi_y\right)\right|}, \tag{42}$$

where ψ_x and ψ_y are the angles made by the x axis and the y axis with a typical direction (θ, φ). Since

$$\cos\psi_x = \sin\theta\cos\varphi, \qquad \cos\psi_y = \sin\theta\sin\varphi, \tag{43}$$

the space factor of the uniform rectangular array of nondirective sources is

$$S^2 = \frac{\sin^2\left(\frac{m\pi a_1}{\lambda}\sin\theta\cos\varphi\right)}{\sin^2\left(\frac{\pi a_1}{\lambda}\sin\theta\cos\varphi\right)}\;\frac{\sin^2\left(\frac{n\pi b_1}{\lambda}\sin\theta\sin\varphi\right)}{\sin^2\left(\frac{\pi b_1}{\lambda}\sin\theta\sin\varphi\right)}. \tag{44}$$

In terms of the dimensions of the array, $(m-1)a_1 = a$ and $(n-1)b_1 = b$, equation 44 becomes

$$S^2 = \frac{\sin^2\left[\frac{\pi a}{\lambda(1-m^{-1})}\sin\theta\cos\varphi\right]}{\sin^2\left[\frac{\pi a}{(m-1)\lambda}\sin\theta\cos\varphi\right]}\;\frac{\sin^2\left[\frac{\pi b}{\lambda(1-n^{-1})}\sin\theta\sin\varphi\right]}{\sin^2\left[\frac{\pi b}{(n-1)\lambda}\sin\theta\sin\varphi\right]}. \tag{45}$$

The radiation intensity of a continuous uniform rectangular array of nondirective sources may be obtained from equation 45 if we make m and n approach infinity. When x is small, $\sin x \simeq x$, and the sines in the denominators may be replaced by their arguments. Of course, if the strength of each source is kept equal to unity, Φ will increase indefinitely, but, if the total strength of the array is finite and has the value p, then the strength of each element is p/mn; multiplying equation 45 by $(p/mn)^2$ and passing to the limit, we find

$$S^2 = p^2 \frac{\sin^2\left(\frac{\pi a}{\lambda}\sin\theta\cos\varphi\right)}{\left(\frac{\pi a}{\lambda}\sin\theta\cos\varphi\right)^2}\;\frac{\sin^2\left(\frac{\pi b}{\lambda}\sin\theta\sin\varphi\right)}{\left(\frac{\pi b}{\lambda}\sin\theta\sin\varphi\right)^2}. \tag{46}$$

A uniform rectangular current sheet (Fig. 5.13) is a continuous array of current elements. Since the radiation intensity of a current element parallel to the x axis is proportional to

$$\sin^2\psi_x = 1 - \cos^2\psi_x = 1 - \sin^2\theta\cos^2\varphi, \tag{47}$$

the radiation intensity of the current sheet is

$$\Phi = S^2(1 - \sin^2\theta\cos^2\varphi). \tag{48}$$

To obtain the factor of proportionality p^2, we shall evaluate Φ_{max} from equation 48 and equate it to its value obtained from the field equations. Since the maximum radiation takes place in the direction of the z axis, where $\theta = 0$ or π, we have

$$\Phi_{max} = p^2. \tag{49}$$

On the other hand, the moment of each current element of area $dx\,dy$ is the product of the current $(I/b)\,dy$ in the element and its length dx. Since in the direction $\theta = 0$ the fields of all the elements add in phase, the amplitude of the magnetic intensity is

$$|H_\varphi| = \int_0^b \int_0^a \frac{I}{2b\lambda r}\,dx\,dy = \frac{Ia}{2\lambda r}\cdot \tag{50}$$

By equation 10 the maximum radiation intensity of the current sheet is

$$\Phi_{\max} = p^2 = 15\pi \left(\frac{aI}{\lambda}\right)^2; \tag{51}$$

hence, the complete expression for the radiation intensity of a uniform current sheet is

$$\Phi = \frac{15\pi a^2 I^2 \sin^2\left(\frac{\pi a}{\lambda}\sin\theta\cos\varphi\right)\sin^2\left(\frac{\pi b}{\lambda}\sin\theta\sin\varphi\right)}{\lambda^2\left(\frac{\pi a}{\lambda}\sin\theta\cos\varphi\right)^2\left(\frac{\pi b}{\lambda}\sin\theta\sin\varphi\right)^2}(1-\sin^2\theta\cos^2\varphi). \tag{52}$$

5.16 Calculation of the radiated power

There are two equivalent methods for calculating the power radiated in a nondissipative medium by a given current distribution: (1) the *Poynting vector method* for obtaining the power carried by the wave; (2) the *radiation resistance method*, or the *induced emf method*, or the *method of moments*, for obtaining directly the work done by the impressed forces against the forces of reaction exerted by the wave. By the radiation resistance method we obtain directly the power transferred from the sources to the wave. By the principle of conservation of energy the two expressions for the radiated power must be equal, that is, if the medium is nondissipative. When the medium is dissipative, then the energy transferred to the wave is gradually dissipated in heat so that the power carried by the wave diminishes with the distance from the source and eventually becomes arbitrarily small; in such media, the Poynting vector method must be replaced by the evaluation of the dissipated power, that is, by the integration of $\frac{1}{2}gEE^*$ over the volume occupied by the wave.

The Poynting vector method was discussed in Section 5.2; it consists of the following steps: (1) the calculation of the distant field produced by the given currents; (2) the calculation of the complex Poynting vector and its real part, given by equation 6 and representing the average flow of power per unit area; and (3) the integration of the

Poynting vector over a closed surface which is usually chosen to be an infinitely large sphere. The amplitude of the distant field varies as $1/r$, and for this reason it is convenient to multiply the Poynting vector by r^2 and thus obtain the power Φ radiated per unit solid angle. This radiation intensity Φ is independent of r, and the final step in the calculation of the radiated power consists of evaluating the double integral (equation 5). This was the earliest method proposed for calculating the radiated power, and it was used successfully to obtain the radiation from antennas with sinusoidally distributed currents.*

5.17 Asymptotic formulas for the radiated power

If the source of power is highly directive, it is easy to obtain an approximate value of P by evaluating the integral over a relatively small range of the solid angle Ω where the integrand is particularly large. For example, let us evaluate the power radiated by a uniform rectangular current sheet (Fig. 5.13) carrying current I parallel to the x axis.

The radiation intensity is given by equation 52. This expression does not change if we substitute $\pi - \theta$ for θ, which is as it should be, since the pattern must be symmetric with respect to the plane of the current sheet. Therefore,

$$P = 2\iint' \Phi \, d\Omega, \tag{53}$$

where the prime is to remind us that the integration is to be extended over only one half of the unit sphere.† The greatest values of Φ occur for small values of θ, when $\sin\theta \simeq \theta$. Let us introduce the "Cartesian coordinates" (x, y) for the area on the unit sphere in the immediate vicinity of the origin P where $\theta = 0$; then, since θ is the distance from the origin (Fig. 5.14),

$$x = \theta\cos\varphi, \qquad y = \theta\sin\varphi, \qquad d\Omega = dx\,dy. \tag{54}$$

Hence, after substitution from equations 52 and 54, equation 53 will become approximately

$$P = \frac{30\pi a^2 I^2}{\lambda^2}\int_{-\theta_1}^{\theta_1} \frac{\sin^2(\pi a x/\lambda)}{(\pi a x/\lambda)^2}\,dx \int_{-\theta_1}^{\theta_1} \frac{\sin^2(\pi b y/\lambda)}{(\pi b y/\lambda)^2}\,dy, \tag{55}$$

where the integration is extended over a square whose side $2\theta_1$ is large enough to take most of the radiation. We have neglected x^2 in the factor $(1 - x^2)$.

* B. van der Pol, On the wave-lengths and radiation of loaded antennae, *Proc. Phys. Soc.* (London), **29**, June 1917, pp. 269–289.

† Recalling that the solid angle $d\Omega$ is an element of area on a sphere whose radius is unity.

We now introduce new variables,

$$u = \frac{\pi a x}{\lambda}, \qquad v = \frac{\pi b y}{\lambda}. \tag{56}$$

The current sheet is highly directive only if a/λ and b/λ are large; the new limits of integration will be large even though θ_1 is small; and the

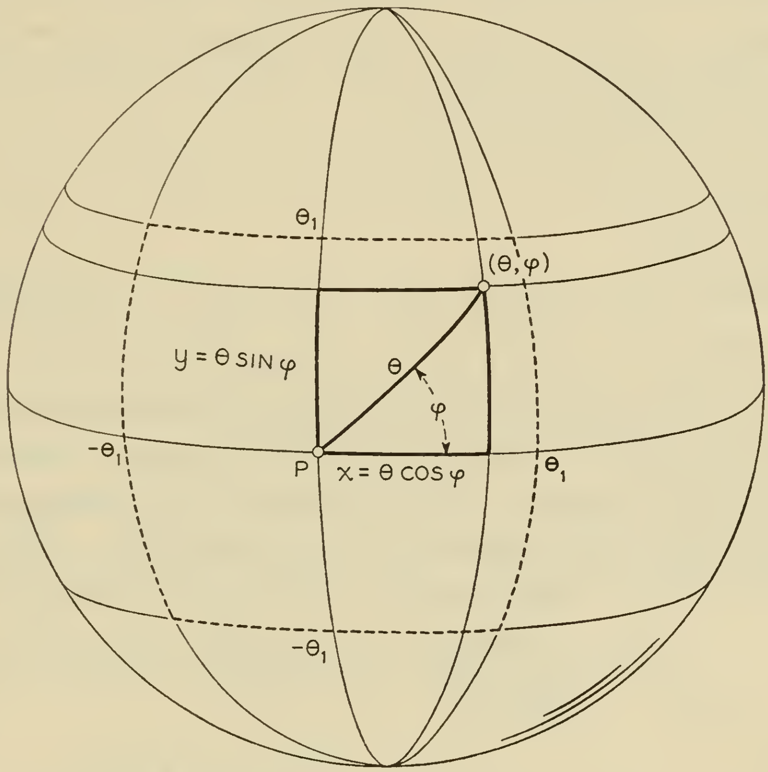


FIG. 5.14 "Rectangular" and "polar" coordinates in a small region of a spherical surface.

integrands diminish rapidly. Thus, we may approximate P by taking the limits of integration infinite,

$$P = \frac{30aI^2}{\pi b}\left(\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\,dt\right)^2 = \frac{120aI^2}{\pi b}\left(\int_0^{\infty} \frac{\sin^2 t}{t^2}\,dt\right)^2. \tag{57}$$

The integral may be evaluated by parts:

$$\begin{aligned} Q &= \int_0^{\infty} \frac{\sin^2 t}{t^2} = \int_0^{\infty} \sin^2 t \, d\left(-\frac{1}{t}\right) \\ &= \left(-\frac{1}{t}\right)\sin^2 t \,\Big|_0^{\infty} + \int_0^{\infty} \frac{2 \sin t \cos t}{t}\,dt \\ &= \int_0^{\infty} \frac{\sin 2t}{t}\,dt = \int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}. \end{aligned} \tag{58}$$

Hence,

$$P = \frac{30\pi a}{b} I^2. \tag{59}$$

The transformations 54 and 56 may be used in other similar problems.

An infinite sheet of current of linear density C generates plane waves on both sides. At the surface of the sheet $H = \frac{1}{2}C$, $E = \frac{1}{2}\eta C$, and the power radiated per unit area is $\frac{1}{4}\eta C^2 = 30\pi C^2$. Hence, the power radiated by an area ab is $30\pi C^2 ab$. If I is the current flowing across b, $C = I/b$, and we have equation 59.

5.18 Energy and power expended in excitation of electromagnetic waves

The impressed forces must do work to maintain a given current distribution against the reaction of the wave produced by it. Let $\tilde{J}$ be the instantaneous current density in a volume element $dS\,ds$, where dS is the element of area perpendicular to $\tilde{J}$ and ds is the element of length in the direction of $\tilde{J}$. If $\tilde{E}$ is the electric intensity of the field, the electromotive force between the end surfaces of the element is $\tilde{E}_s\,ds$; to counteract this emf we must have an impressed emf $-\tilde{E}_s\,ds$. Under the influence of this emf the charge $\tilde{J}\,dS\,dt$ is transferred from one end of the volume element to the other, and the work done in the interval $(0, t)$ is

$$\mathcal{E} = -\int_0^t dt \iiint \tilde{E}_s \tilde{J}\,dS\,ds. \tag{60}$$

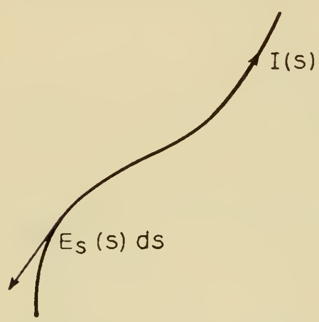


Fig. 5.15 An electric current filament.

Hence, the average power is

$$P = -\tfrac{1}{2}\,\mathrm{re} \iiint E_s J^*\,dS\,ds. \tag{61}$$

This method of calculating the radiated power was suggested by Léon Brillouin* and has been used extensively† since 1929.

For a thin filament (Fig. 5.15), E_s is the same throughout dS, and equation 61 becomes

$$P = -\tfrac{1}{2}\,\mathrm{re} \int E_s I^*\,ds. \tag{62}$$

* Sur l'origine de la résistance de rayonnement, *Radioélectricité*, 3, April 1922, pp. 147–152.

† A. A. Pistolkors, The radiation resistance of beam antennas, *IRE Proc.*, 17, March 1929, pp. 562–579.

R. Bechman, On the calculation of radiation resistance of antennas and antenna combinations, *IRE Proc.*, 19, August 1931, pp. 1471–1480.

The same result can be obtained by integrating the Poynting vector over the surface of the filament, since the integral of the tangential H around the filament must be equal to I.

For future convenience we shall write equation 62 as follows:

$$P = \text{re}(\Psi), \qquad \Psi = -\tfrac{1}{2}\iiint E_s J^*\, dS\, ds = -\tfrac{1}{2}\int E_s I^*\, ds, \tag{63}$$

where Ψ is the *complex power.*

In the following sections we shall illustrate equation 62 by applying it to a current element and to a pair of current elements; then we shall derive another general formula for the power radiated by a given current distribution.

5.19 Radiation drag on an electric current; the radiation resistance

The field of a current element is given by equations 4–82; it consists of two components: one in quadrature with the moment Is of the element and the other in phase or 180° out of phase. The second component is the one we need in order to determine the average reaction of the wave on the current. Assuming that the element is thin, we may obtain this component as the limit of $\text{re}(E_r)$ when r approaches zero; thus,*

$$\begin{aligned}
\text{re}(E_r) &= \frac{60}{r^2}\left(\cos\beta r - \frac{\sin\beta r}{\beta r}\right) I\, ds;\\
\cos\beta r &= 1 - \tfrac{1}{2}(\beta r)^2 + \tfrac{1}{24}(\beta r)^4 - \cdots \\
\sin\beta r &= \beta r - \tfrac{1}{6}(\beta r)^3 + \tfrac{1}{120}(\beta r)^5 - \cdots \\
\text{re}(E_r) &= -\frac{80\pi^2}{\lambda^2}\left[1 - \frac{2\pi^2 r^2}{5\lambda^2} + \cdots\right] I\, ds,
\end{aligned} \tag{64}$$

and in the limit, as r/λ approaches zero,

$$\text{re}(E_r) = \frac{-80\pi^2 I\, ds}{\lambda^2}. \tag{65}$$

Thus, on the average, the electric intensity of the wave *opposes* the flow of current just as a resistance would. The ratio of the emf impressed in phase with the current to the current is called the *radiation resistance* of the current element; hence,

$$R_{\text{rad}} = -\frac{ds\, \text{re}(E_r)}{I} = \frac{80\pi^2\, (ds)^2}{\lambda^2}. \tag{66}$$

* We can always choose the origin of time so that the initial phase of *one* variable is zero. Choosing I as this variable, we make it real so that $I^* = I$.

Thus, we obtain the following expression for the radiated power:

$$P = \tfrac{1}{2}R_{\text{rad}}I^2 = 40\pi^2 \left(\frac{I\,ds}{\lambda}\right)^2. \tag{67}$$

This agrees with equation 16.

5.20 The mutual radiation resistance of two current elements

In the case of two current elements (Fig. 5.16), the complex power is

$$\begin{aligned}\Psi &= \Psi_{11} + \Psi_{12} + \Psi_{21} + \Psi_{22} \\ &= -\tfrac{1}{2}E_{1,s_1}I_1^*\,ds_1 - \tfrac{1}{2}E_{1,s_2}I_2^*\,ds_2 - \tfrac{1}{2}E_{2,s_1}I_1^*\,ds_1 - \\ &\qquad \tfrac{1}{2}E_{2,s_2}I_2^*\,ds_2,\end{aligned} \tag{68}$$

where E_{1,s_1} is the component of the electric field of the first element

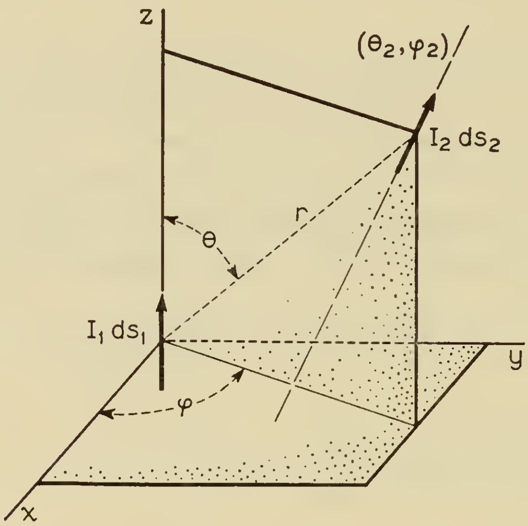


FIG. 5.16 Two current elements.

along I_1, E_{1,s_2} is the component of this field along I_2, etc. The mutual impedances of the elements are defined as follows:

$$Z_{12} = -\frac{E_{1,s_2}\,ds_2}{I_1}, \qquad Z_{21} = -\frac{E_{2,s_1}\,ds_1}{I_2}. \tag{69}$$

Thus, Z_{12} is the emf which must be impressed on the second element in order to sustain the current I_2 against the field of the first element per unit current in the first element. Using equations 4–82, we find that

$$Z_{12} = Z_{21}. \tag{70}$$

When the two elements coincide, the mutual impedance becomes the

radiation self-impedance. Hence equation 68 may be expressed as

$$\Psi = \tfrac{1}{2}Z_{11}I_1I_1^* + \tfrac{1}{2}Z_{12}(I_1I_2^* + I_2I_1^*) + \tfrac{1}{2}Z_{22}I_2I_2^*. \tag{71}$$

The mutual radiation resistance is the real part of the mutual impedance Z_{12}, and the radiation resistance of an element is the real part of its radiation self-impedance,

$$R_{12} = \operatorname{re} Z_{12}, \qquad R_{11} = \operatorname{re} Z_{11}. \tag{72}$$

The quantity in parentheses in equation 71 is real since

$$\begin{aligned} I_1 &= |I_1| \exp(j\vartheta_{I_1}), \qquad I_2 = |I_2| \exp(j\vartheta_{I_2}), \\ I_1I_2^* &= |I_1I_2| \exp j(\vartheta_{I_1} - \vartheta_{I_2}), \\ I_2I_1^* &= |I_1I_2| \exp j(\vartheta_{I_2} - \vartheta_{I_1}), \end{aligned} \tag{73}$$

and, therefore,

$$I_1I_2^* + I_2I_1^* = 2|I_1I_2| \cos \vartheta_{12}, \tag{74}$$

where ϑ_{12} is the phase difference between the currents,

$$\vartheta_{12} = \vartheta_{I_1} - \vartheta_{I_2}. \tag{75}$$

Consequently, the radiated power is

$$P = \tfrac{1}{2}R_{11}|I_1|^2 + R_{12}|I_1I_2| \cos \vartheta_{12} + \tfrac{1}{2}R_{22}|I_2|^2. \tag{76}$$

It is important to note that the mutual radiated power vanishes when $\vartheta = \pi/2$, so that the two current elements operate in quadrature. This

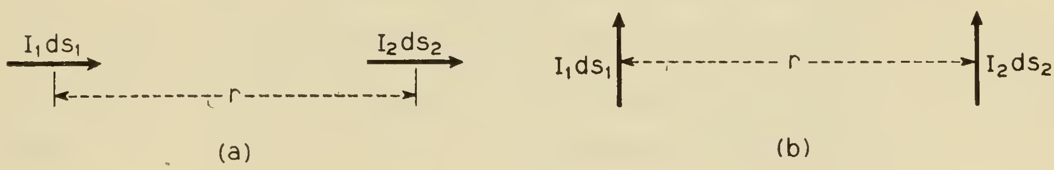


FIG. 5.17 (*a*) Two colinear elements, and (*b*) two parallel elements with their axes perpendicular to the line joining them.

does not mean, however, that such elements radiate independently. It means merely that the *combined radiated power is the same as if the elements were radiating independently.* One of the elements may radiate less than it would by itself; but the other will supply the deficiency.

In the case of two colinear elements (Fig. 5.17*a*), one element is in the E_r field of the other, and, from equations 4–82, we obtain

$$R_{12} = \frac{60\, ds_1\, ds_2}{r^2}\left(\frac{\sin \beta r}{\beta r} - \cos \beta r\right). \tag{77}$$

Similarly, if two parallel elements are perpendicular to the line joining their centers (Fig. 5.17*b*), one element is in the E_θ field of the other, and

$$R_{12} = \frac{60\pi\, ds_1\, ds_2}{\lambda r}\left(\sin \beta r + \frac{\cos \beta r}{\beta r} - \frac{\sin \beta r}{\beta^2 r^2}\right). \tag{78}$$

5.21 The method of moments for calculating the radiated power

It is clear from equation 67 that the power radiated by a current element depends on the moment of the current element per wavelength, Il/λ. This equation and also equations 77 and 78 suggest that it will be convenient to express the mutual radiation resistance in the following form:

$$R_{12} = \frac{K_{12}\, ds_1\, ds_2}{\lambda^2}. \tag{79}$$

The " radiation influence coefficient " K_{12} has the dimensions of a resistance; it depends only on the distance between the elements in wavelengths and on their relative orientation. Let

$$du_m = |I_m| \frac{ds_m}{\lambda} \tag{80}$$

be the moment of a typical element per wavelength. In terms of these quantities the expression 76 for the power radiated by two elements is

$$P = \tfrac{1}{2}K_{11}\, du_1\, du_1 + K_{12} \cos \vartheta_{12}\, du_1\, du_2 + \tfrac{1}{2}K_{11}\, du_2\, du_2. \tag{81}$$

More generally,

$$\begin{aligned} P = \tfrac{1}{2}K_{11}\, &(du_1\, du_1 + du_2\, du_2 + du_3\, du_3 + \cdots) + \\ &K_{12} \cos \vartheta_{12}\, du_1\, du_2 + K_{13} \cos \vartheta_{13}\, du_1\, du_3 + \cdots + \\ &K_{23} \cos \vartheta_{23}\, du_2\, du_3 + K_{24} \cos \vartheta_{24}\, du_2\, du_4 + \cdots + \\ &K_{34} \cos \vartheta_{34}\, du_3\, du_4 + \cdots. \end{aligned} \tag{82}$$

In a more compact form,

$$P = \tfrac{1}{2} \sum_m \sum_n K_{mn} \cos \vartheta_{mn}\, du_m\, du_n, \tag{83}$$

where the summation is extended over all the elements. The factor $\frac{1}{2}$ appears, because in the summation each term for which $m \neq n$ occurs twice. For a continuous distribution of elements, equation 83 becomes

$$P = \tfrac{1}{2}\iint K_{mn} \cos \vartheta_{mn}\, du_m\, du_n. \tag{84}$$

The problem is now reduced to the determination of the influence coefficient K_{mn}. To present the results in a convenient form we intro-

duce the following functions:

$$A\left(\frac{r}{\lambda}\right) = 30\left(\frac{\lambda}{r}\right)^2\left[\frac{\sin(2\pi r/\lambda)}{2\pi r/\lambda} - \cos\frac{2\pi r}{\lambda}\right],$$

$$B\left(\frac{r}{\lambda}\right) = \frac{30\pi\lambda}{r}\left[\sin\frac{2\pi r}{\lambda} + \frac{\cos(2\pi r/\lambda)}{2\pi r/\lambda} - \frac{\sin(2\pi r/\lambda)}{(2\pi r/\lambda)^2}\right], \tag{85}$$

$$S\left(\frac{r}{\lambda}\right) = A\left(\frac{r}{\lambda}\right) + B\left(\frac{r}{\lambda}\right), \qquad T\left(\frac{r}{\lambda}\right) = A\left(\frac{r}{\lambda}\right) - B\left(\frac{r}{\lambda}\right).$$

Then, *for two parallel elements* (Fig. 5.18*a*), we have

$$K_{12}\left(\frac{r}{\lambda}\right) = 2A\left(\frac{r}{\lambda}\right)\cos^2\theta + 2B\left(\frac{r}{\lambda}\right)\sin^2\theta,$$

where the first term is contributed by the E_r field and the second by the E_θ field. Since

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta), \qquad \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta),$$

we have

$$K_{12}\left(\frac{r}{\lambda}\right) = S\left(\frac{r}{\lambda}\right) + T\left(\frac{r}{\lambda}\right)\cos 2\theta. \tag{86}$$

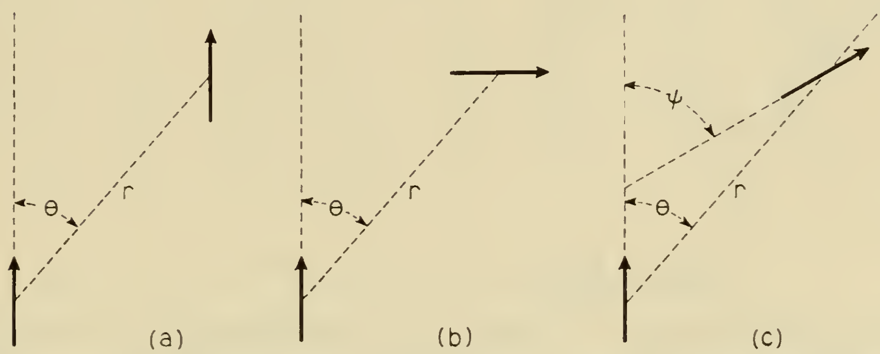


Fig. 5.18 (*a*) Two parallel elements, (*b*) two coplanar perpendicular elements, and (*c*) two coplanar elements.

For two perpendicular elements in the same plane (Fig. 5.18*b*), we have

$$K_{12}\left(\frac{r}{\lambda}\right) = 2A\left(\frac{r}{\lambda}\right)\cos\theta\sin\theta - 2B\left(\frac{r}{\lambda}\right)\sin\theta\cos\theta$$

$$= T\left(\frac{r}{\lambda}\right)\sin 2\theta. \tag{87}$$

For any two elements in the same plane (Fig. 5.18*c*), the parallel and perpendicular components of the unit moment are $\cos\psi$ and $\sin\psi$.

Hence, to obtain K_{12} we multiply equations 86 and 87 by $\cos\psi$ and $\sin\psi$, respectively, and then add,

$$K_{12}\left(\frac{r}{\lambda}\right) = S\left(\frac{r}{\lambda}\right)\cos\psi + T\left(\frac{r}{\lambda}\right)\cos(2\theta - \psi). \tag{88}$$

If one element is perpendicular to the plane containing the other and the line joining their centers, then,

$$K_{12}\left(\frac{r}{\lambda}\right) = 0. \tag{89}$$

Finally, for any two elements, situated as in Fig. 5.16, the component of the unit moment of the second element in the direction of the first is $\cos\theta_2$, and the perpendicular component in the axial plane of the first element is $\sin\theta_2\cos(\varphi - \varphi_2)$. Multiplying equations 86 and 87 by these components and adding, we have

$$K_{12}\left(\frac{r}{\lambda}\right) = S\left(\frac{r}{\lambda}\right)\cos\theta_2 + T\left(\frac{r}{\lambda}\right)[\cos 2\theta\cos\theta_2 + \sin 2\theta\sin\theta_2\cos(\varphi - \varphi_2)]. \tag{90}$$

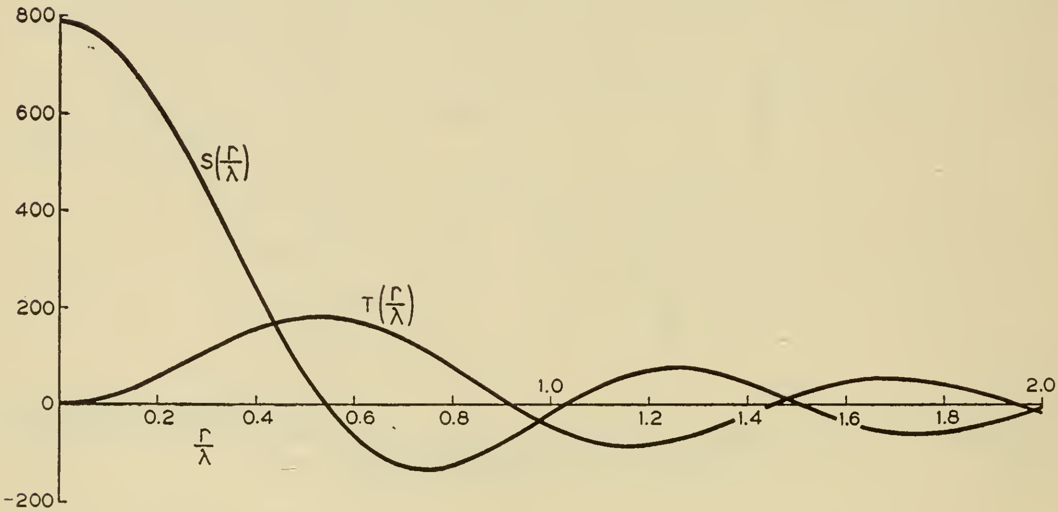


Fig. 5.19 S and T functions defined in equations 85.

The functions $S(r/\lambda)$ and $T(r/\lambda)$ are shown in Fig. 5.19, and a table is given on page 165.

In some instances the integral in equation 84 can be expressed in terms of already tabulated functions; but, in general, it will have to be evaluated numerically. The current distribution is then subdivided into elements, the moment of each element is concentrated in the center of the element, and the double sum (equation 82) is evaluated.

r/λ	$A(r/\lambda)$	$B(r/\lambda)$	$S(r/\lambda)$	$T(r/\lambda)$	r/λ	$A(r/\lambda)$	$B(r/\lambda)$	$S(r/\lambda)$	$T(r/\lambda)$
0	394.8	394.8	789.6	0	0.72	−1.722	−127.7	−129.4	126.0
0.02	394.2	393.5	787.7	0.6112	0.74	−8.320	−123.0	−131.3	114.6
0.04	392.3	389.8	782.1	2.483	0.76	−14.12	−116.7	−130.8	102.6
0.06	389.2	383.6	772.8	5.554	0.78	−19.12	−109.1	−128.3	90.01
0.08	384.9	375.1	760.0	9.796	0.80	−23.35	−100.4	−123.7	77.01
0.10	379.4	364.3	743.7	15.15	0.82	−26.83	−90.58	−117.4	63.75
0.12	372.8	351.2	724.0	21.55	0.84	−29.58	−79.94	−109.5	50.36
0.14	365.1	336.2	701.2	28.90	0.86	−31.64	−68.62	−100.3	36.98
0.16	356.3	319.2	675.5	37.10	0.88	−33.04	−56.80	−89.83	23.76
0.18	346.5	300.5	647.0	46.04	0.90	−33.81	−44.65	−78.46	10.83
0.20	335.9	280.2	616.1	55.61	0.92	−34.02	−32.35	−66.37	−1.676
0.22	324.3	258.7	583.0	65.67	0.94	−33.68	−20.07	−53.75	−13.62
0.24	312.0	235.9	547.9	76.08	0.96	−32.87	−7.979	−40.85	−24.89
0.26	299.0	212.3	511.3	86.70	0.98	−31.63	3.760	−27.87	−35.39
0.28	285.4	188.0	473.3	97.39	1.0	−30.00	15.00	−15.00	−45.00
0.30	271.2	163.2	434.4	108.0	1.05	−24.60	40.04	15.44	−64.64
0.32	256.6	138.2	394.8	118.4	1.1	−17.95	59.34	41.39	−77.29
0.34	241.6	113.2	354.9	128.4	1.15	−10.79	71.70	60.91	−82.49
0.36	226.4	88.52	314.9	137.9	1.2	−3.810	76.60	72.79	−80.41
0.38	211.0	64.28	275.3	146.7	1.25	2.445	74.18	76.62	−71.73
0.40	195.5	40.72	236.3	154.8	1.3	7.552	65.17	72.73	−57.62
0.42	180.1	18.07	198.1	162.0	1.35	11.24	50.86	62.10	−39.61
0.44	164.7	−3.503	161.2	168.2	1.4	13.41	32.87	46.27	−19.46
0.46	149.5	−23.81	125.7	173.3	1.45	14.05	13.06	27.11	0.9958
0.48	134.6	−42.69	91.91	177.3	1.5	13.33	−6.667	6.667	20.00
0.50	120.0	−60.00	60.00	180.0	1.55	11.48	−24.93	−13.45	36.41
0.52	105.8	−75.62	30.19	181.4	1.6	8.796	−39.71	−30.91	48.50
0.54	92.11	−89.46	2.649	181.6	1.65	5.617	−49.88	−44.26	55.50
0.56	78.94	−101.4	−22.49	180.4	1.7	2.284	−54.79	−52.51	57.08
0.58	66.36	−111.5	−45.10	177.8	1.75	−.8909	−54.30	−55.19	53.41
0.60	54.43	−119.5	−65.12	174.0	1.8	−3.640	−48.76	−52.40	45.12
0.62	43.18	−125.6	−82.47	168.8	1.85	−5.762	−38.94	−44.71	33.18
0.64	32.65	−129.8	−97.14	162.4	1.9	−7.132	−26.00	−33.13	18.87
0.66	22.88	−132.0	−109.1	154.9	1.95	−7.702	−11.28	−18.99	3.581
0.68	13.88	−132.4	−118.5	146.2	2.0	−7.500	3.750	−3.750	−11.25
0.70	5.680	−120.9	−125.2	136.6					

5.22 Applications of the method of moments

To illustrate the applications of the method of moments, we shall calculate the power radiated by a sinusoidal current filament a half wavelength long (Fig. 5.20). In Chapter 8 we shall show that this is an approximate current distribution in a thin wire whose length is $\lambda/2$. Assuming that the filament is along the z axis and that its center is at the origin, we have

$$I(z) = I_0 \cos \beta z, \qquad \beta = \frac{2\pi}{\lambda}. \tag{91}$$

In the present case it is possible to evaluate the integral in equation 84,

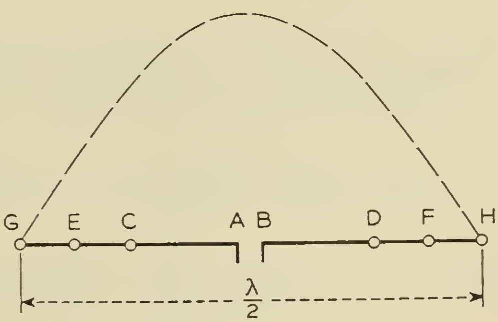


FIG. 5.20 A sinusoidal current filament a half wavelength long.

and we shall do so in Section 11.4; but here we wish to explain the technique that may be used when we are unable to express the integrals in terms of known functions. The technique is essentially a numerical integration consisting of replacing the integral in equation 84 by the double sum in equation 82.

The moment per wavelength of the current between $z = z_1$ and $z = z_2$ is

$$u = I \int_{z_1}^{z_2} \cos \beta z \, d\left(\frac{z}{\lambda}\right) = \frac{I}{2\pi} \sin \beta z \Big|_{z_1}^{z_2} \cdot \tag{92}$$

If we assume that the entire moment is concentrated at the center of the filament, then $z_1 = -\lambda/4$, $z_2 = \lambda/4$, and

$$u = \frac{I}{\pi} \cdot \tag{93}$$

Since $K_{11} = 80\pi^2$, we have

$$P = 40I^2. \tag{94}$$

Let us now subdivide each arm of the antenna into two halves and assume that the moments of the sections GC, CD, DH are concentrated in the centers of these sections. The moments of CD and DH are, respectively:

$$U_{CD} = \frac{I}{\pi\sqrt{2}}, \qquad U_{DH} = \frac{I(\sqrt{2}-1)}{2\sqrt{2}\,\pi} \cdot \tag{95}$$

By equation 82 we have

$$P = 40\pi^2 I^2 \left[\frac{1}{2\pi^2} + \frac{(\sqrt{2}-1)^2}{8\pi^2} + \frac{(\sqrt{2}-1)^2}{8\pi^2}\right] + 685.3 I^2 \frac{\sqrt{2}-1}{2\pi^2} + 429.7 I^2 \frac{(\sqrt{2}-1)^2}{8\pi^2} = 37.0 I^2. \tag{96}$$

The exact value for the assumed distribution is $36.56I^2$.

If the wire is bent as shown in Fig. 5.21, we have

$$P = 40\pi^2 I^2 \left[\frac{1}{2\pi^2} + \frac{(\sqrt{2}-1)^2}{8\pi^2} + \frac{(\sqrt{2}-1)^2}{8\pi^2}\right] + \tfrac{4}{5} I^2 T\left(\frac{\sqrt{5}}{16}\right) \frac{\sqrt{2}-1}{2\pi^2} + I^2[S(\tfrac{1}{4}) - T(\tfrac{1}{4})] \frac{3-2\sqrt{2}}{8\pi^2} = 23.2 I^2. \tag{97}$$

Fairly accurate results may be obtained by the method of moments even with a relatively small number of subdivisions; and, of course,

there is no limit to the accuracy as the number of subdivisions is increased, aside from the uncertainties inherent in the current distribution in any actual antenna.

5.23 Directive reception

The absorbing properties of an antenna are derived from its radiating properties by the reciprocity theorem, of which more will be said later. Thus, the receiving pattern is identical with the radiation pattern.

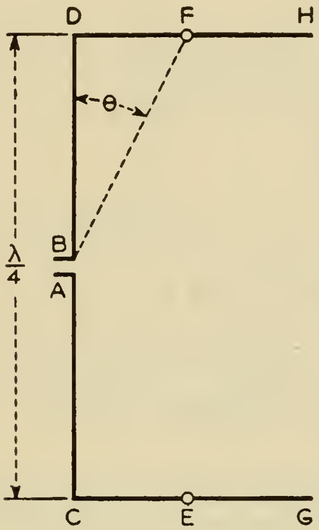


FIG. 5.21 A bent sinusoidal current filament a half wavelength long.

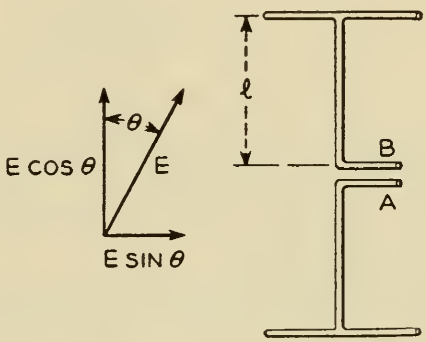


FIG. 5.22 A short antenna, capacitively loaded at each end.

Because of the reciprocity theorem, throughout this book we deal mostly with "transmitting antennas." In some cases — as in the case of the "wave antenna," for example — we shall find it convenient to change our point of view and think of the receiving properties first. In any case, it is helpful to look at antennas from both points of view.

Let us consider a short wire, heavily loaded with capacitance at the ends (Fig. 5.22). The capacitive loading may be in the form of two metal disks. The impressed electric intensity E, making an angle θ with the wire, may be resolved into two components: $E \cos \theta$ parallel to the wire and $E \sin \theta$ perpendicular to it. The perpendicular component induces no voltage between the terminals A and B. The voltage induced by the parallel component is

$$V = 2El \cos \theta. \tag{98}$$

In the case of two receiving elements (Fig. 5.23), the voltages induced across the terminals of each element are

$$V_1 = 2El \cos \theta, \qquad V_2 = 2Ele^{-i\beta d \cos \psi} \cos \theta, \tag{99}$$

where E is the electric intensity at the first element and ψ is the angle between the line joining the elements and the direction $\vec{n}$ of wave propagation. These two voltages may be combined in any phase relationship we wish. If they are combined in phase, our receiving couplet will act as a broadside receiver. If the output of one element is retarded by βd,

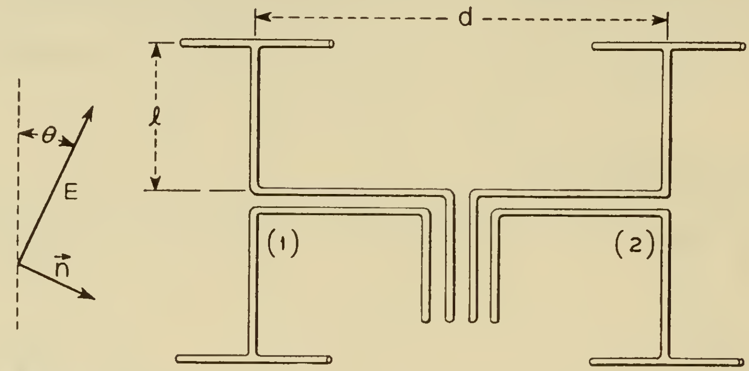


FIG. 5.23 Two loaded antenna elements used as receivers.

we shall have an end-fire receiver. The voltages induced in the individual elements may be amplified, not necessarily equally, before they are combined.

The voltage induced in a short wire without end loading is approximately one half of that induced in a heavily loaded wire; thus,

$$V = El \cos \theta, \tag{100}$$

and the reception pattern is the same as that of a current element.

5.24 Antenna array synthesis

As we have seen in this chapter, it is easy to calculate the radiation pattern of a given distribution of radiating elements. The converse problem of finding an array with prescribed directive properties is much more difficult. This problem is somewhat simplified by the fact that, in most practical applications, we are interested not in the precise details of the pattern but in such features as the width of the main lobe, the level of the largest minor lobe, and the shape of the main lobe. Some progress toward practical solution of these problems can be made by guessing, based upon our knowledge of the various solutions of direct problems; but more direct analytic methods of investigation are needed if we are to make real progress.

In the analysis of an array we are concerned mostly with its space factor (equation 22). Its complex form given by equation 20 is particularly well suited to the analysis of its properties. The following method

has been developed by one of the authors.* We introduce a complex variable

$$z = e^{j\xi} = e^{j\beta l \cos \psi}, \tag{101}$$

where l is the spacing between the elements and ψ is the angle between the axis of the array and a typical direction in space. The complex space factor will then become a polynomial

$$F = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}, \tag{102}$$

the coefficients of which represent the relative amplitudes and phases of the various elements. For example, the polynomial associated with a uniform broadside array is

$$F = 1 + z + z^2 + \cdots + z^{n-1}, \tag{103}$$

and that associated with a uniform end-fire array is

$$F = 1 + kz + k^2 z^2 + \cdots + k^{n-1} z^{n-1}, \qquad k = e^{-j\beta l}. \tag{104}$$

By the fundamental theorem of algebra, any polynomial of the $(n - 1)$th degree has $n - 1$ zeros, and it may be factored; thus,

$$F = a_{n-1}(z - z_1)(z - z_2)(z - z_3)\cdots(z - z_{n-1}), \tag{105}$$

where $z_1, z_2, \cdots z_{n-1}$ are the zeros of F. In the case of the uniform broadside array, for example, we have

$$F = \frac{z^n - 1}{z - 1}. \tag{106}$$

The zeros of the numerator are the various nth roots of unity,

$$z_m = u^m, \qquad u = e^{2\pi j/n}, \qquad m = 0, 1, 2, \cdots, n - 1. \tag{107}$$

Therefore,

$$z^n - 1 = (z - 1)(z - u)(z - u^2)\cdots(z - u^{n-1}), \tag{108}$$

and

$$F = (z - u)(z - u^2)\cdots(z - u^{n-1}). \tag{109}$$

The space factor is the absolute value of F,

$$S = |F| = |a_{n-1}|\,|z - z_1|\,|z - z_2|\cdots|z - z_{n-1}|. \tag{110}$$

Geometrically, the absolute value of the difference $z - z_m$ between two complex numbers is represented by the length of the straight segment joining the points in the complex plane represented by z and z_m. As

* S. A. Schelkunoff, U. S. Patent 2,286,839, filed December 20, 1939, issued June 16, 1942. Also, A mathematical theory of linear arrays, *Bell Sys. Tech. Jour.*, 22, January 1943, pp. 80–107.

seen from equation 101, the various directions are represented by points situated on the unit circle with its center at $z = 0$. Hence, the space factor of an array is represented by the product of the lengths of straight segments joining a typical point on the unit circle to the zeros of the

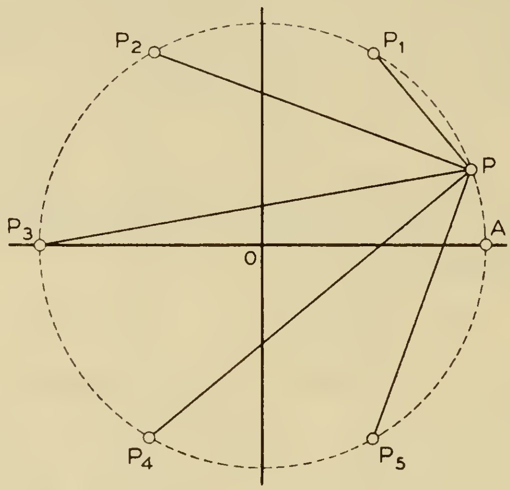


FIG. 5.24 Zeros of a uniform broadside array of six elements.

complex space factor F. For instance, the zeros (equation 107) of a uniform broadside array are represented by points equispaced on the unit circle, the point $z = 1$ being excepted. For a six-element array these zeros are at $P_1, P_2, \cdots, P_5$ in Fig. 5.24. The space factor is thus given by

$$S = (PP_1)(PP_2)(PP_3)(PP_4)(PP_5). \tag{111}$$

It vanishes when P coincides with one of the zeros; it is maximum when P is at A, the point corresponding to $\psi = 90°$ (that is, the direction perpendicular to the axis of the array).

Thus we have the main result: *the space factor of a linear array is characterized completely by an associated polynomial in a complex variable whose range is the unit circle.*

In the case of symmetric arrays, we can substitute a new variable and obtain an associated polynomial in a real variable whose range is $(-1, 1)$. Consider, for instance, the uniform broadside array (equation 103), and assume that n is odd. First we write

$$F = z^{(n-1)/2}(z^{-(n-1)/2} + z^{-(n-3)/2} + \cdots + z^{(n-3)/2} + z^{(n-1)/2}), \tag{112}$$

and note that the absolute value of the first factor is unity, so that it has no effect on the space factor. Thus, we may delete this factor from F. Let

$$w = \tfrac{1}{2}(z + z^{-1}), \tag{113}$$

and take the successive powers,

$$\begin{aligned} w^2 &= \tfrac{1}{4}(z^2 + 2 + z^{-2}), \\ w^3 &= \tfrac{1}{8}(z^3 + 3z + 3z^{-1} + z^{-3}), \cdots. \end{aligned} \tag{114}$$

From these equations we can express $z^m + z^{-m}$ as a polynomial in w; hence, we can express F as a polynomial in w. The new variable w is

$$w = \cos \xi = \cos(\beta l \cos \psi); \tag{115}$$

it is real, and the maximum range of its variation is $(-1, 1)$.

This method was utilized for increasing the directivity of an array without increasing its size and for suppressing the minor lobes.* The largest minor lobe of a uniform array is only 13 db below the level of the major lobe, irrespective of the number of elements. In some applications this is too large. By making the array nonuniform it is possible to reduce this level. Spectacular gains in the directivity may be obtained, but only at the expense of spectacular loss in efficiency, spectacular narrowing of the bandwidth, and spectacular increase in the precision with which the amplitudes and phases of the various elements have to be adjusted. In practice one must be satisfied only with moderate directive gains over uniform arrays. Such gains have actually been obtained.

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* *Ibid.* Also Section 6.20.

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PROBLEMS

5.1–1 Let l be the length of each arm of a very short and very thin antenna ($l \ll \lambda/4$). Assume that the charge is distributed uniformly on each arm, and show that the distant field is

$$E_\theta = \frac{60\pi I_0 l}{\lambda r} j e^{-j\beta r} \sin\theta,$$

where I_0 is the input current.

5.2–1 Show that in pure water (zero conductivity) the radiation intensity is given approximately by

$$\Phi = 20r^2(H_\theta H_\theta^* + H_\varphi H_\varphi^*) = \frac{r^2(E_\theta E_\theta^* + E_\varphi E_\varphi^*)}{80}.$$

5.3–1 Derive equation 15.

5.4–1 Calculate the power radiated by a very short and thin antenna of total length $2l$. Let the input current be I_0.

Ans. $$P = 40\pi^2\left(\frac{I_0 l}{\lambda}\right)^2.$$

5.4–2 Express the distant field of the current element in terms of the power radiated by it.

Ans. $$E_\theta = \frac{(90P)^{1/2}}{r} j e^{-j\beta r} \sin\theta.$$

5.4–3 Show that the expressions for the distant fields of a current element and a short antenna in terms of radiated power are the same.

5.8–1 Find the space factor of two equal sources located at points whose Cartesian coordinates are (0, 0, 0) and (l, 0, 0).

Ans. $$S = 2\cos(\tfrac{1}{2}\beta l \sin\theta \cos\varphi).$$

5.8–2 Find the space factor of two equal sources located at (0, 0, 0) and (0, l, 0).

Ans. $$S = 2\cos(\tfrac{1}{2}\beta l \sin\theta \sin\varphi).$$

5.8–3 Find the space factor of four equal sources located at the corners of a square: (0, 0, 0), (l, 0, 0), (0, l, 0), (l, l, 0). *Hint*: The quickest method is to consider the full array as an array consisting of pairs of sources.

Ans. $$S = 4\cos(\tfrac{1}{2}\beta l \sin\theta \cos\varphi)\cos(\tfrac{1}{2}\beta l \sin\theta \sin\varphi).$$

5.8–4 Find the space factor of eight equal sources located at the corners of a cube with its vertices at (0, 0, 0), (l, 0, 0), (0, l, 0), (0, 0, l), etc.

Ans. $S = 8\cos(\tfrac{1}{2}\beta l \sin\theta\cos\varphi)\cos(\tfrac{1}{2}\beta l\sin\theta\sin\varphi)\cos(\tfrac{1}{2}\beta l\cos\theta).$

5.8–5 Find the space factor of three sources, located at (0, 0, 0), (0, 0, l), (0, 0, $2l$), operating in phase but with amplitudes in the ratio 1, 2, 1.

Ans. $$S = 4\cos^2(\tfrac{1}{2}\beta l\cos\theta).$$

5.8–6 Find the space factor of a linear array of $n + 1$ sources, all operating in phase and with amplitudes proportional to the coefficients in the binomial expansion of $(a + b)^n$. Let l be the distance between the successive elements and ψ the angle between the axis of the array and a typical direction in space.

Ans. $$S = 2^n\cos^n(\tfrac{1}{2}\beta l\cos\psi).$$

5.9–1 Obtain the radiation intensity of an end-fire couplet consisting of two elements parallel to the z axis and arranged along the x axis.

Ans. $$\Phi = 60\pi\left(\frac{I\,dz}{\lambda}\right)^2\cos^2\left[\frac{\pi}{4}(1 - \sin\theta\cos\varphi)\right]\sin^2\theta.$$

5.10–1 What is the radiation intensity of a broadside couplet of two current elements, situated on the z axis and parallel to it?

Ans. $$\Phi = 60\pi\left(\frac{I\,dz}{\lambda}\right)^2\cos^2\left(\frac{\pi l}{\lambda}\cos\theta\right)\sin^2\theta.$$

5.10–2 Find the radiation intensity of a broadside couplet of two current elements parallel to the z axis and arranged along the y axis.

Ans. $$\Phi = 60\pi\left(\frac{I\,dz}{\lambda}\right)^2\cos^2\left(\frac{\pi l}{\lambda}\sin\theta\sin\varphi\right)\sin^2\theta.$$

5.10–3 Obtain the radiation intensity of a broadside couplet of two current elements arranged along the x axis and parallel to it.

Ans. $$\Phi = 60\pi\left(\frac{I\,dx}{\lambda}\right)^2\cos^2\left(\frac{\pi l}{\lambda}\sin\theta\cos\varphi\right)(1 - \sin^2\theta\cos^2\varphi).$$

5.12–1 Find the space factor of a broadside linear array of n elements whose amplitudes are attenuated at the rate of α nepers per unit length.

Ans. $$S^2 = \frac{\left|1 - e^{n(-\alpha+j\beta\cos\psi)l}\right|^2}{\left|1 - e^{(-\alpha+j\beta\cos\psi)l}\right|^2}$$

$$= \frac{1 - 2e^{-n\alpha l}\cos(n\beta l\cos\psi) + e^{-2n\alpha l}}{1 - 2e^{-\alpha l}\cos(\beta l\cos\psi) + e^{-2\alpha l}}$$

$$= e^{-(n-1)\alpha l}\,\frac{\cosh n\alpha l - \cos(n\beta l\cos\psi)}{\cosh\alpha l - \cos(\beta l\cos\psi)}\,.$$

5.13–1 Find the radiation intensity of an end-fire array of elements arranged as in Fig. 5.9*a*.

Ans.
$$\Phi = 15\pi \left(\frac{I\,dz}{\lambda}\right)^2 \frac{\sin^2 \dfrac{n\pi l}{\lambda}(1 - \cos\theta)}{\sin^2 \dfrac{\pi l}{\lambda}(1 - \cos\theta)} \sin^2\theta.$$

5.13–2 Obtain the space factor of an end-fire linear array of n elements whose amplitudes are attenuated at the rate of α nepers per unit length.

Ans.
$$S^2 = \frac{1 - 2e^{-n\alpha l}\cos[n\beta l(1 - \cos\psi)] + e^{-2n\alpha l}}{1 - 2e^{-\alpha l}\cos[\beta l(1 - \cos\psi)] + e^{-2\alpha l}}$$
$$= e^{-(n-1)\alpha l}\frac{\cosh n\alpha l - \cos[n\beta l(1 - \cos\psi)]}{\cosh \alpha l - \cos[\beta l(1 - \cos\psi)]}.$$

5.14–1 Find the space factor of a continuous linear broadside array of sources whose amplitudes are attenuated at the rate of α nepers per unit length. Let l be the length of the array.

Ans.
$$S^2 = \frac{1 - 2e^{-\alpha l}\cos(\beta l\cos\psi) + e^{-2\alpha l}}{\alpha^2 + \beta^2\cos^2\psi}.$$

5.14–2 Solve the preceding problem for an end-fire array.

Ans.
$$S^2 = \frac{1 - 2e^{-\alpha l}\cos[\beta l(1 - \cos\psi)] + e^{-2\alpha l}}{\alpha^2 + \beta^2(1 - \cos\psi)^2}.$$

5.14–3 Find the space factor of a linear broadside array of length l if the amplitude of the sources is maximum at the center (unity, let us say) and drops linearly to a fraction k at the ends.

Ans.
$$S = l\left|\frac{k\sin(\frac{1}{2}\beta l\cos\psi)}{\frac{1}{2}\beta l\cos\psi} + \frac{\frac{1}{2}(1 - k)\sin^2(\frac{1}{4}\beta l\cos\psi)}{(\frac{1}{4}\beta l\cos\psi)^2}\right|$$

5.14–4 Find the radiation intensity of an electric current filament extending along the z axis from $z = -\lambda/4$ to $z = \lambda/4$, assuming that the current is $I(z) = I_0\cos\beta z$.

Ans.
$$\Phi = \frac{15 I_0^2\cos^2(\frac{1}{2}\pi\cos\theta)}{\pi\sin^2\theta}, \qquad \Phi_{\max} = \frac{15 I_0^2}{\pi}.$$

5.14–5 Find the radiation intensity of an electric current filament extending along the z axis from $z = -l$ to $z = l$, assuming that the current is $I(z) = I_0 \sin\beta(l - |z|)$.

Ans.
$$\Phi = \frac{15 I_0^2[\cos(\beta l\cos\theta) - \cos\beta l]^2}{\pi\sin^2\theta}.$$

5.14–6 Find the radiation intensity of a uniform current filament of length l, extended along the z axis.

Ans.
$$\Phi = \frac{15 I_0^2[1 - \cos(\beta l\cos\theta)]}{2\pi\cos^2\theta}\sin^2\theta.$$

5.14–7 Find the radiation intensity of a progressive current wave, traveling with phase constant k from $z = 0$ to $z = l$, $I(z) = I_0 e^{-jkz}$.

Ans.
$$\Phi = \frac{30\pi I_0^2[1 - \cos(k - \beta\cos\theta)l]}{\lambda^2(k - \beta\cos\theta)^2}\sin^2\theta.$$

5.14–8 Obtain the radiation intensity of a current filament extending from $z = 0$ to $z = l$ when $I(z) = I_0 \sin(\pi z/l)$.

Ans.
$$\Phi = \frac{60\pi^3 l^2 I_0^2 \cos^2(\frac{1}{2}\beta l\cos\theta)\sin^2\theta}{\lambda^2(\pi^2 - \beta^2 l^2\cos^2\theta)^2}\cdot$$

5.14–9 Obtain the radiation intensity of a filament extending from $z = 0$ to $z = n\lambda/2$ and divided into n equal parts with identical current distributions. The current from $z = 0$ to $z = \lambda/2$ is $I(z) = I_0 \sin\beta z$; from $z = \lambda/2$ to $z = \lambda$ it is $I(z) = I_0 \sin\beta(z - \frac{1}{2}\lambda)$; etc.

Ans.
$$\Phi = \frac{15 I_0^2}{\pi}\,\frac{\cos^2(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\,\frac{\sin^2(\frac{1}{2}n\pi\cos\theta)}{\sin^2(\frac{1}{2}\pi\cos\theta)}\cdot$$

5.14–10 Find the space factor of a uniform circular array of radius a. Let the array be in the xy plane and the center be at the origin of the coordinate system.

Ans.
$$S = 2\pi a J_0(\beta a\sin\theta).$$

5.15–1 Find the space factor of a uniform cylindrical sheet of sources. Let the radius be a and the height l. Assume that the cylindrical sheet is coaxial with the z axis.

Ans.
$$S = 4\pi a J_0(\beta a\sin\theta)\,\frac{\sin(\frac{1}{2}\beta l\cos\theta)}{\beta\cos\theta}\cdot$$

5.15–2 Find the radiation intensity of a uniform cylindrical electric current sheet. Assume that the current is parallel to the axis of the sheet.

Ans.
$$\Phi = \frac{15\pi C^2}{\lambda^2}S^2\sin^2\theta = \frac{15 I^2}{4\pi\lambda^2 a^2}S^2\sin^2\theta,$$

where S is given in the preceding problem, C is the linear current density of the sheet, and $I = 2\pi aC$ is the total current flowing parallel to the axis of the cylinder.

5.15–3 Find the radiation intensity of a uniform cylindrical current sheet on the assumption that current I is circulating around it. *Hint:* Decompose each current element of moment $(Ia/l)\,d\varphi\,dz$ into its x and y components whose fields can later be added. Note: A simpler method is given in Section 12.1.

Ans.
$$\Phi = \frac{60\pi a^2 I^2 J_1^2(\beta a\sin\theta)\sin^2(\frac{1}{2}\beta l\cos\theta)}{l^2\cos^2\theta}\cdot$$

5.16–1 Using the result of Problem 5.10–1 and equation 5, obtain the power radiated by a broadside couplet of the broadcast type.

Ans.
$$80\pi^2\left[1 + \frac{3}{\beta^2 l^2}\left(\frac{\sin\beta l}{\beta l} - \cos\beta l\right)\right]\left(\frac{I\,ds}{\lambda}\right)^2\cdot$$

5.16–2 Consider two current elements on the z axis, distance l apart. Let their moments be $I_1\,dz_1$ and $I_2\,dz_2$. Assuming that I_1 and I_2 are in phase, calculate the radiation intensity; also the radiated power by integrating equation 5. Note that this gives the mutual radiated power and the mutual radiation resistance.

Ans.

$$\Phi = \frac{15\pi}{\lambda^2}\,[(I_1\,dz_1)^2 + 2I_1I_2\,dz_1\,dz_2\cos(\beta l\cos\theta) + (I_2\,dz_2)^2]\sin^2\theta,$$

$$P = 40\pi^2\left(\frac{I_1\,dz_1}{\lambda}\right)^2 + \frac{60I_1I_2\,dz_1\,dz_2}{l^2}\left(\frac{\sin\beta l}{\beta l} - \cos\beta l\right) + 40\pi^2\left(\frac{I_2\,dz_2}{\lambda}\right)^2.$$

5.16–3 In the preceding problem, assume that the moments are $I_1\,dz_1$ and $I_2\,e^{j\vartheta}\,dz_2$, where I_1 and I_2 are in phase. Find the radiation intensity, and prove that the mutual radiated power differs from the corresponding power in the preceding problem by a factor $\cos\vartheta$.

Ans. $$\Phi = \frac{15\pi}{\lambda^2}\,[(I_1\,dz_1)^2 + 2I_1I_2\,dz_1\,dz_2\cos(\beta l\cos\theta + \vartheta) + (I_2\,dz_2)^2]\sin^2\theta.$$

5.16–4 Calculate the mutual radiated power for two parallel current elements, distance l apart and perpendicular to the line joining their centers. Let their moments be $I_1\,ds_1$ and $I_2\,e^{j\vartheta}\,ds_2$, where I_1 and I_2 are in phase.

Ans. $$P_{12} = \frac{60\pi I_1I_2\,ds_1\,ds_2}{\lambda l}\left(\sin\beta l + \frac{\cos\beta l}{\beta l} - \frac{\sin\beta l}{\beta^2l^2}\right)\cos\vartheta.$$

5.16–5 Using the result of the preceding problem calculate the power radiated by a broadside array of the point-to-point type consisting of three elements of moment $I\,ds$, one-half wavelength apart.

Ans. $$P = 3P_{11} + 2P_{12} + F_{13} = 10(12\pi^2 - 21)\left(\frac{I\,ds}{\lambda}\right)^2.$$

5.16–6 Solve the preceding problem for the case of end-fire operation.

Ans. $$P = 10(12\pi^2 + 27)\left(\frac{I\,ds}{\lambda}\right)^2.$$

5.16–7 Calculate the power radiated by three equal current elements perpendicular to a given plane and located at the vertices of an equilateral triangle whose sides are of length $\lambda/2$.

Ans. $$P = 120(\pi^2 - 3)\left(\frac{I\,ds}{\lambda}\right)^2.$$

5.16–8 Calculate the power radiated by a uniform current filament of length l. See equations 6–52 for the definitions of sine and cosine integrals.

Ans. $$\frac{60\pi l}{\lambda}\left(\mathrm{Si}\,\beta l + \frac{\cos\beta l - 2}{\beta l} + \frac{\sin\beta l}{\beta^2l^2}\right)I_0^2.$$

5.16–9 Calculate the power radiated by the filament in Problem 5.14–4.

Ans. $$15(\log 2\pi + C - \mathrm{Ci}\, 2\pi)I_0^2 = 36.56 I_0^2.$$

5.16–10 Show that, as βl approaches zero, the power radiated by the two elements in Problem 5.16–2 approaches

$$P = 40\pi^2 \left(\frac{I_1\, dz_1 + I_2\, dz_2}{\lambda}\right)^2.$$

Hence, for any current distribution $I(z)$ between $z = -l$ and $z = l$ in which the current elements are in phase and $2l/\lambda$ is small, the approximate radiated power is

$$P = \frac{40\pi^2}{\lambda^2}\left[\int_{-l}^{l} I(z)\, dz\right]^2.$$

5.16–11 Show that, if the approximate formula derived in the preceding problem is applied in Problem 5.16–9, the answer will be $P = 40 I_0^2$.

5.17–1 Calculate the approximate power radiated by a long, uniform current filament, and compare it with the exact expression given in Problem 5.16–8.

Ans. $$\frac{30\pi^2 l}{\lambda} I_0^2.$$

5.20–1 Give the mutual radiation resistance of two colinear elements separated by the following distances: (1) $\lambda/4$, (2) $\lambda/2$, (3) $3\lambda/4$, (4) λ.

Ans. (1) $\dfrac{1920}{\pi}\dfrac{ds_1\, ds_2}{\lambda^2}$, (2) $240\dfrac{ds_1\, ds_2}{\lambda^2}$, (3) $-\dfrac{640}{9\pi}\dfrac{ds_1\, ds_2}{\lambda^2}$,

(4) $-60\dfrac{ds_1\, ds_2}{\lambda^2}$.

5.20–2 Give the mutual radiation resistances of two parallel elements with a common equatorial plane (Fig. 5.17*b*) when the distances between the elements are: (1) $\lambda/4$, (2) $\lambda/2$, (3) $3\lambda/4$, (4) λ.

Ans. (1) $240\pi\left(1 - \dfrac{4}{\pi^2}\right)\dfrac{ds_1\, ds_2}{\lambda^2}$, (2) $-120\dfrac{ds_1\, ds_2}{\lambda^2}$,

(3) $-80\pi\left(1 - \dfrac{4}{9\pi^2}\right)\dfrac{ds_1\, ds_2}{\lambda^2}$, (4) $30\dfrac{ds_1\, ds_2}{\lambda^2}$.

5.20–3 Find the power radiated by an end-fire couplet of two elements one-quarter wavelength apart.

Ans. $$80\pi^2\left(\frac{I\, ds}{\lambda}\right)^2.$$

5.20–4 Find the power radiated by a broadside couplet of the broadcast type by the mutual resistance method, and thus check the answer to Problem 5.16–1.

5.20–5 Find the power radiated by a broadside couplet of the point-to-point type.

Ans. $$80\pi^2\left[1 + \frac{3}{2\beta l}\left(\sin\beta l + \frac{\cos\beta l}{\beta l} - \frac{\sin\beta l}{\beta^2 l^2}\right)\right]\left(\frac{I\, ds}{\lambda}\right)^2.$$

5.21–1 Obtain the influence coefficients of two colinear elements separated by the following distances: (1) $\lambda/4$, (2) $\lambda/2$, (3) $3\lambda/4$, (4) λ.

Ans. (1) $1920/\pi$, (2) 240, (3) $-640/9\pi$, (4) -60.

5.21–2 Obtain the influence coefficients of two parallel elements with a common equatorial plane when the distances between the elements are: (1) $\lambda/4$, (2) $\lambda/2$, (3) $3\lambda/4$, (4) λ.

Ans. (1) $240\pi\left(1 - \frac{4}{\pi^2}\right)$, (2) -120, (3) $-80\pi\left(1 - \frac{4}{9\pi^2}\right)$, (4) 30.

5.21–3 Estimate by the method of moments the power radiated by a current filament extending from $z = -\lambda/2$ to $z = \lambda/2$ when $I(z) = I_0 \sin \beta|z|$.

Ans.
$$80\left(1 + \frac{3}{\pi^2}\right) I_0^2 \simeq 104 I_0^2.$$

5.21–4 Solve the preceding problem for the case in which $I(z) = I_0 \sin \beta z$.

Ans.
$$80\left(1 - \frac{3}{\pi^2}\right) I_0^2 \simeq 56 I_0^2.$$

5.23–1 Consider two receivers at points whose Cartesian coordinates are $(0, 0, 0)$ and $(l, 0, 0)$. Let their receiving patterns be identical. Find the space factor of the array, assuming that the outputs of the receivers are equal and combined in phase. Compare with Problem 5.8–1, and note that the radiating and receiving patterns are identical.

5.23–2 Solve the preceding problem on the assumption that the outputs are equal but that in combining them the output of the receiver at $(l, 0, 0)$ is delayed by ϑ radians.

Ans.
$$S = 2 \cos\left(\tfrac{1}{2}\beta l \sin\theta \cos\varphi - \tfrac{1}{2}\vartheta\right).$$

5.23–3 Solve the preceding problem on the assumption that the voltage output of the receiver at $(0, 0, 0)$ has been doubled before being combined with the output of the other receiver.

Ans.
$$S^2 = 5 + 4 \cos(\beta l \sin\theta \cos\varphi - \vartheta).$$

5.23–4 Restate Problem 5.8–5 for reception, and show by direct calculation that the same space factor is obtained.

6

DIRECTIVITY AND EFFECTIVE AREA

6.1 Directivity of a transmitting antenna

The radiation pattern of an antenna shows the relative power radiated in different directions. When the polarization of the radiated wave is important, it is necessary to obtain separate patterns for the θ and φ components of E.

In comparing two antennas, it is convenient to supplement the information given by their radiation patterns by stating the maximum possible power gain due to their directive properties. This gain of one antenna over another is defined as the power ratio corresponding to equal maximum radiation intensities; that is,

$$g_{21} = \frac{P_1}{P_2} = \frac{\iint \Phi_1 \, d\Omega}{\iint \Phi_2 \, d\Omega}, \qquad \Phi_{1,\max} = \Phi_{2,\max}, \tag{1}$$

is the directive gain of the second antenna over the first. It is convenient to refer all antennas to one simple standard. The simplest standard is an isotropic radiator. The directive gain g with respect to the isotropic radiator is called the *directivity of the antenna*; thus,

$$g = \frac{P_0}{P} = \frac{\iint \Phi_0 \, d\Omega}{\iint \Phi \, d\Omega} = \frac{4\pi\Phi_0}{\iint \Phi \, d\Omega}, \qquad \Phi_0 = \Phi_{\max}. \tag{2}$$

It is often convenient to choose Φ so that $\Phi_{\max} = 1$, when equation 2 becomes

$$g = \frac{4\pi}{P} = \frac{4\pi}{\iint \Phi \, d\Omega}. \tag{2$'$}$$

From this equation it is evident that the *directivity is equal to the ratio of the maximum radiation intensity to the average radiation intensity.*

For example, the radiation intensity of a current element is proportional to $\sin^2\theta$; hence,

$$g = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\varphi} = \frac{3}{2}. \tag{3}$$

If g_{AB} is the directive gain of antenna A with respect to antenna B, and g_{BC} the directive gain of antenna B with respect to antenna C, then,

$$g_{AB}\, g_{BC} = g_{AC}, \tag{4}$$

where g_{AC} is the directive gain of A with respect to C.

The directivity of an antenna and the directive gain of one antenna with respect to another are often expressed in logarithmic units, particularly in decibels; thus,

$$G = 10 \log_{10} g, \qquad G_{AB} = 10 \log_{10} g_{AB}, \tag{5}$$

and

$$G_{AB} + G_{BC} = G_{AC}.$$

Applying equations 2 and 4 to equation 1, we obtain

$$g_{21} = \frac{\Phi_{2,\max}\iint \Phi_1 \, d\Omega}{\Phi_{1,\max}\iint \Phi_2 \, d\Omega} = \frac{P_1\Phi_{2,\max}}{P_2\Phi_{1,\max}}. \tag{6}$$

If $\Phi_{1,\max} = \Phi_{2,\max}$, we obtain equation 1; and, if $P_1 = P_2$, then, $g_{21} = \Phi_{2,\max}/\Phi_{1,\max}$. The ratio of the radiation intensities equals the ratio of the squares of the field intensities (either E or H).

6.2 Efficiency

In the preceding section we have assumed that the antennas are lossless and that no power is dissipated in heat in the conductors and dielectrics involved in their construction. Heat losses introduce efficiency considerations in antenna design. Thus, current elements of different lengths are equally effective radiators and receivers of power if they are lossless, but not otherwise. If the ohmic resistance of the element is R_0, the total power delivered to the element acting as a transmitting antenna is

$$P = \tfrac{1}{2}(R_0 + R_{\text{rad}})I^2. \tag{7}$$

The *efficiency* of this antenna is then

$$\frac{P_{\text{rad}}}{P} = \frac{R_{\text{rad}}}{R_0 + R_{\text{rad}}}. \tag{8}$$

Since R_0 is proportional to the length of the element while R_{rad} is proportional to the square of the length, the shorter elements are not as efficient as the longer ones. In fact, really short elements are exceedingly inefficient.

6.3 Power gain

The total gain or power gain of antenna 2 over antenna 1 is defined as the ratio

$$g_{21} = \frac{P_1}{P_2}, \qquad \Phi_{1,\max} = \Phi_{2,\max}, \tag{9}$$

where P_1 is the power delivered to antenna 1 and P_2 is the power delivered to antenna 2. The *absolute power gain* of a given antenna is taken with reference to a nondissipative isotropic radiator; that is,

$$g = \frac{P_0}{P} = \frac{4\pi\Phi_{\max}}{P_d + \iint \Phi \, d\Omega}, \tag{10}$$

where P_d represents the heat loss in the given antenna.

6.4 Effective area of a receiving antenna

The *effective area of a receiving antenna* is the maximum power that can be received at the terminals of the antenna from a linearly polarized wave* divided by the power per unit area carried by the wave; that is,

$$A_{\text{eff}} = \frac{P_{\text{max rec}}}{E^2/240\pi} = \frac{240\pi P_{\text{max rec}}}{E^2}. \tag{11}$$

When the heat loss is neglected, we have the " directivity area " of the antenna.

For example, let us calculate the effective area of a nondissipative current element of length s: that is, its directivity area. The voltage impressed on the element is maximum when the element is parallel to the electric vector; this voltage is $V = Es$. Let Z_{rad} be the radiation impedance of the element, that is, the impedance due to the reaction of the wave produced by the current I in the element; let Z_{load} be the impedance of the circuit connected in series with the element; then,

$$V = Es = (Z_{\text{rad}} + Z_{\text{load}})I, \qquad I = \frac{Es}{Z_{\text{rad}} + Z_{\text{load}}}. \tag{12}$$

* In which the electric vector at a typical point P is contained in a fixed straight line passing through P.

The received power is then

$$P = \tfrac{1}{2}R_{\text{load}}|I|^2 = \frac{R_{\text{load}}|Es|^2}{2|Z_{\text{rad}} + Z_{\text{load}}|^2}$$

$$= \frac{R_{\text{load}}E^2s^2}{2(R_{\text{rad}} + R_{\text{load}})^2 + 2(X_{\text{rad}} + X_{\text{load}})^2} \cdot \qquad (13)$$

This is maximum if

$$X_{\text{rad}} + X_{\text{load}} = 0, \qquad R_{\text{load}} = R_{\text{rad}}; \qquad (14)$$

then, from equations 13 and 5–66,

$$P_{\text{max}} = \frac{E^2s^2}{8R_{\text{rad}}} = \frac{E^2s^2}{640\pi^2s^2/\lambda^2} = \frac{E^2\lambda^2}{640\pi^2} \cdot \qquad (15)$$

Thus the maximum received power is independent of the length of the element. Substituting from equation 15 in equation 11, we find the *effective area of the nondissipative element,*

$$A = \frac{3\lambda^2}{8\pi} \cdot \qquad (16)$$

This is approximately the area of a square whose diagonal is $\lambda/2$, or the area of a rectangle whose sides are $\lambda/2$ and $\lambda/4$.

The heat loss affects the efficiency of a receiving antenna and reduces its effective area. Expression 13 for the power received by the current element, assuming that its reactance is tuned out, becomes

$$P = \frac{R_{\text{load}}E^2s^2}{2(R_{\text{rad}} + R_0 + R_{\text{load}})^2}, \qquad (17)$$

where R_0 is the ohmic resistance of the element. The maximum power is delivered to R_{load} when

$$R_{\text{load}} = R_{\text{rad}} + R_0, \qquad (18)$$

and this power is

$$P = \frac{E^2s^2}{8(R_{\text{rad}} + R_0)} \cdot \qquad (19)$$

Hence, the effective area of a dissipative current element is

$$A = \frac{3\lambda^2}{8\pi}\,\frac{R_{\text{rad}}}{R_{\text{rad}} + R_0} \cdot \qquad (20)$$

If the efficiency of the receiving antenna is defined as the ratio of the power actually delivered to the load to that which could be delivered in the absence of heat loss, then the efficiency of the current element used as a receiver is the same as its efficiency when used as a transmitter.

6.5 Free-space transmission formulas

Consider two antennas, distance r apart (Fig. 6.1). Let the receiving antenna 2 be oriented in the direction of maximum radiation from the transmitting antenna 1. Assume that the wave generated by the transmitting antenna is linearly polarized at the place occupied by the receiving antenna, and assume that the latter is properly oriented for

FIG. 6.1 Two antennas in free space.

maximum reception of power. To find the ratio of the received power P_{rec} to the power P_{tr} radiated by the transmitting antenna, we reason as follows: (1) If P_{tr} is radiated uniformly in all directions, then the power flowing per unit area at distance r is $P_{tr}/4\pi r^2$; (2) if the power gain of the transmitting antenna is g_1, the power flow at the receiving antenna is increased by a factor g_1; therefore, if the effective area of the receiving antenna is A_2, we have

$$P_{rec} = \frac{P_{tr}}{4\pi r^2} g_1 A_2, \qquad \frac{P_{rec}}{P_{tr}} = \frac{g_1 A_2}{4\pi r^2}. \tag{21}$$

For example, for two nondissipative current elements we have $g_1 = \frac{3}{2}$, $A_2 = 3\lambda^2/8\pi$, and

$$\frac{P_{rec}}{P_{tr}} = \frac{9\lambda^2}{64\pi^2 r^2}. \tag{22}$$

If the receiving antenna is oriented to receive maximum power when the electric vector is in the direction n, different from that of E, then equation 21 becomes

$$\frac{P_{rec}}{P_{tr}} = \frac{g_1 A_2}{4\pi r^2} \cos^2(E, n), \tag{23}$$

where (E, n) is the angle between the two directions.

Similarly, there will be a reduction in the power ratio, represented by a factor Φ/Φ_{max}, when the receiving antenna is not situated in the direction of maximum radiation from the transmitting antenna. When referring to gain or effective area in "a given direction," this factor is included in g or A.

Likewise, if the transmitting antenna generates elliptically polarized waves,* while the receiving antenna is not designed to receive these waves most effectively, there will be a polarization loss factor. For instance, consider two mutually perpendicular equal elements so interconnected that they operate in quadrature (Fig. 6.2). If this antenna is used as a transmitting antenna, its directivity is the same as that of a single element. This occurs because the power radiated by these elements may be calculated as if they were radiating independently (Section 5.20). The distant fields of the elements are in quadrature. Hence, both the radiated power and the maximum radiation intensity are twice as large as they are for a single element. If a single element is used as a receiving antenna (Fig. 6.2), it will receive power only from that component of the total which is parallel to the element. Thus it can receive only one half of the total power that would be available to an antenna designed to receive both polarizations equally effectively.†

FIG. 6.2 A transmitting antenna consisting of two mutually perpendicular elements operating in quadrature, and a single element operating as a receiving antenna.

6.6 Relation between directivity and effective area

If two antennas (Fig. 6.1) radiate linearly polarized waves in the direction (or directions) of most effective radiation, then, using antenna 2 as a transmitting antenna and 1 as a receiving antenna, we have

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{g_2 A_1}{4\pi r^2}. \tag{24}$$

By the reciprocity theorem this ratio must have the same value as in equation 21, and, consequently,

$$g_1 A_2 = g_2 A_1, \qquad \frac{A_2}{A_1} = \frac{g_2}{g_1}. \tag{25}$$

Thus, *the effective areas of two antennas are proportional to their directivities* (*or power gains in the dissipative case*).

Let one antenna be a current element and the other an isotropic radiator; then,

$$\tfrac{3}{2}A_0 = \frac{3\lambda^2}{8\pi}, \qquad A_0 = \frac{\lambda^2}{4\pi}, \tag{26}$$

where A_0 is the effective area of the isotropic radiator.

* That is, waves in which during each cycle the end of the electric vector describes an ellipse.

† Elliptically polarized waves will be considered in more detail in Sections 12.13 and 12.14.

Applying equations 25 to any antenna emitting linearly polarized waves in the direction of maximum radiation in combination with an isotropic radiator, we have the following basic relationship between the effective area and the gain of an antenna,

$$A = \frac{g\lambda^2}{4\pi}, \qquad g = \frac{4\pi A}{\lambda^2}. \tag{27}$$

6.7 Supplementary free-space transmission formulas

Using equation 27, we can express the power ratio (equation 21) either in terms of the directivities of the two antennas or in terms of their effective areas; thus,

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{g_1 A_2}{4\pi r^2} = g_1 g_2 \left(\frac{\lambda}{4\pi r}\right)^2 = \frac{A_1 A_2}{\lambda^2 r^2}. \tag{28}$$

The last expression is particularly convenient when the effective areas of the transmitter *and* the receiver are independent of the wavelength (as in the case of large horns). The next to the last is best when the directivities are independent of the wavelength (as in the case of half-wave antennas). Finally, the first expression is best when the directivity of the transmitter and the effective area of the receiver are independent of the wavelength.

6.8 Directivity of an end-fire couplet

In the end-fire couplet of current elements, the elements are operated in quadrature. We have shown (Section 5.20) that the mutual radiation resistance of such elements is equal to zero; hence, the power P radiated by the couplet is double that radiated by one element. From equation 5–67 we have

$$P = 80\pi^2 \left(\frac{I\,ds}{\lambda}\right)^2. \tag{29}$$

The maximum radiation intensity of the couplet is obtained from equation 5–27,

$$\Phi_{\max} = 60\pi \left(\frac{I\,ds}{\lambda}\right)^2. \tag{30}$$

This becomes unity if

$$\left(\frac{I\,ds}{\lambda}\right)^2 = \frac{1}{60\pi};$$

then,

$$P = \frac{4\pi}{3}.$$

From equation 2 we now find*

$$g = \frac{4\pi}{P} = 3, \tag{31}$$

and from equation 27,

$$A = \frac{3\lambda^2}{4\pi} \cdot$$

6.9 Directivity of a broadside couplet

In the broadcast type of broadside couplet, the current elements are coaxial (Fig. 5.9a). If the elements are a half wavelength apart, the mutual radiation resistance is obtained by assuming $\beta r = \pi$ in equation 5–77; thus,

$$R_{12} = \frac{240\ ds^2}{\lambda^2} \cdot \tag{32}$$

From equation 5–66 we have

$$R_{11} = R_{22} = \frac{80\pi^2\ ds^2}{\lambda^2};$$

from equation 5–76,

$$P = 80\pi^2 \left(\frac{I\ ds}{\lambda}\right)^2 + 240 \left(\frac{I\ ds}{\lambda}\right)^2;$$

from equation 5–36,

$$\Phi_{\max} = 60\pi \left(\frac{I\ ds}{\lambda}\right)^2 = 1, \qquad \left(\frac{I\ ds}{\lambda}\right)^2 = \frac{1}{60\pi}; \tag{33}$$

therefore,

$$g = \frac{3\pi^2}{\pi^2 + 3} = 2.30\cdots, \qquad A = \frac{3\pi\lambda^2}{4(\pi^2+3)} = 0.183\lambda^2. \tag{34}$$

In a point-to-point type of broadside couplet (Fig. 5.9b), the current elements are parallel to each other and perpendicular to the line joining their centers. The radiation intensity is given by equation 5–37 and the mutual radiation resistance by equation 5–78. Thus, for a half-wavelength separation, we find

$$g = \frac{6\pi^2}{2\pi^2 - 3} = 3.54, \qquad A = \frac{3\pi\lambda^2}{2(2\pi^2-3)} = 0.281\lambda^2. \tag{35}$$

* If the data used in this section are not readily available, it is best to use equation 2′ directly. First we obtain Φ as in equation 5–27. We may drop the constant coefficient as it does not affect g; then $\Phi_{\max} = 1$, and g is 4π divided by the integral of Φ.

6.10 A vertical element above perfect ground

To find the effect of the ground on the field of a vertical current element (Fig. 6.3), we introduce its image. The maximum radiation intensity is independent of the height h above the ground and is equal to unity when equation 33 is satisfied. Substituting $r = 2h$ in equation 5–77 and using equation 5–76, we obtain

$$P = 80\pi^2 \left(\frac{I\,ds}{\lambda}\right)^2 + 15\left(\frac{I\,ds}{\lambda}\right)^2 \frac{\lambda^2}{h^2}\left(\frac{\sin 2\beta h}{2\beta h} - \cos 2\beta h\right).$$

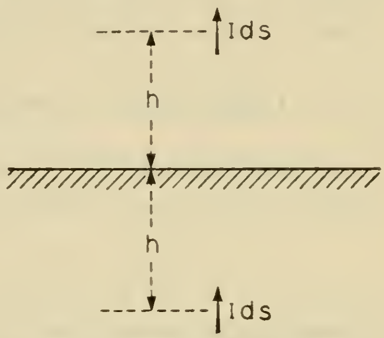

Fig. 6.3 A vertical current element above a perfect ground and its image.

Hence, the directivity of the element and its image is

$$g = 3\left[1 + \frac{3}{4\beta^2h^2}\left(\frac{\sin 2\beta h}{2\beta h} - \cos 2\beta h\right)\right]^{-1}. \tag{36}$$

This represents the increase in the radiation intensity due to the presence of the ground; Fig. 6.4 shows its variation with height as a function of height in wavelengths.

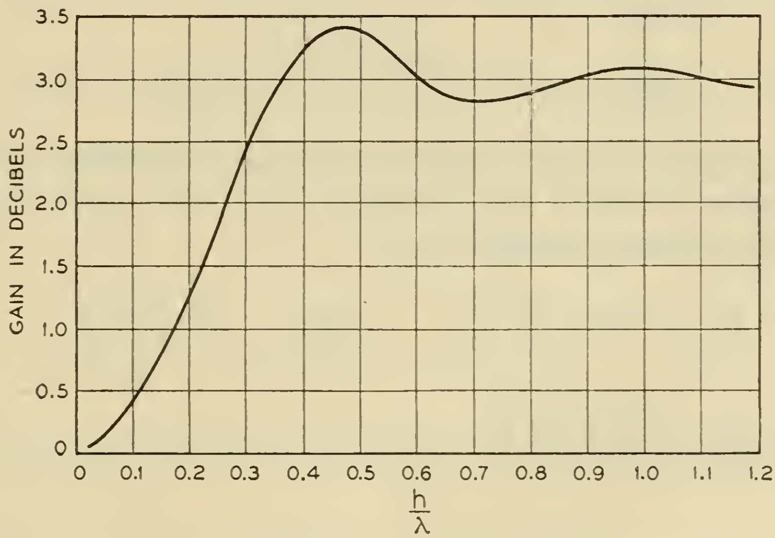

Fig. 6.4 The increase in the directivity of a vertical current element and its image as a function of the height in wavelengths.

If now we reintroduce the ground, the radiated power will be halved while the radiation intensity will remain unchanged. Thus we shall have an added over-all gain of 3 db.

We may also say that, due to the reinforcement of the direct ray by the ground reflected ray, the electric intensity in a direction parallel to the ground is doubled; hence the radiation intensity is quadrupled. High above the ground the radiated power will be the same as in free space; hence, we have a gain of 6 db due to the presence of the ground.

6.11 Directivity of broadside arrays

In a broadside array of n current elements, of either broadcast type or point-to-point type, the radiated power is

$$P = [\tfrac{1}{2}nR_{11} + (n-1)R_{12} + (n-2)R_{13} + \cdots + R_{1n}]I^2, \quad (37)$$

where R_{12} is the mutual radiation resistance between adjacent elements, R_{13} is the mutual radiation resistance between the mth and the $(m+2)$th elements, etc. For either type, the maximum electric intensity is, by equation 5–1,

$$|E_\theta| = \frac{60\pi nI\,ds}{\lambda r}. \quad (38)$$

Substituting in equation 5–10, we find

$$\Phi_{\max} = 15\pi\left(\frac{n\,ds}{\lambda}\right)^2 I^2. \quad (39)$$

Therefore, by equation 2,

$$g = \frac{4\pi\Phi_{\max}}{P} = \frac{60\pi^2(n\,ds/\lambda)^2}{\frac{1}{2}nR_{11} + (n-1)R_{12} + (n-2)R_{13} + \cdots + R_{1n}}. \quad (40)$$

The mutual radiation resistances for the broadcast type of the broadside array are found from equation 5–77; thus, when the distance between the successive elements is $\lambda/2$,

$$R_{12} = 240\left(\frac{ds}{\lambda}\right)^2, \qquad R_{13} = -60\left(\frac{ds}{\lambda}\right)^2,$$

$$R_{14} = \frac{80}{3}\left(\frac{ds}{\lambda}\right)^2, \cdots. \quad (41)$$

Hence,

$$g = \frac{1.5n}{1+\frac{6}{\pi^2}\left(1-\frac{1}{n}\right)-\frac{6}{4\pi^2}\left(1-\frac{2}{n}\right)+\frac{6}{9\pi^2}\left(1-\frac{3}{n}\right)-\cdots\pm\frac{6}{\pi^2(n-1)^2n}}. \quad (42)$$

If n is large, g is proportional to n.

Similarly, for the point-to-point type of the broadside array, we find

$$R_{1m} = (-)^{m-1} \frac{120(ds/\lambda)^2}{(m-1)^2}, \qquad m \neq 1; \tag{43}$$

$$g = 1.5n\left[1 + \sum_{m=2}^{n} (-)^{m-1} \frac{3}{(m-1)^2\pi^2}\left(1 - \frac{m-1}{n}\right)\right]^{-1}. \tag{44}$$

For the corresponding array of end-fire couplets, g will be nearly doubled.

6.12 Directivity of end-fire arrays

The maximum radiation intensity of an end-fire array of n current elements is still given by equation 39; but we must include the effect of relative phases as shown in equation 5–76. If the successive elements are a quarter wavelength apart, then,

$$P = \tfrac{1}{2}nR_{11}I^2 + \sum_{m=2}^{n} (n - m + 1)R_{1m} \cos \frac{(m-1)\pi}{2} I^2.$$

The cosine factor vanishes when m is even and is equal to $(-)^k$ when $m = 2k + 1$; thus,

$$P = \tfrac{1}{2}nR_{11}I^2 + \sum_{k=1}^{k\leq(n-1)/2} (-)^k(n - 2k)R_{1,2k+1}I^2. \tag{45}$$

Consequently, using equation 39 for the value of $\Phi_{\max}$,

$$g = 1.5n\left[1 + \sum_{k=1}^{k\leq(n-1)/2} \frac{3}{k^2\pi^2}\left(1 - \frac{2k}{n}\right)\right]^{-1}. \tag{46}$$

As n increases, g approaches asymptotically the value n,

$$g \sim 1.5n\left[1 + \frac{3}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)\right]^{-1}$$

$$= 1.5n\left[1 + \frac{3}{\pi^2}\,\frac{\pi^2}{6}\right]^{-1} = n. \tag{47}$$

By the same method, we find that g in equation 42 tends to n and in equation 44 to $2n$.

6.13 Directivity of continuous end-fire arrays

When an array is continuous, it may be more convenient to obtain the radiated power by integrating the radiation intensity (see equation 5–5), rather than by integrating the mutual power radiated by two typical differential elements of the array (see equation 5–84). Consider, for instance, a continuous distribution along the z axis of current elements

parallel to the x axis. If the array is to radiate maximum power in the positive z direction, the progressive phase delay along the array must be at least equal to the phase delay of waves in free space. Assuming this condition, we obtain the space factor by letting $k = \beta = 2\pi/\lambda$ in equation 5–41; thus, the radiation intensity of the end-fire continuous array of length l is

$$\Phi = \frac{\sin^2\left(\beta l \sin^2 \dfrac{\theta}{2}\right)}{\beta^2 l^2 \sin^4 \dfrac{\theta}{2}} (1 - \sin^2\theta \cos^2\varphi). \tag{48}$$

A numerical factor has been inserted to make* $\Phi_{max} = 1$. Substituting in equation 5–5 and noting that $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, we have, after integrating with respect to φ,

$$P = \frac{4\pi}{\beta^2 l^2}\left[\int_0^\pi \frac{\sin^2\left(\beta l \sin^2 \dfrac{\theta}{2}\right)}{\sin^4 \dfrac{\theta}{2}}\, d\left(\sin^2\frac{\theta}{2}\right) - 2\int_0^\pi \frac{\sin^2\left(\beta l \sin^2 \dfrac{\theta}{2}\right)}{\sin^2 \dfrac{\theta}{2}} \cos^2\frac{\theta}{2}\, d\left(\sin^2\frac{\theta}{2}\right)\right]. \tag{49}$$

To evaluate this integral, introduce a new variable of integration,

$$t = \beta l \sin^2 \tfrac{1}{2}\theta; \tag{50}$$

thus,

$$P = \frac{4\pi}{\beta l}\int_0^{\beta l} \frac{\sin^2 t}{t^2}\, dt - \frac{8\pi}{\beta^2 l^2}\int_0^{\beta l} \frac{\sin^2 t}{t}\, dt + \frac{8\pi}{\beta^3 l^3}\int_0^{\beta l} \sin^2 t\, dt. \tag{51}$$

The first and second integrals cannot be expressed in terms of "elementary functions"; but they and many other integrals occurring in radiation theory may be expressed in terms of functions known as *sine and cosine integrals* and *defined* as follows:†

$$\mathrm{Si}\, x = \int_0^x \frac{\sin t}{t}\, dt, \qquad \mathrm{Ci}\, x = \int_\infty^x \frac{\cos t}{t}\, dt,$$

$$\mathrm{Cin}\, x = \int_0^x \frac{1 - \cos t}{t}\, dt. \tag{52}$$

* Note that, as x approaches zero, $\sin x$ approaches x.

† *Applied Mathematics*, Chapter 18.

For instance, the first integral in equation 51 may be integrated by parts,

$$\int_0^{\beta l} \frac{\sin^2 t}{t^2}\,dt = -\int_0^{\beta l} \sin^2 t \, d\left(\frac{1}{t}\right) = -\left.\frac{\sin^2 t}{t}\right|_0^{\beta l} + \int_0^{\beta l} \frac{\sin 2t}{t}\,dt$$

$$= -\frac{\sin^2 \beta l}{\beta l} + \mathrm{Si}\, 2\beta l. \tag{53}$$

For the second integral, we have

$$\int_0^{\beta l} \frac{\sin^2 t}{t}\,dt = \frac{1}{2}\int_0^{\beta l} \frac{1-\cos 2t}{t}\,dt = \frac{1}{2}\int_0^{\beta l} \frac{1-\cos 2t}{2t}\,d(2t)$$

$$= \tfrac{1}{2}\,\mathrm{Cin}\, 2\beta l. \tag{54}$$

Therefore,

$$P = \frac{4\pi}{\beta l}\left(\mathrm{Si}\, 2\beta l + \frac{\cos^2 \beta l - \mathrm{Cin}\, 2\beta l}{\beta l} - \frac{\sin 2\beta l}{2\beta^2 l^2}\right). \tag{55}$$

Since $\Phi_{\max} = 1$, the directivity is $g = 4\pi/P$. As x increases, $\mathrm{Si}\, x$ approaches $\pi/2$. Hence, for a long array,

$$g = \frac{2\beta l}{\pi} = \frac{4l}{\lambda}, \qquad G = 10 \log_{10} \frac{l}{\lambda} + 6 \text{ db}. \tag{56}$$

6.14 Directivity of continuous broadside arrays

A uniform current filament of length l along the z axis constitutes a broadside array of the broadcast type. For this array, we find

$$\Phi = \frac{4 \sin^2(\frac{1}{2}\beta l \cos\theta)}{\beta^2 l^2 \cos^2\theta} \sin^2\theta,$$

$$g = \frac{1}{2}\left(\frac{\mathrm{Si}\,\beta l}{\beta l} + \frac{\cos \beta l - 2}{\beta^2 l^2} + \frac{\sin \beta l}{\beta^3 l^3}\right)^{-1}, \tag{57}$$

$$g \to \frac{2l}{\lambda}, \qquad G \to 10 \log_{10} \frac{l}{\lambda} + 3 \text{ db}, \quad \text{as} \quad \frac{l}{\lambda} \to \infty.$$

If the current elements are parallel to the x axis, we have a point-to-point broadside array. Then the space factor is the same as in the preceding case, but the pattern of the element has a different orientation, and

$$\Phi = \frac{4 \sin^2(\frac{1}{2}\beta l \cos\theta)}{\beta^2 l^2 \cos^2\theta} (1 - \sin^2\theta \cos^2\varphi),$$

$$g = \left(\frac{\mathrm{Si}\,\beta l}{\beta l} + \frac{\cos \beta l}{\beta^2 l^2} - \frac{\sin \beta l}{\beta^3 l^3}\right)^{-1}, \tag{58}$$

$$g \to \frac{4l}{\lambda}, \qquad G \to 10 \log_{10} \frac{l}{\lambda} + 6 \text{ db}.$$

6.15 Directivity of continuous rectangular broadside arrays

A uniform rectangular current sheet (Fig. 5.13) constitutes a broadside array whose greatest radiation takes place in the two directions normal to the sheet. The radiation intensity is given by equation 5–52, and, when $\theta = 0$,

$$\Phi = \Phi_{\max} = \frac{15\pi a^2 I^2}{\lambda^2}, \tag{59}$$

where I is the total current. In Section 5.17 we obtained the power radiated by a large current sheet (see equation 5–59). For the directivity and the effective area, we thus find

$$g = \frac{2\pi ab}{\lambda^2}, \qquad A = \tfrac{1}{2}ab. \tag{60}$$

If the current sheet is backed by a reflector which throws the radiation forward, we find that, for the same current in the sheet, Φ is four times as large as the value given by equation 59. On the other hand, P is only twice as large as the value used in equations 60, for there is only one major lobe. Hence,

$$g = \frac{4\pi ab}{\lambda^2}, \qquad G = 10 \log_{10} \frac{ab}{\lambda^2} + 10.99 \text{ db}, \qquad A = ab, \tag{61}$$

and the effective area is equal to the actual area.

6.16 Radiation from progressive current waves on a wire

Suppose that we maintain an electric current from $z = 0$ to $z = l$ whose phase is proportional to z. If the phase delay is $\vartheta = kz$, the ratio of the phase velocity along the wire to that of waves in free space is β/k. Introducing this phase delay into the field (equation 5–1) of a typical current element, we find

$$E_\theta = j\,\frac{60\pi I}{\lambda r_0}\, e^{-j\beta r_0} \sin\theta \int_0^l e^{j(\beta\cos\theta - k)z}\, dz. \tag{62}$$

Integrating as in Section 5.14 and using equation 5–10, we have

$$\Phi = \frac{30\pi I^2[1 - \cos(\beta\cos\theta - k)l]}{\lambda^2(\beta\cos\theta - k)^2} \sin^2\theta. \tag{63}$$

To calculate the radiated power we introduce a new variable $t = (k - \beta\cos\theta)l$; thus, we obtain

$$P = \frac{30\pi l}{\lambda} I^2 \Bigg[\left(1 - \frac{k^2}{\beta^2}\right) \int_{(k-\beta)l}^{(k+\beta)l} \frac{1-\cos t}{t^2}\, dt + \\ \frac{2k}{\beta^2 l} \int_{(k-\beta)l}^{(k+\beta)l} \frac{1-\cos t}{t}\, dt - \frac{1}{\beta^2 l^2} \int_{(k-\beta)l}^{(k+\beta)l} (1-\cos t)\, dt\Bigg]. \tag{64}$$

These integrals can be readily expressed in terms of sine and cosine integrals. The particular case of great importance in connection with rhombic antennas is the case in which the velocity along the wire is equal to that of free-space waves, $k = \beta$. In this case,

$$P = 30I^2\left(\operatorname{Cin} 2\beta l - 1 + \frac{\sin 2\beta l}{2\beta l}\right). \tag{65}$$

The direction of maximum radiation is found by equating $d\Phi/d\theta$ to zero; thus we obtain the following equation:

$$\frac{\tan u}{u} = 2\left(1 - \frac{u}{\beta l}\right), \qquad \frac{l}{\lambda} = \frac{u}{\pi(1 - \cos\theta)}. \tag{66}$$

The greatest maximum corresponds to the smallest root. When the wire is long, this root is substantially independent of l, and it occurs for a value of u about halfway between 1.16 and 1.17. The maximum radiation intensity is unity if

$$I^2 = \frac{1.16\lambda}{30l \sin^2 1.16} = 0.046\,\frac{\lambda}{l}, \tag{67}$$

We finally obtain

$$G = 10 \log_{10} \frac{l}{\lambda} + 5.97 - 10 \log_{10}\left(\log_{10} \frac{l}{\lambda} + 0.915\right). \tag{68}$$

6.17 Directivity and the solid angle occupied by the major lobe

If the radiation were uniform within a given solid angle Ω and were absent in all other directions, the directivity would be given by a simple formula,

$$g = \frac{4\pi}{\Omega}. \tag{69}$$

The assumed ideal pattern is theoretically impossible, although it could be approximated as closely as desired. If Ω is the solid angle occupied by the major lobe in any actual radiation pattern, then we may write

$$g = k\,\frac{4\pi}{\Omega} = \frac{4\pi}{\Omega/k}, \tag{70}$$

where k depends primarily on the shape of the major lobe and to some extent on the minor lobes (assuming, of course, that the minor lobes are small). The coefficient k might be called the *form factor* of the radiation pattern.

For a long end-fire array the angle of the cone enclosing the major lobe is the smallest (nonzero) angle for which the numerator in equa-

tion 48 vanishes,

$$\beta l \left(\frac{\theta}{2}\right)^2 = \pi, \qquad \theta^2 = \frac{2\lambda}{l} \cdot \tag{71}$$

Hence, the solid angle of the major lobe is

$$\Omega = \int_0^{2\pi} \int_0^{\theta} \sin\theta \, d\theta \, d\varphi = 2\pi(1 - \cos\theta) \backsimeq \pi\theta^2 = \frac{2\pi\lambda}{l} \cdot \tag{72}$$

The directivity (equation 56) is thus equal to

$$g = \frac{8\pi}{\Omega} = \frac{4\pi}{\Omega/2} \cdot \tag{73}$$

Thus, the form factor of a long end-fire linear array is 2, and the effective solid angle is half the total solid angle occupied by the major lobe.

For a long broadside array of the broadcast type, the major lobe is contained between two cones whose angles with the z axis are θ and $\pi - \theta$, where θ is the angle, nearest to $\pi/2$, for which Φ in equation 57 vanishes; thus,

$$\tfrac{1}{2}\beta l \cos\theta = \pi, \qquad \cos\theta = \frac{\lambda}{l} \cdot \tag{74}$$

The solid angle occupied by this lobe is

$$\Omega = \int_0^{2\pi} \int_{\theta}^{\pi-\theta} \sin\theta \, d\theta \, d\varphi = 4\pi\cos\theta = \frac{4\pi\lambda}{l} \cdot \tag{75}$$

Since $g = 2l/\lambda$, we again have equation 73. The cross section of the radiation pattern of the end-fire array is blunt compared with the vertical cross section of the pattern of the broadside array; but in the former all cross sections are tapered, whereas in the latter the horizontal cross section is a circle. This accounts for the same over-all form factor.

In a point-to-point broadside array, the total solid angle occupied by the major lobe is the same as in the broadcast type of array, but the directivity is $g = 4l/\lambda$; hence,

$$g = \frac{16\pi}{\Omega} = \frac{4\pi}{\Omega/4}, \tag{76}$$

and the form factor is 4. Since the radiation intensity in the horizontal plane varies as $\sin^2\varphi$ and since the average value of this factor is 1/2, we might say that the effective solid angle is $\Omega/2$; then the equation would again be equation 73.

If we were to perform similar calculations for a two-dimensional broadside array composed of sources inside a given large circle, we should find

$$g = \frac{14.7\pi}{\Omega} = \frac{4\pi}{\Omega/3.7} \cdot \tag{77}$$

The form factor of two-dimensional arrays is thus larger, and their major lobes must be sharper.

6.18 Directivity and radiation resistance

The directivity of an electric current element is 1.5 regardless of its length (see equation 3); but the radiation resistance is proportional to the square of the length. Hence, there is no relation between the directivity of an antenna and its radiation resistance. On the other hand, there is a definite relationship between the efficiency and the power gain of a lossy antenna as we have found in Sections 2 and 3 of this chapter.

6.19 Voltage gain of an unmatched antenna

The voltage gain of an unmatched receiving antenna should not be confused with either its directive gain or its power gain. For example, if an antenna whose length is small compared with one-half wavelength is connected directly to the grid of a vacuum tube, the received voltage is directly proportional to the length of the antenna, while the directive gain remains substantially constant.

6.20 Superdirective antennas

How much directivity can be expected from an antenna of a given size? Is the directivity of an antenna array determined by the number of its elements or by its dimensions? At first it appears that the over-all dimensions are more important than the number of elements. For example, the directivity of a broadside array of the point-to-point type consisting of n current elements is given by equation 44 for the case in which the distance between the adjacent elements is $\lambda/2$. As n increases,

$$g \to 2n. \tag{78}$$

If l is the total length of this array, then $l = (n - 1)\lambda/2$, and

$$g \to \frac{4l}{\lambda} + 2. \tag{79}$$

On the other hand, for a long continuous array, that is, an array with an infinite number of elements, we have, from equation 56,

$$g \to \frac{4l}{\lambda}\cdot \tag{80}$$

Thus, the length in wavelengths appears to be the determining factor, although the directivity of the discrete array is somewhat larger. The effective area of a large uniform rectangular array which is forced to radiate from one face equals its actual area. Hence, the directivity is determined by the area in wavelengths.

Thus, the directivity can be increased by increasing the size of the antenna system. The size, however, is not essential. The directivity of a short antenna is only slightly smaller than the directivity of a half-wave antenna. Similarly, the directivity of a small loop is 1.5, irrespective of the area of the loop. An increase in directivity was obtained from two loops, only a fraction of a wavelength apart.* It is not difficult to prove that theoretically there is no limit to the directivity of an arbitrarily small antenna. The radiation pattern of a current element in the equatorial plane is a circle. Consider two equal parallel elements with a common equatorial plane. If the elements are operated 180° out of phase, the shape of the radiation pattern is that of the figure 8. Hence, the directivity of the pair should be greater than the directivity of a single element. Replace now each element by a pair of elements. The radiation intensity will be multiplied by another figure 8 factor, and the directivity of the system is thus increased. Using the new system for each element of the original array, we increase the directivity still further. In this way we construct systems of elements with the following current distributions:

$$\begin{aligned}
&(1, -1),\\
&(1, -1) \quad + (-1, 1) \qquad = (1, -2, 1),\\
&(1, -2, 1) + (-1, 2, -1) = (1, -3, 3, -1),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\left[1, -n, \frac{n(n-1)}{2!}, -\frac{n(n-1)(n-2)}{3!}, \cdots\right].
\end{aligned}$$

This is not the most effective method of increasing the directivity. By the method explained briefly in Section 5.24 and more fully elsewhere,† it is possible to obtain spectacular increases in directivity (Figs. 6.5 and 6.6). These increases, however, are accompanied by spectacular decreases in efficiency, due to large copper losses in actual antennas. If the power gain is an important consideration, little can be done to improve on the uniform arrays; but, if we are interested in directive discrimination without undue increase in the size of the antenna system, we have the answer in "superdirective antennas." It should also be noted that, the more superdirective the antenna, the narrower is its bandwidth. Also, a much greater precision will be required in the adjustments of the amplitudes and phases of the various elements. In

* H. T. Friis, A new directional antenna system, *IRE Proc.*, 13, December 1925, pp. 685–707.

† S. A. Schelkunoff, U.S. Patent 2,286,839. Also, A mathematical theory of linear arrays, *Bell Sys. Tech. Jour.*, 22, January 1943, pp. 80–107.

spite of these handicaps moderately superdirective arrays have already found applications (see ref. 6).

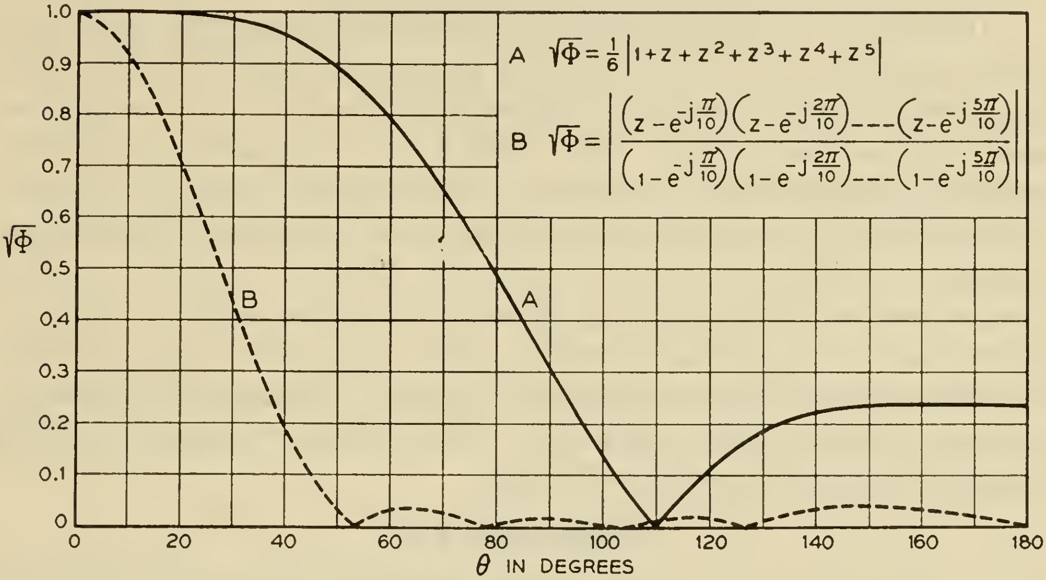

FIG. 6.5 Radiation patterns of end-fire arrays of six elements: curve A is for a uniform array and curve B for a superdirective array.

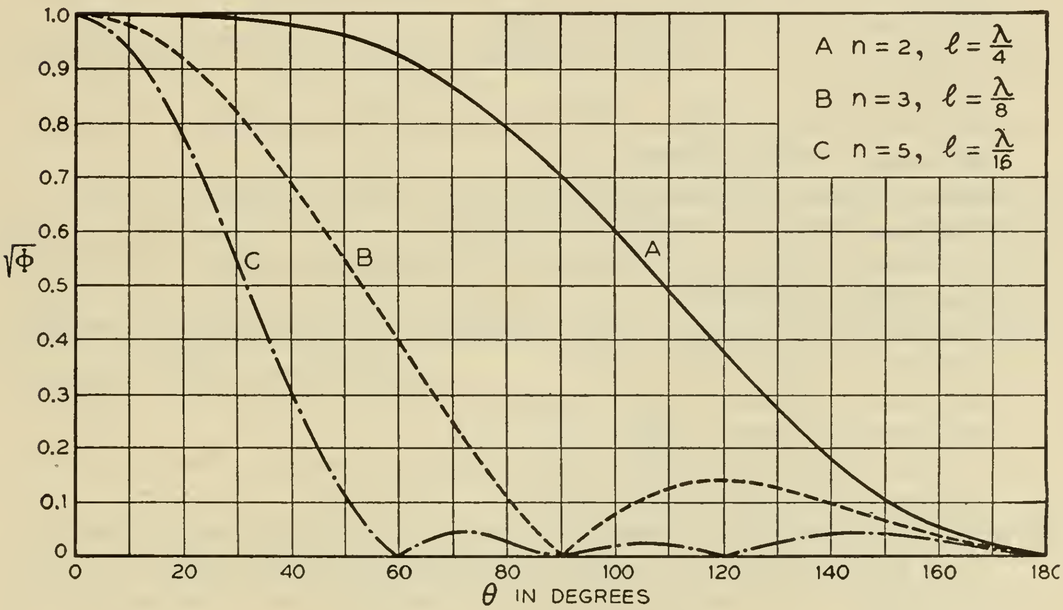

FIG. 6.6 Radiation patterns of three end-fire arrays of the same total length $(n - 1)l = \frac{1}{4}\lambda$ designed according to the principle of superdirectivity.

There is no upper limit to the directivity of superdirective antennas, and, therefore, there is no upper limit to their effective areas *in absence of heat loss.* This is a highly theoretical property since all practical antennas have internal resistances. Nevertheless, this is a property

requiring some explanation, for it implies that a perfectly conducting antenna of infinitesimal dimensions is capable of intercepting from a plane wave the amount of power passing through a very large area.

It is easy to understand that a large resistive sheet can absorb the power incident on it or that the power passing through the aperture of a large slowly tapered horn can eventually be absorbed in a resistance of small physical dimensions. But what is it that enables a perfectly conducting superdirective antenna to collect power from a vast area? The answer is found in the combination of resonance and low radiation resistance. These two factors enable the antenna to create a strong reactive field extending to large distances from the antenna which redirects the power passing through a large area of the incoming plane wave and forces it to flow toward the antenna. Detuned superdirective antennas intercept but little power. Their ohmic resistances tend to diminish the received power still further.

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PROBLEMS

6.1–1 Using the results of Problems 5.10–1 and 5.16–1, obtain the directivity of the broadside couplet.

Ans.
$$g = 3\left[1 + \frac{3}{\beta^2 l^2}\left(\frac{\sin \beta l}{\beta l} - \cos \beta l\right)\right]^{-1}.$$

6.1–2 Using the results of Problems 5.14–4 and 5.16–9, obtain the directivity of the half-wave sinusoidal current filament.

Ans. $g = 1.64, \qquad G = 2.15 \text{ db}.$

6.1–3 Using the results of Problem 5.16–2, obtain the directivity of the array for the case in which the ratio of the moments is 2 and $l = \lambda$.

Ans.
$$g = \frac{27\pi^2}{10\pi^2 - 6}.$$

6.1–4 Using the results of Problems 5.16–2 and 5.16–3, obtain the directivity of the couplet for the case in which the moments are equal, $l = \lambda$ and $\vartheta = \pi/2$.

Ans.
$$g \simeq \frac{45}{16}.$$

6.1–5 Using the result of Problem 5.16–4, obtain the directivity of two equal parallel current elements operating 180° out of phase. Assume that $l \leq \frac{1}{2}\lambda$.

Ans.
$$g = \tfrac{3}{2}(1 - \cos\beta l)\left[1 - \frac{3\lambda}{4\pi l}\left(\sin\beta l + \frac{\cos\beta l}{\beta l} - \frac{\sin\beta l}{\beta^2 l^2}\right)\right]^{-1}.$$

6.1–6. Find the limit of g as l approaches zero.

Ans.
$$g = \frac{15}{4}.$$

6.1–7 Solve Problem 6.1–5 for the case of $l > \lambda/2$.

Ans.
$$3\left[1 - \frac{3\lambda}{4\pi l}\left(\sin\beta l + \frac{\cos\beta l}{\beta l} - \frac{\sin\beta l}{\beta^2 l^2}\right)\right]^{-1}.$$

6.1–8 Consider an end-fire array of n parallel current elements with all elements perpendicular to the line joining their centers. Assume that the distance between the successive elements is $\lambda/4$ and that the current is attenuated from one element to the next, the attenuation factor between the adjacent elements being k. Obtain the approximate gain of the array over a single element by neglecting the interaction between the elements.

Ans.
$$\frac{g}{g_1} = \frac{(1 + k)(1 - k^n)}{(1 - k)(1 + k^n)}.$$

6.1–9 Using the result of the preceding problem, show that, if the directivity of the array of attenuated elements is $g(k)$, then,

$$\frac{g(k)}{g(1)} = \frac{(1 + k)(1 - k^n)}{n(1 - k)(1 + k^n)}.$$

6.1–10 Show that, if the ratio of the current in the last element to that in the first is u, then the result of the preceding problem may be stated as follows:

$$\frac{g(u)}{g(1)} = \frac{(1 + \sqrt[n-1]{u})(1 - u\sqrt[n-1]{u})}{n(1 - \sqrt[n-1]{u})(1 + u\sqrt[n-1]{u})}.$$

6.1–11 Show that, if the total attenuation factor u is kept constant while n approaches infinity, the limit of the result in the preceding problem is

$$\frac{g(u)}{g(1)} = \frac{2(1-u)}{(1+u)\log(1/u)} .$$

Plot this curve and show that, even if the amplitude of the last element drops to one half of the amplitude of the first, the directivity is reduced by only 4 per cent.

6.1–12 Consider a highly directive continuous end-fire array of current elements, and let ψ be the half-angle of the major lobe. Let the maximum radiation intensity be unity. Find the power radiated within the major lobe, the first subsidiary lobe, and the total radiated power. How much power will be radiated if the radiation intensity is unity for all directions inside the cone of small half-angle ψ and negligible for all other directions.

Ans. $1.42\psi^2$, $0.07\psi^2$, $1.57\psi^2$; $3.14\psi^2$.

6.18–1 Prove that, if two antennas have the same directivities and if the maximum radiation intensities corresponding to the same input current are $\Phi_{\max,1}$ and $\Phi_{\max,2}$, then,

$$\frac{R_1}{R_2} = \frac{\Phi_{\max,1}}{\Phi_{\max,2}} ,$$

where R_1 and R_2 are the radiation resistances of the antennas as seen from the input terminals.

6.18–2 Show that a corollary of the preceding theorem is

$$\frac{R_1}{R_2} = \frac{E_{\max,1}{}^2}{E_{\max,2}{}^2} .$$

6.18–3 Prove that, if $\Phi_{\max,1}$ and $\Phi_{\max,2}$ are the maximum radiation intensities of two antennas corresponding to the same input current, then,

$$\frac{R_1}{R_2} = \frac{g_2\Phi_{\max,1}}{g_1\Phi_{\max,2}} ,$$

where R_1 and R_2 are the radiation resistances seen at the input terminals.

7

WAVES OVER GROUND

7.1 Image theory of reflection

Radiation patterns of antennas are affected profoundly by the earth and the atmosphere above it. It is beyond the scope of this book to treat the various effects of the earth and the atmosphere comprehensively. Antennas are usually designed on the basis of their free-space performance. Their actual performance is then evaluated in the light of various

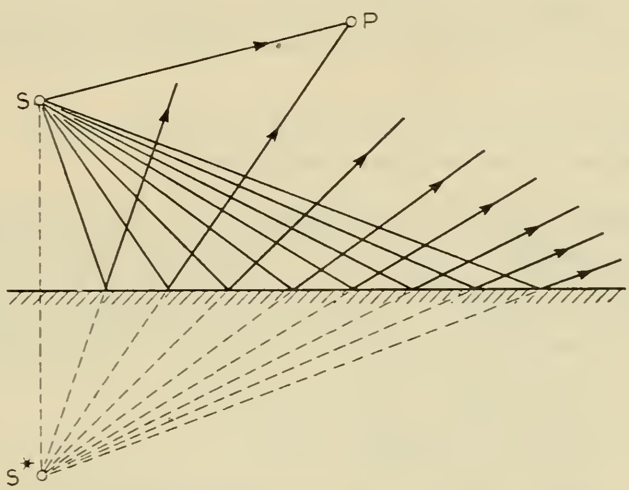

Fig. 7.1 Illustrating the image theory of reflection.

propagation effects. One of the most important of these effects is *ground reflection*. We shall consider only an idealized case in which the ground surface is assumed to be smooth and plane. In Section 4.18 we already considered the case of reflection from a perfectly conducting plane. In this case the field at any point P above the ground plane (Fig. 7.1) equals the sum of the free-space field of the source S and the free-space field of the image source S*. The strength of the image source equals the strength of the actual source except for a possible difference

in phase. Thus, if S is a vertical current element, the image source is a vertical element in phase with it; but, if S is a horizontal element, the image source is a horizontal element 180° out of phase.

There is no simple rule for obtaining the exact field above any actual medium. An approximate field may be obtained as follows. If the source of waves is high above ground, the waves striking ground are substantially plane over any limited area. We shall presently see that the angle at which plane waves are reflected equals the angle of incidence. Thus, the reflected wave appears to come from the image source as in the case of a perfect reflector; but the coefficient of reflection in the general case depends on the angle of incidence. To obtain this result we assumed that the source of waves, or the "transmitter," is high above ground. The receiver, however, can be either high above or close to ground. By reciprocity, we can use the same image rule when the transmitter is close to ground and the receiver is high. The case of transmitter and receiver close to ground requires special consideration.

In this chapter we shall determine the reflection coefficient as a function of the angle of incidence on the assumption that the ground is homogeneous. Important conclusions may be drawn from this restricted solution; in practice, however, reflection coefficients should be measured.

7.2 Calculation of reflection coefficients

Consider a uniform plane wave incident on a plane boundary between two homogeneous media (Fig. 7.2). The angle AOZ between the direction of propagation and the normal to the boundary is called the *angle of incidence*. In radio propagation, however, the complementary angle Δ, the *angle of elevation*, is more convenient. If s is the distance in the direction of wave propagation, then,

$$E^i = E_0{}^i e^{-j\beta_1 s}, \qquad \beta_1 = \omega(\mu_1\varepsilon_1)^{1/2}, \tag{1}$$

where $E_0{}^i$ is the intensity of the incident wave at the point of intersection of the particular incident ray AO and the ground surface. Let us choose the y axis in the direction perpendicular to the direction of propagation and parallel to the ground surface. Then,

$$s = x\cos\Delta - z\sin\Delta. \tag{2}$$

Substituting in equation 1, we obtain

$$E^i = E_0{}^i \exp(-j\beta_x x + j\beta_z z), \tag{3}$$

where

$$\beta_x = \beta_1\cos\Delta, \qquad \beta_z = \beta_1\sin\Delta \tag{4}$$

are the phase constants of propagation in the x and z directions, respectively.

At the ground surface the tangential components of E and H must be continuous. If this requirement is satisfied at some one point, it will not be satisfied at other points unless the phase of the reflected wave is in step with the phase of the incident wave along the entire

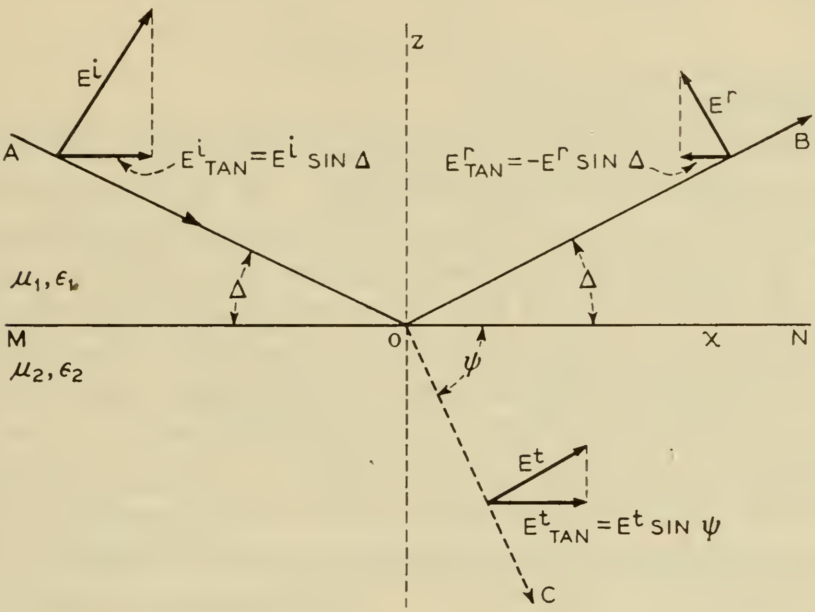

FIG. 7.2 Illustrating the calculation of reflection coefficients.

boundary. Therefore, the phase constant of the reflected wave in the x direction must equal the corresponding phase constant in the incident wave. The reflected wave is traveling away from the interface, and the vertical phase constant changes its sign. Hence,

$$E^r = E_0{}^r \exp(-j\beta_x x - j\beta_z z). \tag{5}$$

This relationship between the phase constants implies that the angles of elevation of the incident and reflected waves are equal.

Let us assume that the transmitted (or "refracted") wave is also a uniform plane wave. If ψ is its angle of elevation,

$$E^t = E_0{}^t \exp(-j\beta_x' x + j\beta_z' z), \tag{6}$$

where

$$\beta_x' = \beta_2 \cos \psi, \qquad \beta_z' = \beta_2 \sin \psi, \qquad \beta_2 = \omega(\mu_2 \varepsilon_2)^{1/2}. \tag{7}$$

Again the continuity of the tangential field components requires that the phase constants in the direction parallel to the ground surface be equal,

$$\beta_x' = \beta_x, \quad \text{or} \quad \beta_1 \cos \Delta = \beta_2 \cos \psi,$$
$$\text{or } (\mu_1 \varepsilon_1)^{1/2} \cos \Delta = (\mu_2 \varepsilon_2)^{1/2} \cos \psi. \tag{8}$$

This equation determines ψ. The *index of refraction* n is defined as follows,

$$n = \left(\frac{\mu\varepsilon}{\mu_v\varepsilon_v}\right)^{1/2}, \tag{9}$$

where μ_v and ε_v refer to vacuum. In terms of refractive indices, equation 8 becomes

$$n_1 \cos \Delta = n_2 \cos \psi. \tag{10}$$

This is Snell's law of refraction.

So far we have made no reference to the relative orientation of E and H with respect to the ground surface. Let us now assume that the E vector is in the *plane of incidence* defined as the plane containing the direction of propagation and the normal to the boundary between the media. Since the H vector must be perpendicular to the E vector and to the direction of propagation, it must be parallel to the ground surface. We can look at the present incident wave either as a uniform plane wave traveling in the direction AO or as a sinusoidal wave pattern moving perpendicularly toward the ground surface. As far as penetration of the wave into ground and reflection from it are concerned, we need consider only this perpendicular propagation, and our problem is the same as that considered in Section 3.6. At the surface of discontinuity in the electromagnetic parameters, the tangential components of E and H must be continuous. As we have seen in Chapter 3, these are analogous to the voltage V and current I in transmission lines. From equation 3–38 we find that the reflection coefficient for $H_{\tan}$ is the negative of the reflection coefficient for $E_{\tan}$. The latter we obtain from equation 3–41, where K is the wave impedance $K_{z,1}$ in the first medium and Z is the wave impedance $K_{z,2}$ in the second medium, both in the direction normal to ground. In the present case $H_{\tan}$ is the total H, and it is convenient to consider its reflection coefficient

$$q = \frac{K_{z,1} - K_{z,2}}{K_{z,1} + K_{z,2}}. \tag{11}$$

This is also the reflection coefficient for the total E, provided the positive direction of the reflected E is taken as in Fig. 7.2 so that E^r, H^r and the direction of propagation form the conventional right-handed system of directions. The wave impedances normal to ground are

$$K_{z,1} = \frac{E^i_{\tan}}{H^i} = \frac{E^i \sin \Delta}{H^i}, \qquad K_{z,2} = \frac{E^t_{\tan}}{H^t} = \frac{E^t \sin \psi}{H^t}. \tag{12}$$

Since

$$E^i = \eta_1 H^i, \qquad E^t = \eta_2 H^t, \tag{13}$$

where η_1 and η_2 are the intrinsic impedances of the two media, and since ψ can be expressed in terms of the given angle Δ from equations 8, the reflection coefficient q can be expressed in terms of Δ and the electromagnetic parameters of the media. In particular, *if the permeabilities of the media are equal,* then,

$$q = \frac{(\varepsilon_2/\varepsilon_1)\sin\Delta - [(\varepsilon_2/\varepsilon_1) - \cos^2\Delta]^{1/2}}{(\varepsilon_2/\varepsilon_1)\sin\Delta + [(\varepsilon_2/\varepsilon_1) - \cos^2\Delta]^{1/2}}. \tag{14}$$

This formula can be generalized to include dissipation. In dissipative media the field equations contain the complex parameter $g + j\omega\varepsilon$ instead of $j\omega\varepsilon$; hence, the formulas for nondissipative media may be made more general if we replace the dielectric constant ε by the "complex dielectric constant" $\varepsilon + (g/j\omega) = \varepsilon - j(g/\omega)$. If the upper medium is air, equation 14 for the *reflection coefficient when H is parallel to the ground surface* becomes

$$q = \frac{(\varepsilon_r - j60g\lambda)\sin\Delta - (\varepsilon_r - \cos^2\Delta - j60g\lambda)^{1/2}}{(\varepsilon_r - j60g\lambda)\sin\Delta + (\varepsilon_r - \cos^2\Delta - j60g\lambda)^{1/2}}, \tag{15}$$

where g is the conductivity of the lower medium and ε_r is its dielectric constant relative to air.

If E is parallel to the ground surface and H is in the plane of incidence (as E is in Fig. 7.2), then we find

$$q = \frac{K_{z,2} - K_{z,1}}{K_{z,2} + K_{z,1}}, \tag{16}$$

where the wave impedances normal to ground are

$$K_{z,1} = \frac{E^i}{H^i_{\tan}} = \frac{E^i}{H^i \sin\Delta} = \eta_1 \csc\Delta, \qquad K_{z,2} = \eta_2 \csc\psi. \tag{17}$$

For waves entering a *nonmagnetic* dissipative medium from air, we find

$$q = \frac{\sin\Delta - (\varepsilon_r - \cos^2\Delta - j60g\lambda)^{1/2}}{\sin\Delta + (\varepsilon_r - \cos^2\Delta - j60g\lambda)^{1/2}}. \tag{18}$$

If $\Delta = 90°$, both E and H are parallel to the ground surface. This is a special case of either of the two preceding cases, and the reflection coefficient may be obtained from either equation 15 or 18. Indeed the same value is obtained *except* for the algebraic sign. The difference in sign is caused by the conventions with respect to positive directions of field intensities. In equation 18 the incident and reflected E vectors are parallel; hence, it is natural that the same positive direction is chosen for both vectors. Similarly, in equation 15 the H vectors are parallel so that it is equally natural to choose the same positive direction for them. This leads to the particular relation between positive

directions of E^r and E^i which is exhibited in Fig. 7.2. When $\Delta = 90°$, the positive direction of E^r is directly opposite to the positive direction of E^i. Note that the convention exhibited in Fig. 7.2 is convenient for small angles of elevation.

7.3 Discussion of ground-reflection coefficients

Ground-reflection coefficients depend on four parameters: ground conductivity g, relative dielectric constant ε_r, wavelength λ in free space, and the angle of elevation Δ of the incident waves. If $\varepsilon_r \gg 1$, which is true for most dry and wet soils, these parameters may be grouped, and equations 15 and 18 may be expressed in new forms which exhibit more clearly the essentials of ground reflection. We define the *quality factor* Q of ground, the *critical wavelength* λ_c for which the quality factor is unity, and the *critical angle of elevation* Δ_c as follows:

$$Q = \frac{\omega\varepsilon}{g}, \qquad \lambda_c = \frac{\varepsilon_r}{60g}, \qquad \sin \Delta_c = \varepsilon_r^{-1/2}\left[1 + \left(\frac{\lambda}{\lambda_c}\right)^2\right]^{-1/4}. \tag{19}$$

The following table gives an indication of the order of magnitude of the critical wavelength:*

	Dry Soil	Wet Soil	Sand	Sea Water
g	0.015	0.015	0.002	5
ε_r	10	30	10	78
λ_c	11	33	83	0.26

The critical angle Δ_c is never greater than $1/\varepsilon_r^{1/2}$ radians. For perfect conductors λ_c and Δ_c are equal to zero. If $\lambda \gg \lambda_c$, then Δ_c is small; otherwise, Δ_c is fairly large. When $\lambda \ll \lambda_c$, the critical angle is nearly independent of λ and typical values are:

Dry Soil	Wet Soil	Sand	Sea Water
0.322(18°26′)	0.183(10°31′)	0.316(18°26′)	0.113(6°30′)

The *complex index of refraction* is often used in literature on wave propagation. It is defined as the ratio σ/σ_v of the intrinsic propagation constant of the medium to the intrinsic propagation constant in vacuum; hence,

$$n^2 = \frac{j\omega\mu(g + j\omega\varepsilon)}{j\omega\mu_v j\omega\varepsilon_v} = \mu_r\varepsilon_r\left(1 - j\frac{\lambda}{\lambda_c}\right). \tag{20}$$

In radio propagation it is usually assumed that $\mu_r = 1$. This equation enables us to identify ε_r and λ/λ_c when n^2 is given.

* For typical values of ground constants for waves several meters long, see C. B. Feldman, The optical behavior of the ground for short radio waves, *IRE Proc.*, 21, June 1933, pp. 764–801.

If $\varepsilon_r \gg 1$, we may neglect $\cos^2 \Delta$ in equations 15 and 18. Then, if the H vector is horizontal (and the E vector is in a vertical plane), the reflection coefficient is

$$q = -\left(1 - \frac{\sin \Delta}{\sin \Delta_c} e^{-j\varphi}\right)\left(1 + \frac{\sin \Delta}{\sin \Delta_c} e^{-j\varphi}\right)^{-1}, \qquad \varphi = \tfrac{1}{2} \tan^{-1} \frac{\lambda}{\lambda_c}; \quad (21)$$

and, if the E vector is horizontal,

$$q = -(1 - e^{j\varphi} \sin \Delta \sin \Delta_c)(1 + e^{j\varphi} \sin \Delta \sin \Delta_c)^{-1}. \quad (22)$$

If E is horizontal, the reflection coefficient approaches negative unity steadily as Δ varies from 90° to 0°.

But, if H is horizontal, the ratio of $\sin \Delta$ to $\sin \Delta_c$ may be either large or small compared with unity, and the reflection coefficient may be comparable to either positive unity or negative unity, depending on

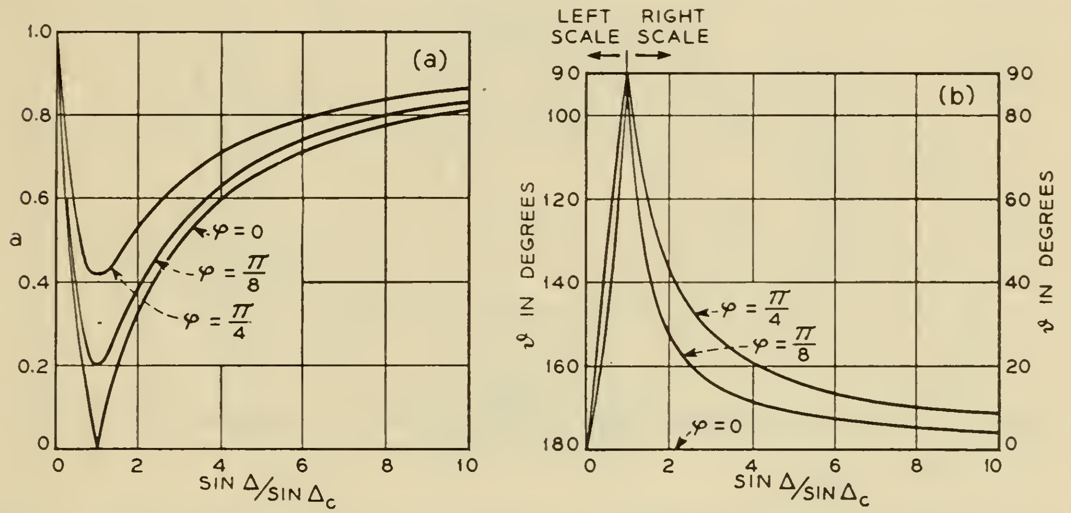

FIG. 7.3 Universal curves for the amplitude a and the negative phase ϑ of the reflection coefficient for waves over ground in which the H vector is horizontal.

whether the angle of elevation is large or small. Figure 7.3 shows the amplitude a and the *negative* phase ϑ — so that $q = a \exp(-j\vartheta)$ — as functions of the angle of elevation and parameter φ defined in equations 21. The amplitude of the reflection coefficient is minimum at the critical angle, and the phase is $-90°$. For perfect dielectrics ($\varphi = 0$) there is actually no reflection at the critical angle, which in this case is known as the *Brewster angle*. In any case, $a_{\min} = \tan \frac{1}{2}\varphi$. Since φ never exceeds $\pi/4$, the greatest value of this minimum is $\tan(\pi/8) = 0.414 \cdots$. The upper and lower curves represent the extreme cases.

7.4 Radiation patterns

To obtain the field of a transmitter at point P_1 (Fig. 7.4), we have to add the field intensities of the primary or "direct" wave and the ground-

reflected wave. At point P_2, so far from P_1 that the polar angles as observed from P_1 and its image Q_1 are substantially equal, the total field of a vertical current element of moment Is is

$$E_\theta = E_\theta{}^p + E_\theta{}^r = \frac{60\pi jIs}{\lambda}\left[\frac{e^{-j\beta r}}{r} + q\,\frac{e^{-j\beta(r_1+r_2)}}{r_1+r_2}\right]\sin\theta. \tag{23}$$

As r increases, the effect of the path difference $r_1 + r_2 - r$ for the direct and ground-reflected waves on the amplitude of the field decreases and

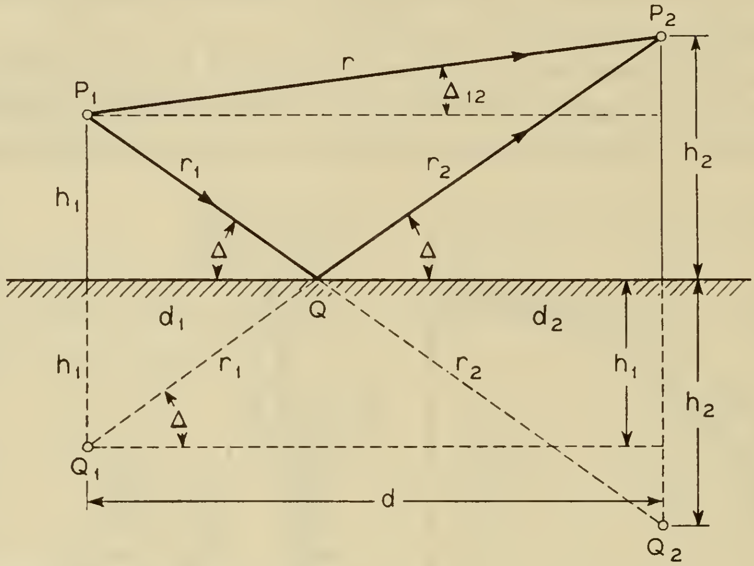

FIG. 7.4 Direct and ground-reflected rays between a transmitter at P_1 and a receiver at P_2.

eventually becomes negligible. Thus, if r is very large compared with h_1 and h_2, equation 23 becomes

$$E_\theta = E_\theta{}^p(1 + qe^{-j\beta(r_1+r_2-r)}). \tag{24}$$

To obtain the radiation pattern we must let r approach infinity. The path difference becomes equal to the projection of P_1Q_1 on Q_1P_2; therefore,

$$E_\theta = E_\theta{}^p(1 + qe^{-2j\beta h_1 \cos\theta}). \tag{25}$$

In the case of infinite conductivity $q = 1$ for all angles. In the horizontal plane $\theta = 90°$, the exponential factor equals unity, and the resultant electric intensity is twice as large as the primary intensity. On the other hand, for *any* finite conductivity, at grazing incidence, $q = -1$; hence, the resultant field vanishes, and the radiation pattern has a null. Figure 7.5 illustrates the profound effect of finite ground conductivity on the radiation pattern of a vertical current element.

The curves were obtained by Feldman* for the case in which $\varepsilon_r = 7$ and $\lambda/\lambda_c = 3/7$.

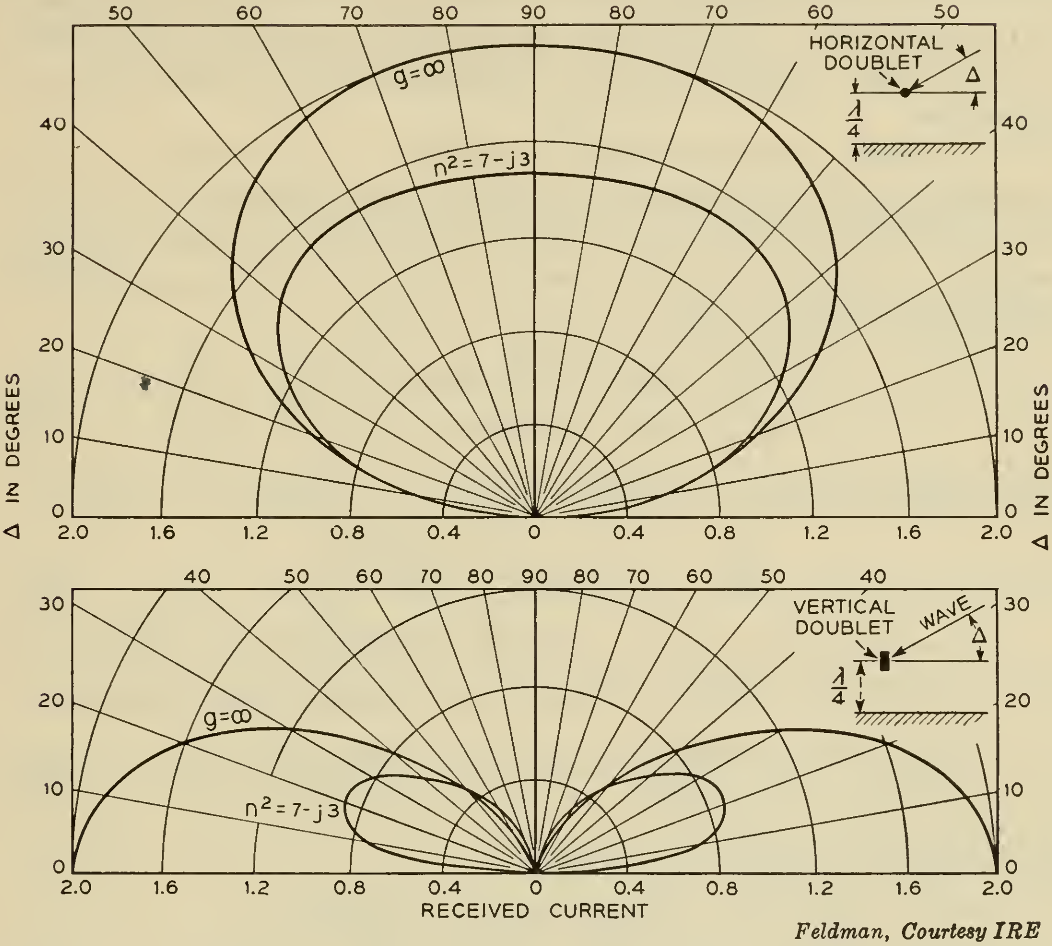

FIG. 7.5 Comparison between the radiation patterns over ground with infinite conductivity and over ground with finite conductivity.

7.5 Low-angle radio transmission

The null in the horizontal direction in the radiation pattern of the vertical current element does not mean that the field is identically zero. Radiation patterns are intended to represent distributions of the major component of the electric intensity, the one whose amplitude varies inversely as the distance from the transmitter. To obtain the radiation pattern, the electric intensity is multiplied by r, and the limit is taken as r approaches infinity. In passing to the limit the terms varying inversely as the square of the distance from the transmitter disappear. Hence, at any finite distance from the transmitter, the radiation pattern gives useful information about the field distribution everywhere except in the thin funnel-shaped holes whose axes are the null directions in the

* *Ibid.*

pattern. In these holes the residual electric intensity varies inversely as the square of the distance from the transmitter. Round each hole there is a transition region where the intensity of the field rises gradually to the main level. In the vicinity of the null directions, it is impossible to plot a single radiation pattern, independent of the distance from the transmitter.

To obtain the transmission formula for low altitudes, we must use more accurate expressions for q and the path difference $r_1 + r_2 - r$ in equation 24 than we had to use in calculating the main radiation pattern. For large horizontal distance d between the transmitter and receiver (Fig. 7.4) we have

$$\begin{aligned} r &= [d^2 + (h_2 - h_1)^2]^{1/2} \simeq d + \tfrac{1}{2}d^{-1}(h_2 - h_1)^2, \\ r_1 + r_2 &= [d^2 + (h_1 + h_2)^2]^{1/2} \simeq d + \tfrac{1}{2}d^{-1}(h_1 + h_2)^2. \end{aligned} \tag{26}$$

Substituting in equation 24 and dropping the subscript θ, we obtain

$$\frac{E}{E^p} = 1 + q \exp\left(\frac{-j4\pi h_1 h_2}{\lambda d}\right) \cdot \tag{27}$$

For *angles small compared with the critical angle* Δ_c, equation 21 for the reflection coefficient, when E is in a vertical plane, becomes

$$q \simeq -1 + \frac{2 \sin \Delta}{\sin \Delta_c} e^{-j\varphi} \simeq -1 + \frac{2(h_1 + h_2)}{d \sin \Delta_c} e^{-j\varphi}. \tag{28}$$

Substituting in equation 27, we find

$$\begin{aligned} \frac{E}{E^p} = 2j \sin\left(\frac{2\pi h_1 h_2}{\lambda d}\right) \exp\left(\frac{-j2\pi h_1 h_2}{\lambda d}\right) + \\ \frac{2(h_1 + h_2)}{d \sin \Delta_c} \exp\left(-j\varphi - j\frac{4\pi h_1 h_2}{\lambda d}\right) \cdot \end{aligned} \tag{29}$$

If $2\pi h_1 h_2 \ll \lambda d$, then,

$$\frac{E}{E^p} = j\frac{4\pi h_1 h_2}{\lambda d} + \frac{2(h_1 + h_2)}{d \sin \Delta_c} e^{-j\varphi}. \tag{30}$$

If h_1 and h_2 are large compared with λ, the first term is dominant, and the electric intensity is proportional to the heights of the transmitter and receiver above ground. At greater heights the height gain begins to decrease, and we must use equation 29. The field intensity reaches its first maximum when

$$4h_1 h_2 = \lambda d, \qquad h_2 = \frac{\lambda d}{4h_1} \cdot \tag{31}$$

This maximum value is twice the intensity of the primary field. We

must remember, however, that this is true only to the extent to which $2(h_1 + h_2)/d \sin \Delta_c$ is negligible compared with unity, so that $q \simeq -1$. In general, the maximum value is smaller and should be determined from equation 27.

If $2h_1h_2 \ll \lambda d$ and $h_1 \ll h_2$, equation 30 becomes

$$\frac{E}{E^p} = j\frac{4\pi h_1}{\lambda}\Delta; \tag{32}$$

that is, the field intensity is proportional to the angle of elevation of the receiver.

If $h_1 = h_2 = 0$, equation 27 gives $E = 0$. It must be recalled, however, that our method of obtaining the reflection coefficient is invalid when both the transmitter and the receiver are very close to ground. In this case it can be shown* that

$$\frac{E}{E_0} = -\frac{jn^4\lambda}{(n^2-1)\pi d} + \frac{2n^2(h_1+h_2)}{(n^2-1)^{1/2}d}, \qquad \frac{2\pi d}{\lambda} \gg 3|n|^2, \tag{33}$$

where n is the complex index of refraction. This extends equation 30 to the immediate vicinity of ground.

7.6 Wave tilt at grazing incidence

At great distances from a vertical antenna above perfect ground, E is normal to ground. We have seen that finite ground conductivity has the effect of reducing E considerably. In addition it causes a tilt in the E vector. The component of E tangential to ground is utilized in the design of "wave antennas" which we shall consider in Chapter 15. From the second of equations 12, we find the normal component of impedance, $K_n = \eta \sin \psi$. For grazing incidence equation 7, when generalized to include the ground conductivity, becomes $j\beta_v = \sigma \cos \psi$. Hence,

$$K_n = \eta\left[1 - \left(\frac{\beta_v}{\sigma}\right)^2\right]^{1/2} \simeq \eta, \tag{34}$$

where η is the intrinsic impedance of ground. Hence, the ratio of the horizontal component of E to the vertical component is

$$\frac{E_H}{E_V} = \frac{\eta}{120\pi}. \tag{35}$$

For long waves $\omega\varepsilon \ll g$, and

$$\frac{E_H}{E_V} = \left(\frac{j}{60g\lambda}\right)^{1/2}. \tag{36}$$

* Using Sommerfeld's integral on p. 433 of *Electromagnetic Waves*.

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8

ANTENNA CURRENT

8.1 Electrical properties of antennas

One of the most practical questions with regard to a transmitting antenna is: What is the strength of the field at a given distance for a given radiated power? Another question is: What is its radiation pattern? We shall also want to know the input impedance of the antenna in order to match the antenna to the source of power, and thus insure that the radiated power is maximum. In broad-band communication we are also interested in the antenna impedance as a function of the frequency, either for making the best choice of several antennas or for designing a corrective network.

We shall be able to answer all these questions if we know the antenna current. Of course, to obtain the input impedance we need only the input current for a given input voltage; but, unfortunately, in most cases we do not know how to evaluate the input current without first obtaining the entire current distribution. However, we need to know the entire current in order to answer the other practical questions. As we shall see, we need not know this distribution very accurately. On the other hand we must know the input current much more accurately if we are to obtain satisfactory information about the impedance. This happens to be the most difficult of all antenna problems, except in a relatively few cases.

Even in the simplest cases, the exact current distribution in conductors is complex. Approximations are usually unavoidable, and it is essential that the relative importance of various factors affecting the current distribution be well understood. Then, as we have just noted, the answers to some questions are relatively insensitive to the errors in the current distribution, whereas the answers to other questions are very sensitive. Thus, we must find out when we can consider that a rough approximation is satisfactory and when we should seek a better approximation.

8.2 Effect of current distribution on the radiation pattern and the radiated power

The radiation pattern and the power radiated by the antenna are insensitive to errors in the assumed form of current distribution. To obtain the field intensity in any given direction we must add the contributions from all parts of the antenna, those parts where the current is large and those where it is small. If we make the maximum amplitude of the usual sinusoidal approximation equal to the maximum amplitude of the exact current, the errors are relatively large *only where the current is small.* These errors are submerged in the integration except in those directions in which the contributions from the various parts of the antenna tend to cancel. *The antenna current must be known more accurately if we are interested in the minima in the radiation pattern; but it need not be known accurately otherwise.*

The radiation patterns of highly directive antenna arrays are determined chiefly by the space factors, and the errors in the distribution of currents in the individual elements are even less important.

The radiated power may be obtained by integrating the radiation intensity Φ. Since Φ is proportional to the square of the field, the errors for those directions in which the field is small (and for which the errors are relatively large) become less important. In addition, these errors will be submerged in the integration. Thus, *we may expect a very good approximation to the radiated power even from a relatively poor approximation to the antenna current.**

8.3 Effect of current distribution on the antenna input resistance

Suppose that we have obtained the radiated power in terms of the maximum current amplitude,

$$P = \tfrac{1}{2}R_a I_{\max}^2, \tag{1}$$

where R_a is called the *radiation resistance with reference to the current antinode.* If R_i is the input resistance, this power may also be expressed as

$$P = \tfrac{1}{2}R_i I_i^2, \tag{2}$$

where I_i is the amplitude of the input current. Equating 1 and 2, we find

$$R_i = R_a \frac{I_{\max}^2}{I_i^2}. \tag{3}$$

* Later we shall find that the difference between the true antenna current and the sinusoidal approximation is largely in quadrature with the main current, and for this reason the effect of the difference on the radiated power is still further reduced.

In obtaining this equation, we have tacitly assumed that all power delivered to the antenna is radiated. Some of it, of course, is dissipated in heat in the conductors and insulators. Equation 3 gives only the resistance due to radiation. Since the dissipated power is usually small, it does not affect the radiated power and may be added to it to obtain a more general equation:

$$\tfrac{1}{2}R_iI_i^2 = \tfrac{1}{2}R_aI_{\max}^2 + \tfrac{1}{2}R_dI_{\max}^2, \qquad R_i = (R_a + R_d)\frac{I_{\max}^2}{I_i^2}\cdot \tag{4}$$

Equation 3 is exact; but its greatest usefulness comes in making approximations. In the preceding section we have concluded that the radiated power is relatively insensitive to the shape of the current distribution; thus, we can obtain a satisfactory approximation for R_a (see Chapter 13) from even a rough approximation to the current. On the other hand, R_i is sensitive to errors in the input current, especially when the latter is small, so that even small absolute errors lead to large relative errors. Equation 3 may be used to check R_i; *it should never be used to check* R_a.

Later in this chapter and particularly in Chapter 13 we shall see how to use equation 3 to obtain certain limit values of the input resistance.

8.4 Factors affecting current distribution

The factors affecting current distribution may be grouped as follows:

1. Effects of size, shape, and proximity of conductors.
2. Effects of discontinuities, such as sudden changes in the radius of the antenna, or sudden divergence of previously closely spaced wires.
3. Effects due to the distribution of generators or to the impressed field.
4. Effects due to radiation of power.
5. Effects due to the resistance of conductors.
6. Effects of proximity between the antenna terminals.

In Chapter 4 we found that the voltage and current associated with the TEM waves on conical conductors are distributed sinusoidally. In such conductors the effects of radiation and of discontinuities on the current may be expressed in terms of so-called higher-order waves. This method of analysis requires advanced mathematical techniques and is outside the scope of this book. In Chapter 4 we also found that the voltage and current associated with the *principal waves* (almost TEM waves) on other than conical wires are governed by the transmission equations 4–33, in which the distributed inductance and capaci-

tance per unit length are slowly varying functions of position along the wire. In such circumstances the voltage and current are approximately sinusoidal, with correction terms depending on the relative deviations of the distributed parameters from their mean values; in addition, there will be other correction terms due to radiation and possible discontinuities. Thus, we anticipate that *the approximate form of current distribution in wires or wire networks is sinusoidal,* with some deviation caused by the above-mentioned factors. The sinusoidal approximation was first discovered by Pocklington.*

This conclusion is predicated on the assumption that *the source of power is localized.* The current distribution due to several localized sources or to a continuous distribution of the impressed field is obtained by superposition of sinusoidally distributed currents, and thus will not appear as sinusoidal, even approximately. *The current in a receiving antenna, for example, is quite different from the current in the same antenna used as a transmitting antenna.*

In this chapter we shall first derive the limiting forms of antenna current as the radius of the antenna approaches zero and then consider more closely the principal effects of the various factors enumerated at the beginning of this section. For this purpose we need to find general expressions for the electric intensity in the vicinity of the antenna in terms of the antenna current. We shall find it expedient to introduce new conceptions. One is the conception of the *quasistatic potential or scalar electric potential* V, which depends directly on the charge distribution and from which we can find *a part* of the total electric intensity by taking the negative gradient in the same way as the *total* electric intensity is obtained in electrostatics. The only difference between the static potential and the quasistatic is that the latter contains a phase retardation factor which depends on the distance between an element of charge and a typical point in the field. This potential is often called the "electrostatic potential," even though the field may be varying at a frequency of a billion cycles per second. We have no serious objection to this usage, provided the difference between static and variable fields is fully understood. We prefer, however, the term "quasistatic potential," which implies its close relationship to the static potential and at the same time suggests that there is a difference.

The *difference* between the total electric intensity of a variable field and the negative gradient of the quasistatic potential is found to vanish with the frequency of the field. Thus, it is due solely to the motion of charge: that is, to the electric current. For this reason the difference will be called the *dynamic component* F of the electric intensity.

* H. E. Pocklington, Electrical oscillations in wires, *Camb. Phil. Soc. Proc.*, 9, October 25, 1897, pp. 324–332.

We shall then express these auxiliary functions in terms of the current and charge, and concentrate more specifically on the field in the vicinity of a thin wire. It is this field that determines the current distribution. Then we shall be ready to consider applications of the general theory to specific cases. Because of the complexity of the exact current distributions we shall resort to the method of successive approximations. We shall find that, *in the first approximation, the current in a thin antenna is distributed exactly as in a transmission line* with a similar distribution of impressed voltage, as shown, for instance, in Figs. 8.5 and 8.6. We shall find that the same rule can be applied to find the second approximation *if we impress on the antenna certain compensating electric fields,* which may be calculated from the first approximation to the current. Theoretically, this process could be continued indefinitely; practically, the integrations soon become very involved and have to be performed numerically. Even if tables of auxiliary functions were prepared, there would still be left an excessive amount of numerical work in combining numerous terms in the solution of any specific case. Fortunately, it is rarely necessary to carry the analysis beyond the second approximation.

The fairly complicated analysis in the following sections, 5 to 16, is needed for two reasons: (1) to establish a firm basis for the approximate evaluation of current distributions, and (2) to develop the equations needed in more accurate calculations. This analysis is unnecessary if we are interested only in the general approximate shape of the current distribution on thin wires, since this may be determined from elementary considerations of electrostatic and electromagnetic induction. Let q, I, V, be, respectively, the charge per unit length, the current, and the potential. We shall think of the potential in the usual electrostatic sense. Its use in the present situation is justified on the ground that the electric charge on a thin wire produces strong electric forces in the immediate vicinity, and that these forces are proportional to the local charge density, the effect of distant charges being relatively small. The difference ΔI between the current leaving an element of the wire and that entering it must equal the time rate of decrease of the charge $q\,\Delta z$ on the element of length Δz; that is, ΔI equals $-j\omega q\,\Delta z$. Since V and q are approximately proportional, we have

$$\Delta I \propto -j\omega V\,\Delta z;$$

or, in the limit,

$$\frac{\partial I}{\partial z} \propto -j\omega V.$$

As the current I varies with time, it produces an opposing emf of self-induction largely proportional to the time rate of change of the local

value of the current. This emf produces a potential difference between the ends of the element. Hence,

$$\Delta V \propto -j\omega I \,\Delta z.$$

In the limit,

$$\frac{\partial V}{\partial z} \propto -j\omega I.$$

The coefficients of proportionality cannot be determined by such a simple argument as above; but we can say that these coefficients are constant along the wire except near the ends. Thus, the approximate equations for the distribution of current and potential are of the same form as ordinary transmission line equations. Hence, the current in thin antennas must be approximately sinusoidal, as shown in Figs. 8.5 and 8.6.

In essence, this is the pattern of analysis which we adopt in the following sections; we shall give, however, a precise meaning to the conception of " potential " in the case of nonstatic fields.

8.5 Quasistatic and dynamic components of electric intensity

Any current distribution may be subdivided into current elements o moment $p = J\,d\tau$, where J is the current density and $d\tau$ is the volume occupied by the element. Thus we may substitute this moment for Is in equations 4–82 for the field of the current element and integrate this field over the given current distribution. The integration must be vectorial, which, in effect, requires that we obtain the Cartesian components of the field from equations 4–82 before integration. The integrands are seen to be complicated.

As already indicated in the preceding section, we can simplify our problem materially by introducing certain auxiliary functions, the scalar electric potential V, and the dynamic component F of the electric intensity. We shall prove that there exist relatively simple functions, a scalar function V and a vector function F, such that

$$E_s = F_s - \frac{\partial V}{\partial s}, \tag{5}$$

or

$$E = F - \operatorname{grad} V. \tag{6}$$

The second term we shall call the *quasistatic component* of E and denote it by G. The reason for the name will soon become apparent.

A clue to this representation of E is found in the fact that an *electro-*

static field may be expressed as the gradient of a potential V,

$$E = -\operatorname{grad} V, \qquad E_s = -\frac{\partial V}{\partial s}, \tag{7}$$

given by

$$V = \iiint \frac{q_\tau \, d\tau}{4\pi\varepsilon r}, \tag{8}$$

where r is the distance between the center (x', y', z') of a typical element of charge, $q_\tau \, d\tau$, and a typical point (x, y, z); thus,

$$r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}, \tag{9}$$

and q_τ is the density of charge. Equation 8 follows at once from the expression 4–62 for the potential of a point charge q,

$$V_0 = \frac{q}{4\pi\varepsilon r} \cdot \tag{10}$$

We need only replace q by $q_\tau \, d\tau$ and integrate over the space occupied by the charge.

Equation 7 implies that the integral of E round a closed curve is zero,

$$\oint E_s \, ds = -\oint \frac{\partial V}{\partial s} \, ds = -V \Big|_A^B = 0, \tag{11}$$

where A is the point from which we start and B is the same point at the end of the round. According to Maxwell's equations, this equation can be true for any closed curve only if the field is static. Otherwise, the variable E will produce a magnetic field H, and the variable H requires that

$$\oint E_s \, ds = -\mu \frac{\partial}{\partial t} \iint H_n \, dS, \tag{12}$$

or, in the steady state,

$$\oint E_s \, ds = -j\omega\mu \iint H_n \, dS. \tag{13}$$

This equation suggests, however, that we might be able to express the variable E as the sum of *two* components, one derivable from a potential and the other vanishing with the frequency.

To avoid the use of advanced mathematical theorems, and to keep in more intimate contact with the physical aspects of our problem, we shall construct the auxiliary functions V, F synthetically and then prove that our final results are correct. Every term in the expressions 4–82 for the field of an electric current element contains a phase factor

$\exp(-j\beta r)$ depending on the distance. This suggests that we include this factor in equation 10 for the potential of a point charge,

$$V_0 = \frac{qe^{-j\beta r}}{4\pi\varepsilon r}. \tag{14}$$

This is a quasistatic potential of the point charge in the sense that it reduces to the static potential as the frequency vanishes, and that it is proportional to the charge in the same way as the static potential. There is, however, an extra factor: the phase retardation factor $\exp(-j\beta r)$. In an electric current element there are two equal and opposite charges (Figs. 4.13 *b*, *c*, *d*), and its potential is, therefore,

$$V = \frac{qe^{-j\beta r_1}}{4\pi\varepsilon r_1} - \frac{qe^{-j\beta r_2}}{4\pi\varepsilon r_2}. \tag{15}$$

We already know E for the current element; therefore, we can find the corresponding dynamic component F from equations 6 and 15,

$$F = E + \text{grad } V = E - G, \tag{16}$$

with the aid of equations 4–82. In the next section we shall carry out the details of the calculation and show that *the dynamic component of the electric intensity is parallel to the element* and is given by

$$F = -j\omega\mu \frac{Ise^{-j\beta r}}{4\pi r}, \tag{17}$$

where Is is the moment of the element.

We now return to the idea expressed in the first paragraph of this section and apply it to the formation of V and F for an arbitrary distribution of charge and current. Replacing q by $q_\tau \, d\tau = q_\tau \, dx' \, dy' \, dz'$, where q_τ is the volume density of charge, and integrating,

$$V = \iiint \frac{q_\tau(x', y', z')e^{-j\beta r}}{4\pi\varepsilon r} \, dx' \, dy' \, dz'. \tag{18}$$

Similarly, from equation 17, we have

$$F = -j\omega\mu \iiint \frac{J(x', y', z')e^{-j\beta r}}{4\pi r} \, dx' \, dy' \, dz'. \tag{19}$$

This is a vector equation, so that, for a typical Cartesian component of the current density J, we have

$$F_z = -j\omega\mu \iiint \frac{J_z(x', y', z')e^{-j\beta r}}{4\pi r} \, dx' \, dy' \, dz'. \tag{20}$$

Having obtained V and F, we can evaluate E from equations 5 or 6, that is,

$$E_s = F_s - \frac{\partial V}{\partial s}, \qquad E = F - \text{grad } V. \tag{21}$$

Another auxiliary function, the *magnetic vector potential* A, has normally been used in place of the dynamic component F of the electric intensity. These two functions are simply related:

$$F = -j\omega\mu A, \qquad A = -\frac{F}{j\omega\mu}. \tag{22}$$

There is some advantage in using A rather than F, since A approaches a constant limit as ω approaches zero whereas F vanishes with ω. On the other hand, F possesses a more definite physical meaning while A remains primarily an auxiliary mathematical function. In this book we shall use mostly F but not to the complete exclusion of A. The expression for a typical Cartesian component of A in terms of the corresponding component of the current density J is found by substituting equation 22 in equation 20,

$$A_z = \iiint \frac{J_z(x', y', z')e^{-j\beta r}}{4\pi r}\, dx'\, dy'\, dz'. \tag{23}$$

8.6 An electric current element

We shall now fill in the missing details in the proof of the final equations in the preceding section. We have to show that the already known equations 4–82 may be expressed in the alternative form 21, where F is given by equation 17, and V, for each end of the element, by equation 14. Since the time rate of increase of the charge at the upper end of the element must equal the current flowing to it, we have a relation between q and I:

$$\frac{\partial q}{\partial t} = I, \qquad j\omega q = I, \qquad q = \frac{I}{j\omega}. \tag{24}$$

At the lower end we have the charge $-q$.

Without loss of generality we may assume that the element is parallel to the z axis. Let the element extend from the lower charge $-q$ at some point $(x', y', z' - \frac{1}{2}s)$ to the charge $+q$ at the upper end $(x', y', z' + \frac{1}{2}s)$, where s is the infinitesimal length of the element. The corresponding scalar potential is given by equation 15 where r_1 and r_2 are, respectively, the distances from the upper and the lower charge (Fig. 4.13). Since, however, s is infinitesimal, we can evaluate the

potential in equation 15 by differentiation,

$$V = s \frac{\partial}{\partial z'} \left(\frac{q e^{-j\beta r}}{4\pi\varepsilon r} \right), \tag{25}$$

where r is given by equation 9. For future convenience we introduce the following wave function:

$$\psi = \frac{e^{-j\beta r}}{4\pi r}; \tag{26}$$

then, in view of equation 24,

$$V = \frac{qs}{\varepsilon} \frac{\partial \psi}{\partial z'} = \frac{Is}{j\omega\varepsilon} \frac{\partial \psi}{\partial z'}. \tag{27}$$

From equation 9 for r we observe that a positive increment in z' has the same effect as a negative increment in z; hence, the derivative with respect to z' equals the negative derivative with respect to z, and

$$V = -\frac{Is}{j\omega\varepsilon} \frac{\partial \psi}{\partial z}. \tag{28}$$

We shall now evaluate V for a current element at the origin. First we note that V depends on z only through r; hence,

$$V = -\frac{Is}{j\omega\varepsilon} \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial z}. \tag{29}$$

For the element at the origin,

$$r^2 = x^2 + y^2 + z^2, \qquad 2r \frac{\partial r}{\partial z} = 2z, \qquad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta. \tag{30}$$

Introducing this value in equation 29 and noting that

$$\frac{\partial \psi}{\partial r} = -\frac{j\beta \exp(-j\beta r)}{4\pi r} - \frac{\exp(-j\beta r)}{4\pi r^2}, \tag{31}$$

we have

$$V = \frac{\eta Is}{4\pi r} \left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \cos\theta. \tag{32}$$

Taking the negative derivatives in the r and θ directions, we obtain the quasistatic components of E:

$$\begin{aligned} G_r &= -\frac{\partial V}{\partial r} = \frac{j\omega\mu Is}{4\pi r} e^{-j\beta r} \cos\theta + \frac{\eta Is}{2\pi r^2} \left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \cos\theta, \\ G_\theta &= -\frac{\partial V}{r\,\partial\theta} = \frac{\eta Is}{4\pi r^2} \left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \sin\theta. \end{aligned} \tag{33}$$

Subtracting these expressions from equations 4–82 in accordance with equation 16, we obtain the dynamic components of the electric intensity of an electric current element of moment Is:

$$F_r = E_r - G_r = -j\omega\mu Is\psi \cos\theta, \qquad \psi = \frac{\exp(-j\beta r)}{4\pi r}, \tag{34}$$
$$F_\theta = E_\theta - G_\theta = j\omega\mu Is\psi \sin\theta.$$

Evaluating the components F_z and F_ρ in the directions parallel to and perpendicular to the element, we find

$$\begin{aligned} F_z &= F_r \cos\theta - F_\theta \sin\theta = -j\omega\mu Is\psi, \\ F_\rho &= F_r \sin\theta + F_\theta \cos\theta = 0. \end{aligned} \tag{35}$$

Hence, *the dynamic component of the electric intensity produced by the current element is parallel to it and depends only on the distance from the element.*

Thus, we have succeeded in resolving the electric intensity of the current element into two components, one of which is parallel to the element and is given by the simple equation 35, while the other is obtained by taking the negative derivative of the *scalar potential* given by equation 32. The first component vanishes as $\omega \to 0$ while the second approaches the static value of E. Having shown that equation 6 applies to a typical current element, we use the principle of superposition to obtain the general equations at the end of the preceding section.

8.7 Infinitely thin current filaments

Linear antennas form an important class of antennas; in these the longitudinal dimension is much larger than the dimensions of the normal cross section, and the latter are small compared with $\lambda/4$. The current in such antennas is largely parallel to the long dimension. Because of the skin effect, the current is distributed on the surface. It is sometimes convenient to idealize a long, thin current filament and approximate it by an infinitely thin filament; when, however, we are interested in the field in the immediate vicinity of the given filament, it is essential that we consider its actual dimensions.

In the case of an infinitely thin current filament (Fig. 8.1), the triple integrals in equations 18 and 19 for V and F become single integrals. Thus the volume element of charge $q_\tau\, dx'\, dy'\, dz'$ may be expressed as $q(s)\, ds$, where $q(s)$ is the linear charge density; similarly, the moment of the current element $J\, dx'\, dy'\, dz'$ becomes $I(s)\, ds$. Hence,

$$V = \int_{s_1}^{s_2} \frac{q(s)e^{-j\beta r}}{4\pi\varepsilon r}\, ds, \qquad F_z = -j\omega\mu \int_{s_1}^{s_2} \frac{I(s)\cos(s, z)e^{-j\beta r}}{4\pi r}\, ds, \tag{36}$$

where (s, z) is the angle between the tangent to the current filament and the direction of a typical coordinate axis. The net current leaving an element ds must be equal to the time rate of decrease of the charge; therefore,

$$\frac{\partial I}{\partial s}\, ds = -\frac{\partial q}{\partial t}\, ds, \qquad \frac{\partial I}{\partial s} = -\frac{\partial q}{\partial t}. \tag{37}$$

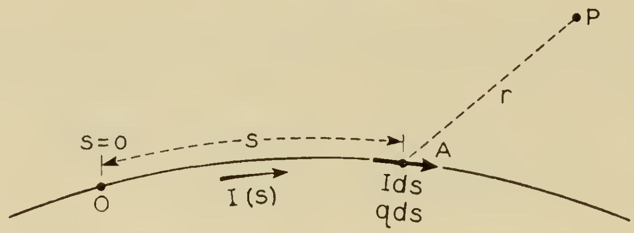

FIG. 8.1 A curved, infinitely thin current filament.

When I and q are exponential functions of time,

$$\frac{\partial I}{\partial s} = -j\omega q. \tag{38}$$

Similarly, at the beginning of the current filament $s = s_1$, and at its end, $s = s_2$, we have

$$I(s_1) = -j\omega Q_1, \qquad I(s_2) = j\omega Q_2, \tag{39}$$

where Q_1 and Q_2 are the end charges. If the end charges are different from zero, the corresponding potentials must be added to the integral for V in equation 36. These terms are not included in the integral when the integral is interpreted in the usual " Riemann sense "; they are included only if we agree that at the end points q is infinite while $q\, ds$ is finite and equal to Q, that is, if we interpret the integrals in the " Stieltjes sense."

Although we often find it convenient to express the field in terms of the two auxiliary functions, V and F, we should note that these functions are not independent. Thus the quasistatic potential of a current element parallel to the z axis is given by equation 28 and the dynamic component of the electric intensity by equations 35. Eliminating the wave function ψ, we have

$$V_{(z)} = -\frac{1}{\beta^2}\frac{\partial F_z}{\partial z}, \tag{40}$$

where the parenthetical subscript associated with V is to remind us that the element is parallel to the z axis, and is not to be interpreted as an indication that V is a vector. In accordance with the principle of superposition, this equation is true for any system of current elements

parallel to the z axis. For systems of elements parallel to the other two axes of a rectangular coordinate system, we have

$$V_{(x)} = -\frac{1}{\beta^2}\frac{\partial F_x}{\partial x}, \qquad V_{(y)} = -\frac{1}{\beta^2}\frac{\partial F_y}{\partial y}. \tag{41}$$

Since the potentials are added algebraically, the potential of the entire system of elements is

$$V = -\frac{1}{\beta^2}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = -\frac{1}{\beta^2}\operatorname{div} F. \tag{42}$$

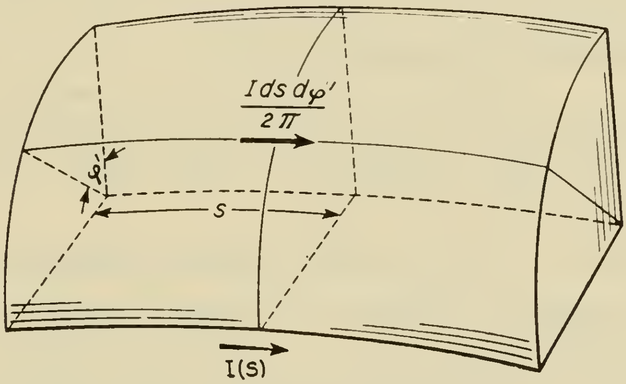

FIG. 8.2 A curved current filament of finite radius.

8.8 Thin current filaments

A thin current filament (Fig. 8.2) may be resolved into infinitely thin filaments of angular density $I(s)/2\pi$. Hence, instead of equations 36, we have

$$V = \frac{1}{2\pi\varepsilon}\int_{s_1}^{s_2}\int_0^{2\pi} \psi\, q(s)\, ds\, d\varphi',$$
$$F_z = -\frac{j\omega\mu}{2\pi}\int_{s_1}^{s_2}\int_0^{2\pi} \psi\, I(s)\cos(s, z)\, ds\, d\varphi', \tag{43}$$

where ψ is given by equation 26, that is,

$$\psi = \frac{e^{-j\beta r}}{4\pi r}, \tag{44}$$

and F_z is a typical Cartesian component of F.

We shall stress once more that the potential is determined by the charge density. As shown by the continuity equation 38, the charge density and the current are not independent; but the relation between them does not invalidate the preceding statement. At a given point of a thin filament, V is determined primarily by the density of charge at

that point; and, similarly, F is determined primarily by the current at that point. This is due to the fact that ψ is large only in the region where $z' - z$ is small. And, of course, it is perfectly possible for one of the quantities, $q(s)$ or $I(s)$, to be large at some point and for the other to be small. Further, when r is small, the retardation βr in the phase of ψ is small. Hence, the forces affecting the distribution of charge on thin wires will be mainly electrostatic, together with the forces of magnetic induction, as is the case in low-frequency transmission lines.* The thinner the wires, the more nearly true this is. The importance of these remarks will be appreciated more fully when we develop a method for evaluating the first-order high-frequency effects from the results based on the low-frequency approximations.

8.9 Exact equations for the distribution of quasistatic potential and the dynamic component of the electric intensity on a system of parallel thin wires

Consider a system of parallel thin wires of arbitrary lengths. The currents are constrained to flow in one direction; namely, parallel to the wires. The dynamic component of E will thus be parallel to the wires; and, if the wires are parallel to the z axis, the longitudinal field due to the currents in the wires is

$$E_z = F_z - \frac{\partial V}{\partial z} \cdot \tag{45}$$

If $E_z^i(z)$ is the electric intensity impressed on a typical wire, other than that due to the currents in the given system of wires, then the total electric intensity on the surface of this wire is $E_z + E_z^i(z)$. This must equal the product of the internal impedance of the wire and the current; thus, on the surface of the wire,

$$E_z + E_z^i(z) = Z_i \, I(z), \tag{46}$$

where Z_i is the internal impedance per unit length. At high frequencies,

$$Z_i = \frac{1}{2\pi a}\left(\frac{\pi\mu f}{g}\right)^{1/2}(1 + j), \tag{47}$$

where a is the radius of the wire. Substituting from equation 45 in

* John R. Carson, Electromagnetic theory and the foundations of electric circuit theory, *Bell Sys. Tech. Jour.*, 6, January 1927, pp. 1–17; The rigorous and approximate theories of electrical transmission along wires, *Bell Sys. Tech. Jour.*, 7, January 1928, pp. 11–25.

equation 46 and rearranging the terms, we obtain

$$\frac{\partial V}{\partial z} = F_z + E_z^i(z) - Z_i I. \tag{48}$$

We also have equation 40,

$$\frac{\partial F_z}{\partial z} = -\beta^2 V. \tag{49}$$

For perfectly conducting wires, equations 48 and 49 become

$$\frac{\partial V}{\partial z} = F_z + E_z^i(z), \qquad \frac{\partial F_z}{\partial z} = -\beta^2 V. \tag{50}$$

If the impressed electric intensity is restricted to small portions of the wires, then, *in regions free from the impressed field,*

$$\frac{\partial V}{\partial z} = F_z, \qquad \frac{\partial F_z}{\partial z} = -\beta^2 V. \tag{51}$$

Eliminating first F_z and then V, we have

$$\frac{\partial^2 V}{\partial z^2} = -\beta^2 V, \qquad \frac{\partial^2 F_z}{\partial z^2} = -\beta^2 F_z. \tag{52}$$

The solutions of these equations are sinusoidal.

In transmitting antennas the impressed voltage is localized in small regions. Hence, *in transmitting antennas the quasistatic potential and the dynamic component of the electric intensity are distributed sinusoidally.* Even the solutions of the more general system of equations 50 are relatively simple. Unfortunately, the current, which is a quantity of primary interest to us, does not enter equations 50 and 52 directly; it is merely related to F_z in a rather complicated manner, as shown by equations 43. Presently we shall replace these equations by much simpler approximate equations; but first we shall extend the present exact equations* to bent wires.

8.10 Exact equations for a system of bent wires

For a system of bent wires, equations 50 become

$$\frac{\partial V}{\partial s} = F_s + E_s^i(s), \qquad \frac{\partial F_s}{\partial s} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = -\beta^2 V, \tag{53}$$

where s is measured along the curved axis of a typical wire and x, y at right angles to s and to each other.

* Or rather "nearly exact" since we are neglecting the transverse currents on the wires.

The first equation expresses the condition that the total electric intensity tangential to a perfectly conducting wire vanishes, and the second is the general equation 42 connecting V and F. For imperfectly conducting wires we should subtract $Z_i I$ on the right side of the first equation in 53.

The second equation in 53, and also in 52, may be replaced by the continuity equation 38,

$$\frac{\partial I}{\partial s} = -\frac{\partial q}{\partial t} = -j\omega q, \tag{54}$$

expressing the fact that the current leaving an infinitesimal segment of the wire must equal the time rate of decrease of the charge on the segment. The linear charge density $q(s)$ is related to the potential as shown by equation 43.

8.11 Boundary conditions at junctions in a network of wires

The boundary conditions at a junction of several wires are:

1. *The sum of the currents entering (or leaving) a junction is zero.*
2. *The potential is continuous.*

The first of these conditions is merely an expression of the principle of conservation of charge. To prove the second we integrate the first equation in the set of equations 53 between two points A and B. Thus,

$$V(B) - V(A) = \int_{AB} F_s\, ds + \int_{AB} E_s^i\, ds. \tag{55}$$

By the mean value theorem,

$$V(B) - V(A) = F_s(s_1)\,\Delta s + E_s^i(s_2)\,\Delta s, \tag{56}$$

where Δs is the distance between A and B on the surface of the conductor and s_1, s_2 are two points between A and B. As A approaches B, Δs approaches zero and $V(B)$ approaches $V(A)$.

8.12 Boundary conditions across a localized generator

The boundary conditions across a localized generator are:

1. The current entering the generator through one terminal equals approximately the current leaving through the other terminal.
2. The potential rise across the generator is approximately equal to the impressed voltage.

The current entering the generator may not be exactly equal to the current leaving it because of a possible accumulation of charge within the generator. As the dimensions of the generator decrease, this charge

will also of necessity decrease. The second condition follows from equation 55 since the last term represents the impressed voltage. The next to the last term decreases as the size of the generator approaches zero.

In this discussion we have assumed that the path of integration is taken directly across the gap AB between the terminals of the generator and not through its interior; that is, the last term in equation 55 represents the voltage appearing between the terminals and not the total internal voltage. The latter is greater than the former by an amount equal to the voltage drop across the internal impedance of the generator.

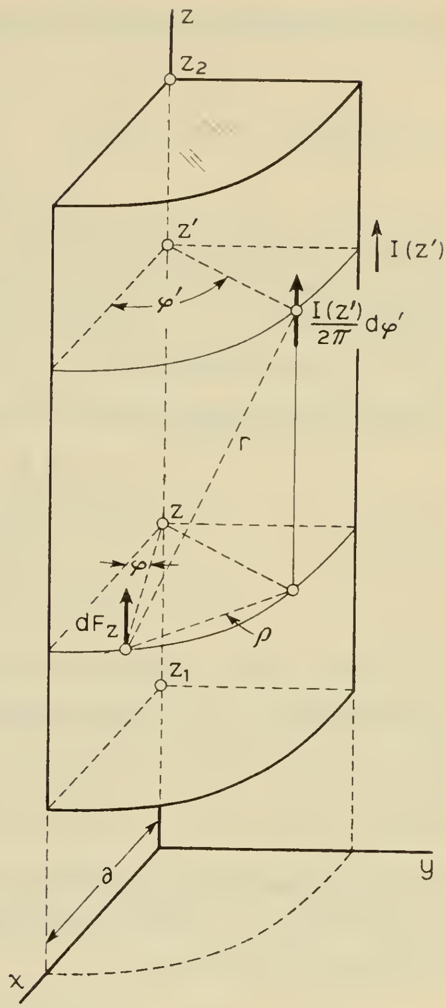

FIG. 8.3 A cylindrical current shell.

8.13 Asymptotic expressions for the field at the surface of a thin current filament

We shall now consider the field produced by a current distributed on the surface of a cylindrical shell of small radius a (Fig. 8.3). Let the cylindrical coordinates of a typical element of charge (and current) be (a, φ', z') and those of another typical point on the surface of the cylinder (a, φ, z). All the current elements are parallel, so that the angle $(s, z) = (z', z)$ appearing in equation 43 vanishes and its cosine is unity. Therefore, *on the surface of the cylinder*, extending from $z = z_1$ to $z = z_2$, equations 43 become, in view of equation 44,

$$
\begin{aligned}
V &= \frac{1}{\varepsilon}\int_{z_1}^{z_2}\int_0^{2\pi} \frac{q(z')\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi' - \\
&\qquad j\frac{1}{\varepsilon}\int_{z_1}^{z_2}\int_0^{2\pi} \frac{q(z')\sin\beta r}{8\pi^2 r}\,dz'\,d\varphi', \\
F_z &= -j\omega\mu\int_{z_1}^{z_2}\int_0^{2\pi} \frac{I(z')\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi' - \\
&\qquad \omega\mu\int_{z_1}^{z_2}\int_0^{2\pi} \frac{I(z')\sin\beta r}{8\pi^2 r}\,dz'\,d\varphi',
\end{aligned}
\tag{57}
$$

where the distance r between two typical points on the cylinder is

$$r = [(z' - z)^2 + 4a^2 \sin^2 \tfrac{1}{2}(\varphi' - \varphi)]^{1/2}. \tag{58}$$

When $z' = z$, we have

$$r = 2a\,|\sin \tfrac{1}{2}(\varphi' - \varphi)|. \tag{59}$$

Since a is small, this distance is small. Thus, the integrands in the first term of the above expressions for V and F_z are particularly large in the vicinity of $z' = z$. In fact, they increase indefinitely as a approaches zero. The integrands in the second terms remain finite. Hence, we have the following asymptotic equations:

$$\begin{aligned} V &\sim \frac{1}{\varepsilon}\int_{z_1}^{z_2}\int_0^{2\pi} \frac{q(z')\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi', \\ F_z &\sim -j\omega\mu\int_{z_1}^{z_2}\int_0^{2\pi} \frac{I(z')\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi'. \end{aligned} \tag{60}$$

Let us now assume *that $q(z')$ and $I(z')$ may be expanded in power series about $z' = z$, with remainder terms,*

$$\begin{aligned} q(z') &= q(z) + (z' - z)\,q'(z) + \tfrac{1}{2}(z' - z)^2\,q''(z) + \cdots + R_1(z', z), \\ I(z') &= I(z) + (z' - z)\,I'(z) + \tfrac{1}{2}(z' - z)^2\,I''(z) + \cdots + R_2(z', z). \end{aligned} \tag{61}$$

Such expansions exist except at the ends of the antenna or at a potential discontinuity, where the charge density is infinite. Substituting in equations 60, we have

$$\begin{aligned} V \sim \frac{1}{\varepsilon}\,q(z)&\int_{z_1}^{z_2}\int_0^{2\pi} \frac{\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi' + \\ &\frac{1}{\varepsilon}\,q'(z)\int_{z_1}^{z_2}\int_0^{2\pi} \frac{(z' - z)\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi' + \cdots, \\ F_z \sim -j\omega\mu\,I(z)&\int_{z_1}^{z_2}\int_0^{2\pi} \frac{\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi' - \\ &j\omega\mu\,I'(z)\int_{z_1}^{z_2}\int_0^{2\pi} \frac{(z' - z)\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi' + \cdots. \end{aligned} \tag{62}$$

Again we note that, as the radius of the current filament approaches zero, the first terms increase indefinitely while the other terms remain finite. Hence,

$$\begin{aligned} V &\backsim \frac{1}{\varepsilon}\,q(z)\int_{z_1}^{z_2}\int_0^{2\pi} \frac{\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi', \\ F_z &\sim -j\omega\mu\,I(z)\int_{z_1}^{z_2}\int_0^{2\pi} \frac{\cos\beta r}{8\pi^2 r}\,dz'\,d\varphi'. \end{aligned} \tag{63}$$

That part of the impressed field which is needed to compensate for F_z equals $-F_z$; the ratio

$$Z = \frac{-F_z}{I(z)} \tag{64}$$

will be called the series impedance per unit length of the wire. Thus,

$$Z = j\omega\mu \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\cos \beta r}{8\pi^2 r}\, dz'\, d\varphi'. \tag{65}$$

The ratio $q(z)/V(z)$ will be called the capacitance C per unit length; thus,

$$\frac{1}{C} = \frac{1}{\varepsilon} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\cos \beta r}{8\pi^2 r}\, dz'\, d\varphi'. \tag{66}$$

The expression

$$Y = j\omega C \tag{67}$$

will be called the shunt admittance per unit length of the wire. The asymptotic equations 63 may then be written as

$$F_z \sim -ZI, \qquad q \sim CV. \tag{68}$$

If the current filament is curved, we may proceed as above and obtain asymptotic expressions similar to equations 63,

$$\begin{aligned} V &\sim \frac{1}{\varepsilon}\, q(s) \int_{s_1}^{s_2} \int_0^{2\pi} \frac{\cos \beta r}{8\pi^2 r}\, ds'\, d\varphi', \\ F_s &\sim -j\omega\mu\, I(s) \int_{s_1}^{s_2} \int_0^{2\pi} \frac{\cos \beta r}{8\pi^2 r} \cos(s', s)\, ds'\, d\varphi', \end{aligned} \tag{69}$$

where r is the distance between two typical points on the surface of the filament.

8.14 Asymptotic equations for the potential and current in a thin wire

Substituting from equations 68 into equations 50 and noting that

$$q = -\frac{1}{j\omega} \frac{\partial I}{\partial z}, \tag{70}$$

we have (changing the partial derivatives into ordinary derivatives since q and I are functions of z only),

$$\frac{dV}{dz} = -ZI + E^i(z), \qquad \frac{dI}{dz} = -YV. \tag{71}$$

Where there is no impressed field, we have

$$\frac{dV}{dz} = -ZI, \qquad \frac{dI}{dz} = -YV. \tag{72}$$

These are ordinary transmission line equations.

Since Z and Y are purely imaginary, there is no dissipation of power. This means that equations 72 do not take into consideration the radiation of power. This is not surprising since these equations have been obtained on the assumption that the radius approaches zero. In this case the energy density in the vicinity of the wire, for a given current, increases indefinitely; so does the total stored energy. Hence, if we keep the stored energy constant, the current must diminish as the radius approaches zero. Consequently, the distant field and, therefore, the radiated power will approach zero with the radius of the antenna. Calculations would show, however, that the approach to the limit is extremely slow, and that even for the thinnest wires available in practice the radiation is not negligible. Thus, we shall not be satisfied with the first approximation obtainable directly from equations 72; and in Chapter 13 we shall discuss a method of using equations 72 to obtain a second approximation which takes radiation into account.

From equations 72 we obtain the secondary parameters of the wire, the characteristic impedance Z_0, and the propagation constant Γ. Thus, from equations 65, 66, and 67, we find

$$\begin{aligned} Z_0 = \sqrt{\frac{Z}{Y}} &= \sqrt{\frac{\mu}{\varepsilon}} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\cos \beta r}{8\pi^2 r}\, dz'\, d\varphi' \\ &= 30 \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\cos \beta r}{2\pi r}\, dz'\, d\varphi', \end{aligned} \tag{73}$$

and

$$\Gamma = \sqrt{ZY} = j\omega\sqrt{\mu\varepsilon} = j\beta = j\,\frac{2\pi}{\lambda}\cdot \tag{74}$$

Thus, the propagation of waves along the wire takes place with the velocity of light.

In the next section we shall find

$$Z_0(z) = 60\left(\log \frac{\lambda}{2\pi a} + 0.116\right) + 30[\operatorname{Ci} \beta(z - z_1) + \operatorname{Ci} \beta(z_2 - z)]. \tag{75}$$

The Ci function is large for small values of the argument and approaches zero for large values. Hence, the terms in square brackets are important near the ends of the wire but not elsewhere. The corresponding

expressions for the primary parameters are

$$Z = j\omega L, \qquad Y = j\omega C, \qquad C = \frac{\mu\varepsilon}{L}, \tag{76}$$

$$L = \frac{\mu}{2\pi}\left(\log\frac{\lambda}{2\pi a} + 0.116\right) + \frac{\mu}{4\pi}\left[\mathrm{Ci}\,\beta(z - z_1) + \mathrm{Ci}\,\beta(z_2 - z)\right].$$

As a approaches zero the constant term in the above equations increases, while the term depending on z is unaltered. Thus, we make the final step and drop the variable terms in L, C, and Z_0. *The limiting forms of current and potential distributions are thus sinusoidal.*

8.15 Evaluation of the characteristic impedance of a wire*

We shall now evaluate the characteristic impedance given by equation 73. The expression 58 for r we shall write as follows:

$$r = [(z' - z)^2 + \rho^2]^{1/2}, \qquad \rho = 2a \sin \tfrac{1}{2}(\varphi' - \varphi). \tag{77}$$

First we shall integrate with respect to z'. If we write

$$\frac{\cos \beta r}{r} = \frac{1}{r} - \frac{1 - \cos \beta r}{r}, \tag{78}$$

we find that the last term is small when r is small. Hence, in this term we may approximate r by $|z' - z|$; for, if $|z' - z|$ is much larger than ρ, then ρ is negligible; and, if $|z' - z|$ is comparable to ρ or less than ρ, then r is small and the entire term is small. Therefore,

$$\frac{\cos \beta r}{r} \simeq \frac{1}{r} - \frac{1 - \cos \beta(z' - z)}{|z' - z|}. \tag{79}$$

The denominator in the second term equals $z' - z$ when z' is greater than z; otherwise, it equals $z - z'$. Therefore, in integrating equation 79 we must divide the range of integration into two intervals, one from $z' = z_1$ to $z' = z$ and the other from $z' = z$ to $z' = z_2$. Thus,

$$\begin{aligned}\int_{z_1}^{z_2} \frac{\cos \beta r}{r}\, dz' &= \int_{z_1}^{z_2} \frac{dz'}{[(z'-z)^2+\rho^2]^{1/2}} - \int_{z_1}^{z_2} \frac{1-\cos \beta(z'-z)}{|z'-z|}\, dz' \\ &= \int_{z_1}^{z_2} \frac{dz'}{[(z'-z)^2+\rho^2]^{1/2}} - \int_{z_1}^{z} \frac{1-\cos \beta(z'-z)}{z-z'}\, dz' - \\ &\qquad \int_{z}^{z_2} \frac{1-\cos \beta(z'-z)}{z'-z}\, dz'. \end{aligned} \tag{80}$$

* In antenna theory the value of the "characteristic impedance" depends somewhat on the particular method of analysis (see Section 13.14).

In the first and last integrals, we substitute $z' - z = t$; in the second, $z - z' = t$. Thus, we find

$$\int_{z_1}^{z_2} \frac{\cos \beta r}{r}\, dz' = \int_{z_1 - z}^{z_2 - z} \frac{dt}{[t^2 + \rho^2]^{1/2}} - \int_0^{z - z_1} \frac{1 - \cos \beta t}{t}\, dt - \int_0^{z_2 - z} \frac{1 - \cos \beta t}{t}\, dt. \tag{81}$$

To integrate the first term, we note that

$$d[t + (t^2 + \rho^2)^{1/2}] = dt + t(t^2 + \rho^2)^{-1/2}\, dt = \frac{[t + (t^2 + \rho^2)^{1/2}]\, dt}{(t^2 + \rho^2)^{1/2}}; \tag{82}$$

hence,

$$\frac{dt}{(t^2 + \rho^2)^{1/2}} = \frac{d[t + (t^2 + \rho^2)^{1/2}]}{t + (t^2 + \rho^2)^{1/2}}. \tag{83}$$

Integrating, we obtain

$$\int_{z_1 - z}^{z_2 - z} \frac{dt}{(t^2 + \rho^2)^{1/2}} = \log[t + (t^2 + \rho^2)^{1/2}] \Big|_{z_1 - z}^{z_2 - z}$$

$$= \log \frac{z_2 - z + [(z_2 - z)^2 + \rho^2]^{1/2}}{z_1 - z + [(z_1 - z)^2 + \rho^2]^{1/2}}. \tag{84}$$

The denominator of the logarithm in equation 84 is small because $z_1 - z$ is negative, whereas the second term is positive and of nearly equal magnitude. A more convenient form is obtained if we multiply the numerator and the denominator of the fraction by

$$z - z_1 + [(z - z_1)^2 + \rho^2]^{1/2}.$$

Thus, we find

$$\int_{z_1 - z}^{z_2 - z} \frac{dt}{(t^2 + \rho^2)^{1/2}}$$

$$= \log \frac{1}{\rho^2} \{z_2 - z + [(z_2 - z)^2 + \rho^2]^{1/2}\} \{z - z_1 + [(z - z_1)^2 + \rho^2]^{1/2}\}$$

$$\simeq \log \frac{4(z_2 - z)(z - z_1)}{\rho^2}. \tag{85}$$

The final approximation is good except very close to the ends of the wire.

The second and third terms in equation 81 are cosine integrals.

Hence,

$$\int_{z_1}^{z_2} \frac{\cos \beta r}{r}\, dz'$$

$$= \log \frac{4(z_2 - z)(z - z_1)}{\rho^2} - \operatorname{Cin} \beta(z - z_1) - \operatorname{Cin} \beta(z_2 - z)$$

$$= 2 \log \frac{\lambda}{2\pi\rho} + 2 \log 2 - 2C + \operatorname{Ci} \beta(z - z_1) + \operatorname{Ci} \beta(z_2 - z), \quad (86)$$

where $C = 0.577\cdots$ is Euler's constant. The last two terms are small compared with the first except near the ends of the wire. Hence, if the end effect is neglected, the integral depends only on λ and the radius a of the wire that occurs in ρ.

To obtain Z_0 as defined by equation 73, we have to integrate equation 86 with respect to φ'; in effect, we have to obtain the average value of the integral round the circumference of the cylinder,

$$\operatorname{av} \int_{z_1}^{z_2} \frac{\cos \beta r}{r}\, dz' = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \int_{z_1}^{z_2} \frac{\cos \beta r}{r}\, dz'. \quad (87)$$

The only term depending on φ' is $\log \rho$. Since the average value is independent of φ, we may let $\varphi = 0$ in equation 77. Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} \log \rho\, d\varphi' = \frac{1}{2\pi} \int_0^{2\pi} \log 2a\, d\varphi' + \frac{1}{2\pi} \int_0^{2\pi} \log \sin \tfrac{1}{2}\varphi'\, d\varphi'$$

$$= \log 2a - \log 2 = \log a. \quad (88)$$

Thus, the *average value of* $\log \rho$ *round the circumference of the circle equals the logarithm of the radius of the circle.* We shall frequently use this result.

Thus we have found

$$\frac{1}{2\pi} \int_{z_1}^{z_2} \int_0^{2\pi} \frac{\cos \beta r}{r}\, dz'\, d\varphi'$$

$$= 2 \log \frac{\lambda}{2\pi a} + 2 \log 2 - 2C + \operatorname{Ci} \beta(z - z_1) + \operatorname{Ci} \beta(z_2 - z). \quad (89)$$

Substituting in equation 73 we obtain equation 75.

8.16 Asymptotic form of the potential and current distribution on thin wires

The numerical results obtained in the preceding section indicate that, whether a wire is straight or curved, whether it is isolated or surrounded

by other wires, the relative variations in the distributed parameters Z, Y decrease with the decreasing radius of the wire. The effects due to the curvature and the environment are independent of a and thus become smaller in comparison with the term involving $\log(\lambda/2\pi a)$, which expresses the effect of the size of the wire. Hence, the asymptotic equations governing the distribution of potential and current on all wires, straight or curved, are

$$\frac{dV}{ds} = -j\omega LI + E^i(s), \qquad \frac{dI}{ds} = -j\omega CV, \tag{90}$$

where L and C are constants based on the average value of the integral in the preceding section (equation 89). Thus,

$$\begin{aligned} L &= \frac{\mu}{2\pi}\left[\log\frac{\lambda}{2\pi a} + 0.116 + \operatorname{Ci}\beta(z_2 - z_1) - \frac{\sin\beta(z_2 - z_1)}{\beta(z_2 - z_1)}\right], \\ C &= \frac{\mu\varepsilon}{L}, \\ Z_0 &= \sqrt{\frac{L}{C}} \\ &= 60\left[\log\frac{\lambda}{2\pi a} + 0.116 + \operatorname{Ci}\beta(z_2 - z_1) - \frac{\sin\beta(z_2 - z_1)}{\beta(z_2 - z_1)}\right]. \end{aligned} \tag{91}$$

In antenna theory we are interested primarily in wires whose length is comparable to $\lambda/2$ or greater than $\lambda/2$, so that the last two terms tend to become negligible as a decreases. If $\lambda \gg 2\pi(z_2 - z_1)$, we have approximately

$$\operatorname{Ci}\beta(z_2 - z_1) \simeq \log\beta(z_2 - z_1) + 0.577.$$

Hence,

$$\begin{aligned} Z_0 &= 60\left[\log\frac{\lambda}{2\pi a} + \log\frac{2\pi(z_2 - z_1)}{\lambda} + \log 2 - 1\right] \\ &= 60\left[\log\frac{2(z_2 - z_1)}{a} - 1\right]. \end{aligned} \tag{92}$$

Thus, at sufficiently low frequencies Z_0 is independent of λ. Nevertheless, $\log(\lambda/2\pi a)$ is ultimately the dominant term if λ and $z_2 - z_1$ are kept fixed while a is permitted to approach zero. It is very important to remember that Z_0 is a function of three parameters: the wavelength λ, the length of the wire $z_2 - z_1$, and the radius a. No general statement can be made with regard to its order of magnitude unless some of these parameters are kept fixed. Because of the approximations in-

volved in equation 92, a should always be substantially smaller than either of the other two parameters.

Along any portion of the wire where the impressed field vanishes, the general solution of equations 90 is

$$I(s) = A \cos \beta s + B \sin \beta s,$$
$$V(s) = -\frac{1}{j\omega C}\frac{dI}{ds} = -jZ_0A \sin \beta s + jZ_0B \cos \beta s. \tag{93}$$

These are the asymptotic forms of current and potential distribution. The constants of integration are determined from the boundary conditions at the generator, at the junctions, and at the ends of the wire.

8.17 Coordinates on split wires

Linear antennas are not continuous wires; they have terminals and junctions with the feed lines. If the antenna is disconnected from the feed line, it becomes a discontinuous structure. Even when we imagine

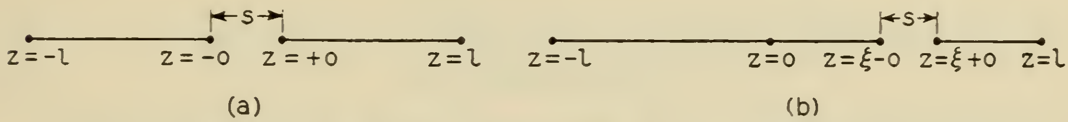

FIG. 8.4 Coordinates on split linear segments.

a small-sized generator between the terminals, as we often do in the sectional analysis of all transmission systems, we still have a discontinuous physical structure, since the properties of the region occupied by the generator are quite different from those of the external region. It is sometimes convenient to use continuous coordinates on such discontinuous line segments (Fig. 8.4).

Thus, in Fig. 8.4a the origin of the coordinate system is split and is represented by two points $z = -0$ and $z = +0$, irrespective of the distance s between these points. Similarly, in a segment split off the origin (Fig. 8.4b), one end of the gap is given by $z = \xi - 0$ and the other by $z = \xi + 0$. The direction from $z = \xi - 0$ to $z = \xi + 0$ should coincide with the positive z direction.

Split coordinates are chiefly useful in describing the potential and current distribution in linear antennas. When considering the field outside the antenna, the coordinates should be changed to the usual system. If, for instance, the origin of the usual system is in the center of the gap (Fig. 8.4a), the transformation equations are:

$$z' = z + \tfrac{1}{2}s, \qquad z > 0;$$
$$z' = z - \tfrac{1}{2}s, \qquad z < 0. \tag{94}$$

If the origin of the normal system is at $z' = 0$, (Fig. 8.4b), then,

$$\begin{aligned} z' &= z, \qquad & z < \xi; \\ z' &= z + s, \qquad & z > \xi. \end{aligned} \tag{95}$$

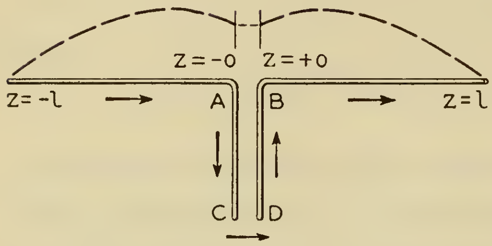

FIG. 8.5 Asymptotic current distribution in a symmetrically fed dipole antenna.

8.18 Asymptotic current distributions in dipole antennas

Asymptotic current distributions are simple and easy to determine. As our first example we shall take an antenna consisting of two wires making a 180° angle. We assume that this antenna is connected to a two-wire transmission line (Fig. 8.5). The generator is connected across the terminals C, D. In each antenna arm the current is sinusoidally distributed; hence,

$$\begin{aligned} I(z) &= A_1 \cos \beta z + B_1 \sin \beta z, \qquad & z > +0; \\ &= A_2 \cos \beta z + B_2 \sin \beta z, \qquad & z < -0. \end{aligned} \tag{96}$$

At the floating ends* $z = \pm l$, the current must vanish in the limit,†

$$I(l) = I(-l) = 0. \tag{97}$$

The network is symmetric, and, if the generator is balanced with respect to the feeder line at C, D, the currents in AC and BD must be equal and opposite; hence, the antenna current entering A must equal that leaving B,

$$I(-0) = I(+0). \tag{98}$$

From equations 96 and 97, we have

$$A_1 \cos \beta l + B_1 \sin \beta l = 0, \qquad A_2 \cos \beta l - B_2 \sin \beta l = 0. \tag{99}$$

From the first equation we can express B_1 in terms of A_1 or both in terms of some new constant I_1. The second equation is similar. Thus,

$$\begin{aligned} A_1 &= I_1 \sin \beta l, \qquad & B_1 &= -I_1 \cos \beta l; \\ A_2 &= I_2 \sin \beta l, \qquad & B_2 &= I_2 \cos \beta l. \end{aligned} \tag{100}$$

Substituting in equation 96, we find

$$\begin{aligned} I(z) &= I_1 \sin \beta(l - z), \qquad & z > +0; \\ &= I_2 \sin \beta(l + z), \qquad & z < -0. \end{aligned} \tag{101}$$

* Free ends or ends not connected to any other conductor or generator.

† The end effect will be considered in Section 8.23.

Using equation 98, we find $I_2 = I_1 = I_0$, and

$$\begin{aligned} I(z) &= I_0 \sin \beta(l - z), \qquad && z > +0; \\ &= I_0 \sin \beta(l + z), && z < -0. \end{aligned} \tag{102}$$

Exactly the same asymptotic form of current distribution is obtained for a pair of inclined wires (Fig. 8.6*a*) or a pair of parallel wires

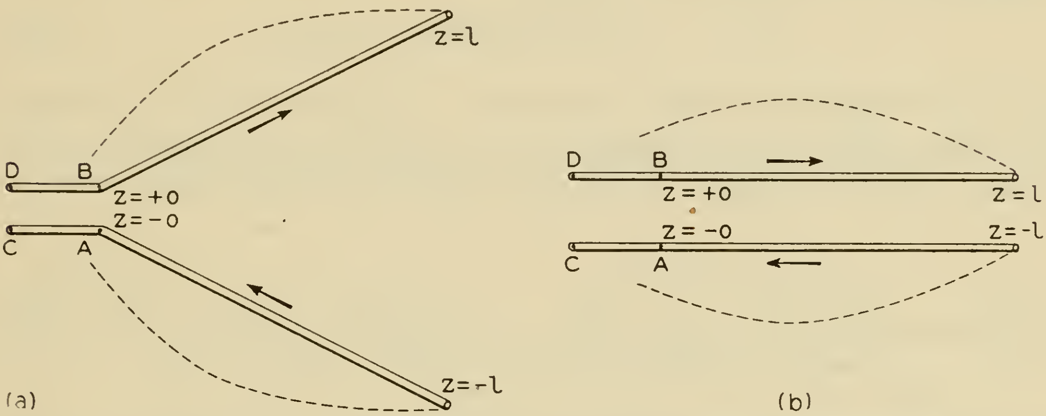

FIG. 8.6 Asymptotic current distribution (*a*) in a symmetrically fed V antenna and (*b*) in a parallel pair.

(Fig. 8.6*b*); the reason is that in all these cases the boundary conditions at the floating ends and across the generator are the same. Because of the cancelation of distant fields produced by closely spaced oppositely directed current elements, the parallel pair will radiate much less than the diverging wires; and its current distribution will be less affected by radiation.

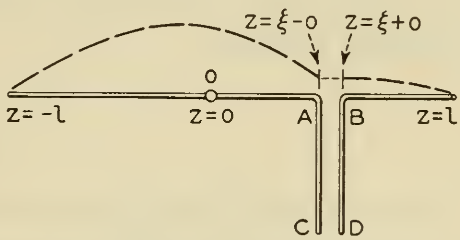

FIG. 8.7 Asymptotic current distribution in an asymmetrically fed dipole antenna.

In the asymmetric case (Fig. 8.7) we shall simplify the procedure by starting with sinusoidal functions which vanish at $z = \pm l$; thus,

$$\begin{aligned} I(z) &= I_1 \sin \beta(l - z), \qquad && z > \xi; \\ &= I_2 \sin \beta(l + z), && z < \xi. \end{aligned} \tag{103}$$

From the continuity of current at $z = \xi$, we have

$$I_1 \sin \beta(l - \xi) = I_2 \sin \beta(l + \xi). \tag{104}$$

To satisfy this equation, we write

$$I_1 = A \sin \beta(l + \xi), \qquad I_2 = A \sin \beta(l - \xi). \tag{105}$$

Substituting in equation 103, we obtain

$$\begin{aligned} I(z) &= A \sin \beta(l + \xi) \sin \beta(l - z), \qquad & z > \xi; \\ &= A \sin \beta(l - \xi) \sin \beta(l + z), \qquad & z < \xi. \end{aligned} \tag{106}$$

The asymmetric case requires further consideration when the length of the antenna equals an integral number of wavelengths. If, for example, $2l = \lambda$, equation 106 gives

$$I(z) = -A \sin \beta\xi \sin \beta z = I_0 \sin \beta z \tag{107}$$

for either positive or negative values of z. This distribution is shown in Fig. 8.8*a*. The current vanishes at $z = 0$ and flows in opposite directions on the two sides of this point. The form of distribution seems to

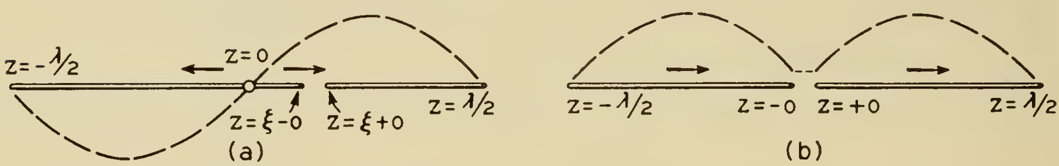

FIG. 8.8 Asymptotic current distributions in full-wave dipole antennas: (*a*) asymmetrically fed; (*b*) symmetrically fed.

be independent of the position $z = \xi$ of the generator. Suppose, however, that the generator is in the center. By symmetry we must have the distribution of the form shown in Fig. 8.8*b*. Likewise, from equation 102, we have

$$\begin{aligned} I(z) &= I_0 \sin \beta z, \qquad & z > 0; \\ &= -I_0 \sin \beta z, \qquad & z < 0. \end{aligned} \tag{108}$$

There appears to be a sudden change in the form of current distribution as the generator is shifted to one side from the exact center. We should, however, remember that we are dealing with limiting forms, and some peculiarities are to be expected. In addition to the above sinusoidal terms we shall have complementary terms. For very thin wires the latter are small; but at points where the principal current is nearly equal to zero even small complementary terms will be important. The transition from the form in Fig. 8.8*b* to that in Fig. 8.8*a* is gradual. The rapidity of transition increases as the radius of the antenna decreases.

The complementary terms, being in the nature of correction terms, are frequently obtained in one way or another by using the principal terms. In the above-mentioned singular cases the question will arise as to which of the two forms should be used as the basis for calculating the complementary terms. This difficulty may be overcome by considering the asymmetric feed in Fig. 8.7 as the resultant of the symmetric feed in

Fig. 8.9*a* and the antisymmetric feed in Fig. 8.9*b*. At $z = -\xi$ the resultant impressed voltage vanishes so that the entire impressed voltage is at $z = \xi$. If the length of the antenna is an integral number of wavelengths, it will be found that the symmetric component for a given voltage is much larger than the antisymmetric component when ξ is small; but, as ξ increases, this relationship is gradually reversed.

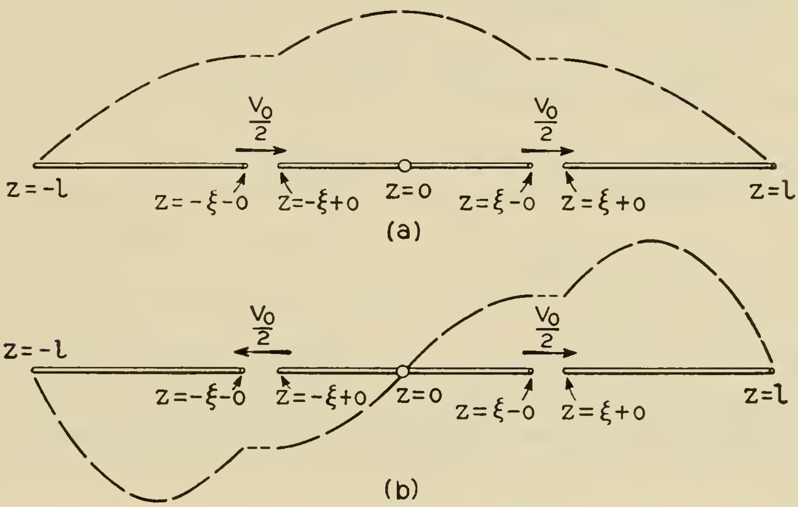

FIG. 8.9 Asymptotic current distributions (*a*) in a symmetrically fed antenna and (*b*) in an antisymmetrically fed antenna.

The following are the asymptotic expressions for the two modes of current distribution:

Symmetric mode (Fig. 8.9*a*):

$$\begin{aligned} I(z) &= A \cos \beta\xi \sin \beta(l - z), & \xi \le z \le l; \\ &= A \sin \beta(l - \xi) \cos \beta z, & -\xi \le z \le \xi; \\ &= A \cos \beta\xi \sin \beta(l + z), & -l \le z \le -\xi. \end{aligned} \tag{109}$$

Antisymmetric mode (Fig. 8.9*b*):

$$\begin{aligned} I(z) &= A \sin \beta\xi \sin \beta(l - z), & \xi \le z \le l; \\ &= A \sin \beta(l - \xi) \sin \beta z, & -\xi \le z \le \xi; \\ &= -A \sin \beta\xi \sin \beta(l + z), & -l \le z \le -\xi. \end{aligned} \tag{110}$$

The derivation is left to the reader.

8.19 Asymptotic current distribution in reflecting antennas

By a *reflecting antenna* we mean simply a wire in an electromagnetic field. In a *receiving antenna* there is a passive impedance, a "load," in place of the generator of a transmitting antenna. Reflecting antennas

are receiving antennas in which the terminals of the load are short-circuited.

To obtain the current in a reflecting antenna, we note first that the current in an infinitely long wire in a uniform field parallel to the wire is constant. In a finite wire the current must vanish at the ends, which must, therefore, act as virtual sources of waves on the wire. The electric charge associated with the uniform current would tend to concentrate at the ends, and the increasing force due to it would oppose the flow of charge. To summarize: (1) the incident uniform field may be considered as a uniform distribution of generators producing a uniform

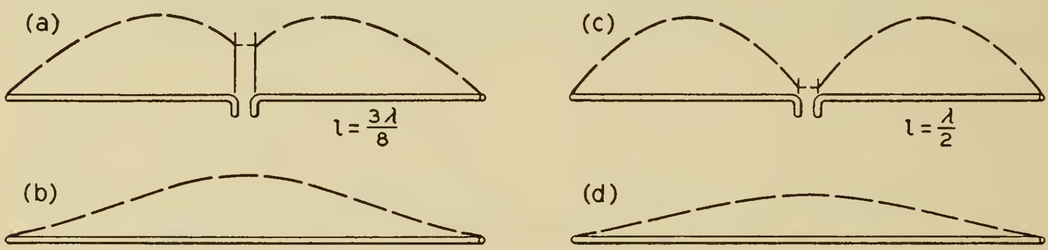

FIG. 8.10 Comparison between asymptotic current distributions in transmitting antennas (a, c) and the corresponding reflecting antennas (b, d).

current; (2) the ends of the wire act as point generators, and the current distribution due to them is, according to the preceding pages, sinusoidal. The total current in a wire extending from $z = -l$ to $z = l$ due to a uniform field parallel to the wire is, thus,

$$I(z) = A + B \cos \beta z. \tag{111}$$

There can be no sine term because $I(-z) = I(z)$. Since $I(-l) = I(l) = 0$, we have

$$A = -B \cos \beta l, \qquad B = -\frac{A}{\cos \beta l}\cdot \tag{112}$$

Hence,

$$I(z) = B(\cos \beta z - \cos \beta l) = A\,\frac{\cos \beta l - \cos \beta z}{\cos \beta l}\cdot \tag{113}$$

In terms of the current at the center of the wire, $z = 0$, we have

$$I(z) = I(0)\,\frac{\cos \beta z - \cos \beta l}{1 - \cos \beta l}\cdot \tag{114}$$

Figure 8.10 illustrates the difference in the forms of current distribution in transmitting and reflecting antennas. When $l = \lambda/4$, these current distributions are alike — to the present order of approximation.

8.20 Asymptotic current distribution in receiving antennas

In a receiving antenna there is a load Z. The voltage across this load is ZI, and the load acts as a virtual generator with an internal emf equal to $-ZI$. Therefore, the current in a receiving antenna in response to a uniform field parallel to the antenna will be the sum of a current of the form peculiar to the reflecting antenna and a current of the form peculiar to the transmitting antenna. Thus, for a load in the center, $z = 0$, the current is

$$I(z) = I_0 \sin \beta(l - |z|) + B(\cos \beta z - \cos \beta l). \tag{115}$$

The constants I_0 and B depend on the impressed field, the length of the wire, the wavelength, the radius of the wire, and the impedance of the load.

8.21 Asymptotic potential distribution in dipole antennas

To obtain the asymptotic distribution of potential we differentiate the current and multiply the result by an appropriate factor, as in equation 93. Thus, from equation 102 we find that, for symmetrically fed dipole antennas,

$$\begin{aligned} V(z) = -\frac{1}{j\omega C}\frac{dI}{dz} &= -jZ_0 I_0 \cos \beta(l - z), \qquad z > 0; \\ &= jZ_0 I_0 \cos \beta(l + z), \qquad z < 0. \end{aligned} \tag{116}$$

Since the sine and cosine waves are displaced with respect to each other by one-quarter wavelength, the maximum potentials will be at the floating ends of the antenna and at the other current nodes. Points of zero potential will coincide with points of maximum current amplitude.

8.22 Effects of radiation on the antenna current

We have already discussed the fact that the asymptotic current distributions do not take radiation into account. Equations 102 and 116 provide another demonstration, since the input impedance obtained from them is a pure reactance,

$$Z_i = \frac{V(+0) - V(-0)}{I(0)} = -2jZ_0 \cot \beta l. \tag{117}$$

In Section 5.19 we have found that the wave excited by a current element produces an electric intensity, one component of which is 180° out of phase with the current. Thus, the wave introduces a drag on the moving charge generating it. This component does not vary much with the distance from the element as long as this distance is substantially smaller than $\lambda/2$. Hence, the reaction field due to radiation from an-

tennas that are not too long is approximately constant, and the current induced by it in the antenna will be of the same shape as in the reflecting antenna. Hence, a more accurate form of the transmitting antenna current is

$$I(z) = I_0 \sin \beta(l - |z|) + jkI_0(\cos \beta z - \cos \beta l), \tag{118}$$

where j has been introduced to exhibit the fact that radiation introduces a quadrature component into the antenna current. The coefficient k slowly decreases as the radius of the antenna decreases.

When $l = \lambda/4$, the two terms merge into one,

$$I(z) = (1 + jk)I_0 \cos \beta z. \tag{119}$$

If $l = \lambda/2$,

$$I(z) = I_0 \sin \beta|z| + jkI_0(1 + \cos \beta z). \tag{120}$$

Hence in this case it is the second term that is important when z is small.

The second term in equation 118 is the "feed current" which supplies the radiated power to the antenna. In Section 11.18 we shall return to this subject and supply the missing information about the magnitude of k.

8.23 End effects in dipole transmitting antennas

In deriving the asymptotic form of the current in dipole antennas, we assumed that at the floating ends the current vanished (see equation 97). This is the true boundary condition in the limit, as the radius of the wire vanishes. Our object, however, is to obtain a working approximation for antennas with finite dimensions, and we must make an allowance for the charge on the flat ends of the wire. Since these flat ends or "caps" are small, the accumulated charge may be calculated from electrostatic equations. The capacitance of a disk of radius a is $8\varepsilon a$; hence, the capacitance of one face of the disk equals $4\varepsilon a$. Thus, the current entering the cap at $z = l$ is $4j\omega\varepsilon a\, V(l)$, where $V(l)$ is the potential of the end $z = l$. We shall thus write

$$I_{\text{cap}} = j\omega C_{\text{cap}} V(l) = 4j\omega\varepsilon a\, V(l). \tag{121}$$

There is also a "fringing effect" near the ends of the wire. The electric lines of force become crowded toward the ends and bulge out as illustrated in Fig. 8.11a. This is due to a higher concentration of charge near the ends, which implies a greater capacitance per unit length. To understand this we note that at any intermediate point a charged particle is subject to a force exerted by the charged particles on both sides of it, while at the end it is acted upon only by the charge on one side. That is, if we assume that the charge is distributed uniformly along the wire, the potential at the intermediate points is twice as large as the potential at

the end. Now near the ends the potential must be substantially constant, since its gradient is proportional to the current and the current is small. Hence, more charge will be pushed toward the ends, which is another way of saying that near the ends the capacitance per unit length is greater.

The sinusoidal form of potential and current distribution is characteristic of a uniform distribution of the inductance and capacitance. Hence, we must expect a deviation from the sinusoidal form near the

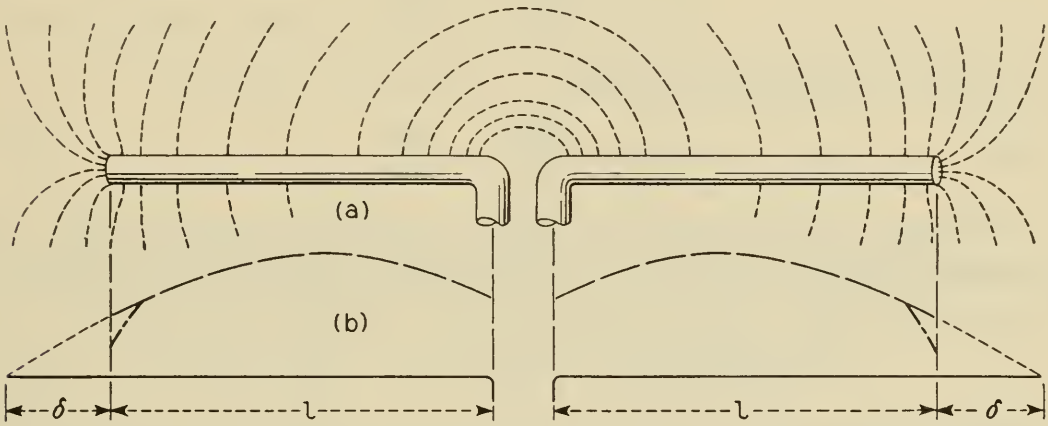

Fig. 8.11 The capacitance of the flat ends of a wire and the excess of capacitance per unit length near the ends over that of the intermediate sections of the wire combine to increase the effective length of the wire.

ends of the antenna. However, the end effect is large only in the immediate vicinity of the ends and can be represented, therefore, by a lumped capacitance at the ends, added to the cap capacitance. That is, the boundary condition at the ends, as far as our principal current component is concerned, is

$$I(l) = j\omega(C_t + C_{\text{cap}})\,V(l), \tag{122}$$

where C_t is the effective capacitance due to the fringing and C_{cap} is the capacitance of the cap.

These end effects effectively lengthen the antenna as shown by the dashed curve for the current in Fig. 8.11*b*. To obtain the effective extension δ in the length of each arm, we equate the end capacitance to the capacitance $C\delta$, where C is the average capacitance per unit length as given by equations 91. Thus,

$$C\delta = C_t + C_{\text{cap}}, \qquad \delta = \frac{C_t + C_{\text{cap}}}{C}. \tag{123}$$

The fringing effect we shall obtain in Chapter 13. The part of δ

due to the cap capacitance may be obtained from equation 121. Thus,

$$\delta_{\text{cap}} = \frac{C_{\text{cap}}}{C} = \frac{\omega C_{\text{cap}}}{\omega C}. \tag{124}$$

Since

$$\omega C = \frac{\omega\sqrt{LC}}{\sqrt{L/C}} = \frac{2\pi}{\lambda Z_0},$$
$$\omega\varepsilon = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{\mu/\varepsilon}} = \frac{2\pi}{\eta\lambda}, \tag{125}$$

we have, in view of equation 121,

$$\delta_{\text{cap}} = \frac{4aZ_0}{\eta} = \frac{aZ_0}{30\pi}. \tag{126}$$

Taking the end effects into account, we replace the expressions 102 for the asymptotic current in a symmetrically fed dipole antenna by

$$\begin{aligned} I(z) &= I_0 \sin\beta(l + \delta - z), && 0 < z < l; \\ &= I_0 \sin\beta(l + \delta + z), && -l < z < 0. \end{aligned} \tag{127}$$

Although δ is small, it has a direct effect on the position of the resonant and antiresonant points. For very thin antennas the effect of the cap capacitance becomes negligible, but the fringing effect remains noticeable even for the smallest practicable dimensions.

8.24 Current distribution in inductively and capacitively loaded antennas

If an impedance Z is inserted in a wire between $z = \xi - 0$ and $z = \xi + 0$ (Fig. 8.12), the potential drop is

$$V_{AB} = V(\xi - 0) - V(\xi + 0) = Z\,I(\xi). \tag{128}$$

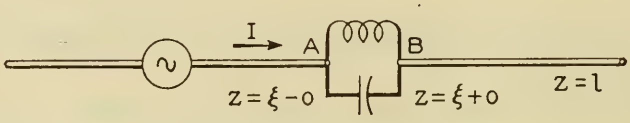

FIG. 8.12 An antenna loaded with an antiresonant circuit.

Let us begin at an open end, $z = l$, for instance. Expressing the end effect by an effective extension of the antenna to $z = l + \delta$, we have

$$I(z) = A \sin\beta(l + \delta - z), \qquad \xi < z < l. \tag{129}$$

For the potential, we find

$$V(z) = -jZ_0 A \cos\beta(l + \delta - z), \qquad \xi < z < l. \tag{130}$$

To the left of $z = \xi - 0$, we have another sinusoidal form:

$$\begin{aligned} I(z) &= B \sin \beta z + D \cos \beta z, \qquad z < \xi, \\ V(z) &= jZ_0 B \cos \beta z - jZ_0 D \sin \beta z. \end{aligned} \tag{131}$$

As implied by equation 128, we assume that the capacitance of the impedor Z to the antenna is negligible so that the current entering the impedor equals that leaving it,

$$B \sin \beta\xi + D \cos \beta\xi = A \sin \beta(l + \delta - \xi). \tag{132}$$

From equations 128, 130, 131, and 132, we have

$$B \cos \beta\xi - D \sin \beta\xi + A \cos \beta(l + \delta - \xi) = \frac{Z}{jZ_0} A \sin \beta(l + \delta - \xi). \tag{133}$$

From the last two equations we can express B and D in terms of A. Equations 131 will hold until we reach another discontinuity; then the above procedure is repeated until we reach the other end where we use the appropriate boundary condition given by equation 122.

8.25 Asymptotic forms of current distribution in loosely coupled networks of conductors

We shall now consider networks of conductors such as the *shunt-excited* antennas shown in Figs. 8.13*a* and *b*. In this section we shall consider only those parts of the wire network that are loosely coupled, so that the

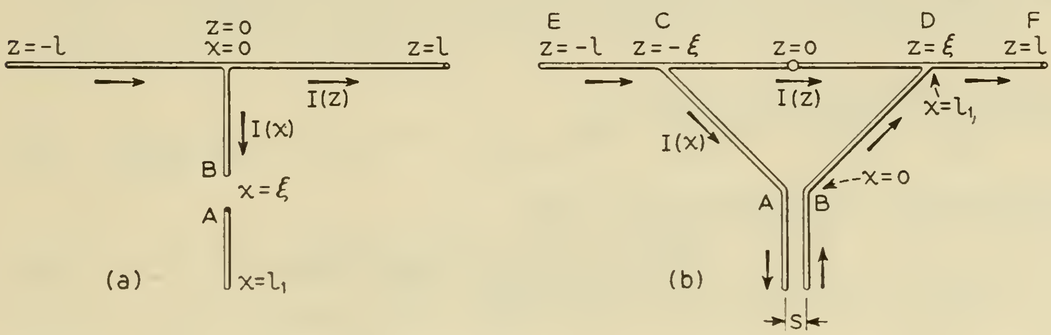

FIG. 8.13 Shunt-excited antennas.

characteristic impedance of each wire is not seriously affected by the proximity to the other wires; that is, we shall exclude the parallel pair feeding the antenna in Fig. 8.13*b*. We shall also assume that the radii of the wires are the same so that their characteristic impedances are the same. In this case the *continuity of the potential at each junction of the network implies the continuity of the derivative of the current.*

For the network in Fig. 8.13*a*, we write

$$\begin{aligned} I(z) &= I_1 \sin \beta(l - z), & 0 \le z \le l; \\ &= I_2 \sin \beta(l + z), & -l \le z \le 0; \\ I(x) &= I_3 \cos \beta x + I_4 \sin \beta x, & 0 \le x \le \xi; \\ &= I_5 \sin \beta(l_1 - x), & \xi \le x \le l_1. \end{aligned} \tag{134}$$

From the continuity of current at $x = \xi$, we have

$$I_3 \cos \beta\xi + I_4 \sin \beta\xi = I_5 \sin \beta(l_1 - \xi). \tag{135}$$

The total current leaving the junction, $z = 0$, $x = 0$, must vanish, and the derivatives of the currents in the various branches must be continuous; therefore,

$$\begin{aligned} I_1 \sin \beta l - I_2 \sin \beta l + I_3 &= 0, \\ -I_1 \cos \beta l = I_2 \cos \beta l &= I_4. \end{aligned} \tag{136}$$

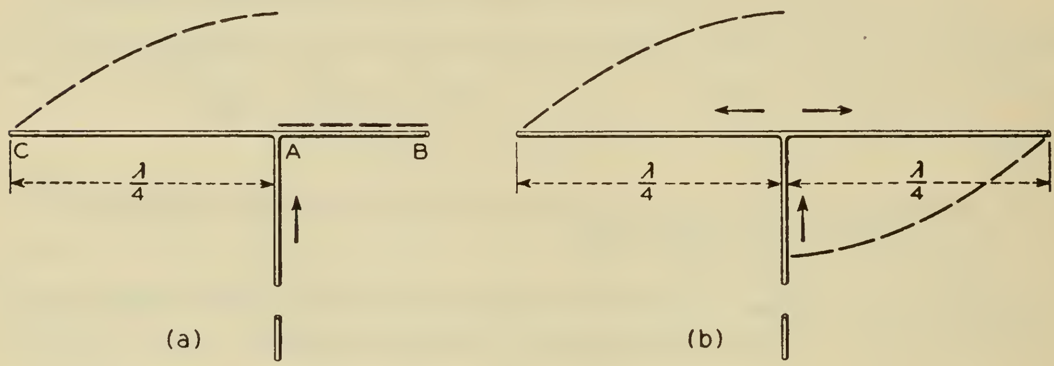

Fig. 8.14 *Asymptotic* current distribution in networks of wires when the length of one branch is $\lambda/4$ or an odd multiple of $\lambda/4$: (*a*) when the other branch is not an odd multiple of $\lambda/4$; (*b*) when the other branch *is* an odd multiple of $\lambda/4$.

From these equations, we have

$$I_2 = -I_1, \qquad I_4 = -I_1 \cos \beta l, \qquad I_3 = -2I_1 \sin \beta l. \tag{137}$$

Substituting in equation 135, we find

$$I_5 = -\frac{2 \sin \beta l \cos \beta\xi + \cos \beta l \sin \beta\xi}{\sin \beta(l_1 - \xi)} I_1. \tag{138}$$

Thus, all the coefficients have been expressed in terms of I_1, and the shape of the asymptotic current distribution has been obtained.

Similarly, for the shunt-excited antenna in Fig. 8.13*b* we find

$$\begin{aligned} I(x) &= A[\sin \beta\xi \cos \beta(l - \xi) \sin \beta(l_1 - x) - \cos \beta l \cos \beta(l_1 - x)]; \\ I(z) &= A \cos \beta(l - \xi) \cos \beta z, \qquad 0 \le |z| \le \xi; \\ &= A \sin \beta\xi \sin \beta(l - |z|), \qquad \xi \le |z| \le l. \end{aligned} \tag{139}$$

The details of the derivation are left to the reader.

Exceptional cases arise when the length of some branch of the network equals $(2n + 1)\lambda/4$, where n is an integer. Thus, in the case shown in Fig. 8.14*a*, the derivative of the current in AC at A vanishes. Now, if the length of AB is different from a quarter wavelength or an odd number of quarter wavelengths, the derivative of the current in it will be different from zero unless the current is identically equal to zero. Thus, the asymptotic form of the current distribution is that shown by

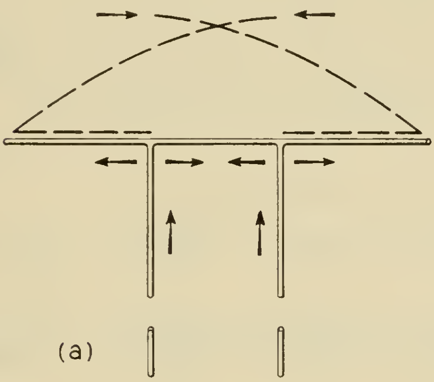

FIG. 8.15 (*a*) Symmetric and (*b*) antisymmetric modes of shunt excitation.

the dotted curve in Fig. 8.14*a* for all lengths of AB except the odd multiples of $\lambda/4$; but, if AB is equal to $\lambda/4$, then the current will be of the form shown in Fig. 8.14*b*. This discontinuity in the form exists only in the asymptotic case. In any actual case the transition is continuous, although it may be very rapid when the wires are thin.

In calculating the correction terms in such exceptional cases, it is necessary to consider the asymmetric feed in Fig. 8.14*a* as the result of superposing the two feeds in Fig. 8.15.

8.26 Potential and current in thin, tightly coupled parallel wires

Consider now two thin parallel wires of equal length l (Fig. 8.16). Either we may write equations 43 for the total potential and total dynamic component of E at a typical point of either wire, or else we may split these quantities into components associated with the currents and charges in each wire. The latter method is more convenient. Let V_1 and V_2 be the potentials of two opposite points A, B, and let

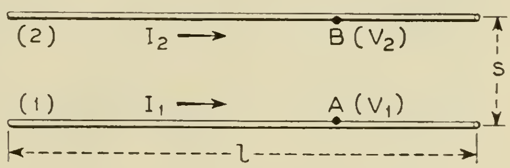

FIG. 8.16 Parallel wires.

$$V_1 = V_{11} + V_{12}, \qquad V_2 = V_{21} + V_{22}, \tag{140}$$

where V_{11} is the potential at A due to the charge on wire 1, V_{12} the potential at A due to the charge on wire 2, etc. Similarly, we split the dynamic components,

$$F_{z,1} = F_{z,11} + F_{z,12}, \qquad F_{z,2} = F_{z,21} + F_{z,22}. \tag{141}$$

If the spacing s *between the axes of the wires is small,* and if we use the same approximations as in Section 8.16, we have

$$\begin{aligned} F_{z,11} &= -j\omega L_{11} I_1, \\ \frac{F_{z,12}}{I_2} &= \frac{F_{z,21}}{I_1} = -j\omega L_{12}, \qquad F_{z,22} = -j\omega L_{22} I_2, \end{aligned} \tag{142}$$

where

$$\begin{aligned} L_{11} &= \frac{\mu}{2\pi}\left(\log\frac{\lambda}{2\pi a_1} + 0.116 + \mathrm{Ci}\,\beta l - \frac{\sin\beta l}{\beta l}\right), \\ L_{22} &= \frac{\mu}{2\pi}\left(\log\frac{\lambda}{2\pi a_2} + 0.116 + \mathrm{Ci}\,\beta l - \frac{\sin\beta l}{\beta l}\right), \\ L_{12} &= \frac{\mu}{2\pi}\left(\log\frac{\lambda}{2\pi s} + 0.116 + \mathrm{Ci}\,\beta l - \frac{\sin\beta l}{\beta l}\right). \end{aligned} \tag{143}$$

Equations 90 may now be written

$$\begin{aligned} \frac{dV_1}{dz} &= -j\omega L_{11} I_1 - j\omega L_{12} I_2 + E_1{}^i(z), \qquad L_{11}\frac{dI_1}{dz} + L_{12}\frac{dI_2}{dz} = -j\omega\mu\varepsilon V_1, \\ \frac{dV_2}{dz} &= -j\omega L_{12} I_1 - j\omega L_{22} I_2 + E_2{}^i(z), \qquad L_{12}\frac{dI_1}{dz} + L_{22}\frac{dI_2}{dz} = -j\omega\mu\varepsilon V_2. \end{aligned} \tag{144}$$

If the *wires are of the same radius,* $L_{22} = L_{11}$, then we can obtain separate equations connecting $V_1 - V_2$ with $I_1 - I_2$ and $V_1 + V_2$ with $I_1 + I_2$. Thus,

$$\begin{aligned} \frac{d(V_1 - V_2)}{dz} &= -j\omega(L_{11} - L_{12})(I_1 - I_2) + [E_1{}^i(z) - E_2{}^i(z)], \\ \frac{d(I_1 - I_2)}{dz} &= -\frac{j\omega\varepsilon\mu}{L_{11} - L_{12}}(V_1 - V_2). \end{aligned} \tag{145}$$

Similarly,

$$\begin{aligned} \frac{d(V_1 + V_2)}{dz} &= -j\omega(L_{11} + L_{12})(I_1 + I_2) + [E_1{}^i(z) + E_2{}^i(z)], \\ \frac{d(I_1 + I_2)}{dz} &= -\frac{j\omega\varepsilon\mu}{L_{11} + L_{12}}(V_1 + V_2). \end{aligned} \tag{146}$$

If the impressed field is such that $I_2 = -I_1$, we have only the "push-pull" mode of propagation given by equations 145. In this case the equations are identical with the usual equations for two-wire lines, since

$$2(L_{11} - L_{12}) = \frac{\mu}{\pi} \log \frac{s}{a}, \tag{147}$$

$V_1 - V_2$ is the transverse voltage between A and B, and $\frac{1}{2}(I_1 - I_2) = I_1$ is the longitudinal current in one wire. If $I_2 = I_1$, we have only the "push-push" mode given by equations 146.

Denoting the mean potential of A and B by V and the total current by I,

$$V = \tfrac{1}{2}(V_1 + V_2), \qquad I = I_1 + I_2, \tag{148}$$

we find

$$\frac{dV}{dz} = -j\omega LI + \tfrac{1}{2}[E_1{}^i(z) + E_2{}^i(z)], \qquad \frac{dI}{dz} = -\frac{j\omega\mu\varepsilon}{L} V, \tag{149}$$

where

$$\begin{aligned} L &= \tfrac{1}{2}(L_{11} + L_{12}) \\ &= \frac{\mu}{2\pi}\left(\log \frac{\lambda}{2\pi\sqrt{as}} + 0.116 + \operatorname{Ci} \beta l - \frac{\sin \beta l}{\beta l}\right). \end{aligned} \tag{150}$$

Hence, the equations for the push-push mode on parallel wires are the same as those for a single wire whose radius is the geometric mean of the radius of each wire and the interaxial spacing.

If the radii are unequal, there are still two independent modes of propagation; but the magnitudes of the currents in the wires are no longer equal in either mode. For small inequalities it is best to subdivide I_1 and I_2 into weakly coupled push-pull and push-push modes.

8.27 Networks, some of whose parts are strongly coupled

The simplest example of a wire network, part of which is strongly coupled and the rest weakly coupled, is a two-wire line feeding an antenna (Fig. 8.17). Both in the line and in the antenna the current and potential distributions are substantially sinusoidal; but the sinusoids are not the continuations of each other. Later we shall find, for instance, that the antenna will be resonant when its length is a few per cent shorter than one-half wavelength and that its input resistance will be approximately 73 ohms. Suppose that the characteristic impedance of the parallel pair is equal to this resistance; then the waves in the parallel pair will be progressive. In the antenna, on the other hand, the waves are very nearly stationary. If we start increasing the dis-

tance between the parallel wires, we shall upset the impedance match, and standing waves will begin to appear in these wires. When the distance is large, the principal current in each arm of the antenna will be a sinusoidal continuation of the principal current in the corresponding member of the parallel pair. On the other hand, if the separation between the wires is reduced so that the characteristic impedance of the two-wire line is much smaller than 73 ohms, the current in the line will approximate that existing when the line is open at A, B. This illustrates the effect produced by the proximity of wires carrying equal and opposite currents.

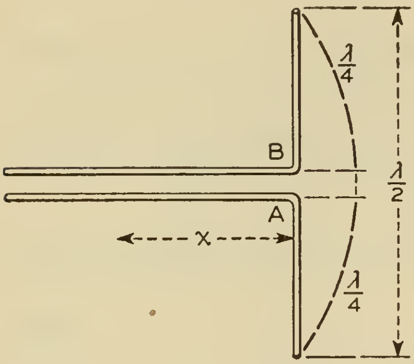

FIG. 8.17 A two-wire transmission line feeding a half-wave antenna.

At this point we should stress that, in order *to determine properly the connection between the current and potential distributions in those parts of the network that are strongly coupled with those in the weakly coupled parts, it is, in general, essential to use more accurate expressions for the current in the weakly coupled parts than the asymptotic forms* so far studied. To understand this we recall that at a junction $x = 0$, $z = 0$ (Fig. 8.17) the current and potential must be continuous; hence their ratio must be continuous. This means that the impedance seen by the two-wire line must equal the input impedance of the antenna. The ratio between the incident and reflected waves in the two-wire line will thus depend on the ratio of the antenna impedance to the characteristic impedance of the two-wire line. The radiation usually has an important effect on the impedance of the antenna. At resonance, for example, the impedance would be zero were it not for radiation; because of the radiation the impedance is 73 ohms. If the characteristic impedance of the two-wire line is much larger than 73 ohms, then the antenna will act approximately as a short circuit; but, if the characteristic impedance of the line is much smaller than 73 ohms, then the line will be almost open electrically. This change from one to the other of two opposite conditions is due entirely to radiation which is not included in the asymptotic current distributions studied so far. To summarize: By the methods developed in this chapter we are able to find the forms of potential and current distributions in the strongly coupled parts of a wire network and in the weakly coupled parts. To connect the two distributions we must calculate the feed current due to the radiation from the weakly coupled part

of the network; this is equivalent to the calculation of the antenna impedance.

8.28 Effects of a sudden change in the radius of the antenna

The distant field of an antenna increases with its length, and it is found particularly desirable that the length of each arm be comparable to $\lambda/2$. At the lower broadcast frequencies, around 550 kc/sec, this length is of the order of 900 ft. For mechanical reasons a tower of this height must have a fairly large cross section near the base; it is also desirable

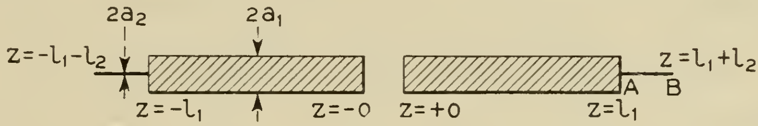

FIG. 8.18 An antenna composed of two wires of different diameters.

that near the top its cross section should be small. Some such towers with thin poles on their tops have actually been built and found unsatisfactory as far as extension in effective length is concerned. Theoretically it is not difficult to see the reason.

Consider an antenna made up of two sections of different diameters (Fig. 8.18). In this study we shall assume that the current at the end $z = l_1 + l_2$ vanishes; hence,

$$\begin{aligned} I(z) &= A \sin \beta(l_1 + l_2 - z), \qquad l_1 \leq z \leq l_1 + l_2; \\ V(z) &= -jZ_0''A \cos \beta(l_1 + l_2 - z). \end{aligned} \tag{151}$$

In the thick portion,

$$\begin{aligned} I(z) &= B \sin \beta(l_1 - z) + C \cos \beta(l_1 - z), \qquad 0 < z \leq l_1; \\ V(z) &= -jZ_0'B \cos \beta(l_1 - z) + jZ_0'C \sin \beta(l_1 - z). \end{aligned} \tag{152}$$

If we neglect the capacitance of the junction between the thick and thin parts of the antenna, then,

$$I(l_1 - 0) = I(l_1 + 0), \qquad V(l_1 - 0) = V(l_1 + 0). \tag{153}$$

Using these conditions, we can express B and C in terms of A. Thus, we obtain

$$\begin{aligned} I(z) &= A\left[\sin \beta l_2 \cos \beta(l_1 - z) + \frac{Z_0''}{Z_0'} \cos \beta l_2 \sin \beta(l_1 - z)\right] \\ &= A\left[\sin \beta(l_1 + l_2 - z) + \frac{Z_0'' - Z_0'}{Z_0'} \cos \beta l_2 \sin \beta(l_1 - z)\right], \\ &\qquad\qquad 0 < z \leq l_1. \end{aligned} \tag{154}$$

If we now assume

$$Z_0'' = 3.5Z_0', \qquad l_1 = 4l_2, \tag{155}$$

we find that the current is distributed as shown by the solid curve in Fig. 8.19. The dotted curve corresponds to the case of equal impedances. Thus, very little current will flow in the thin part of the antenna unless the entire antenna is thin.*

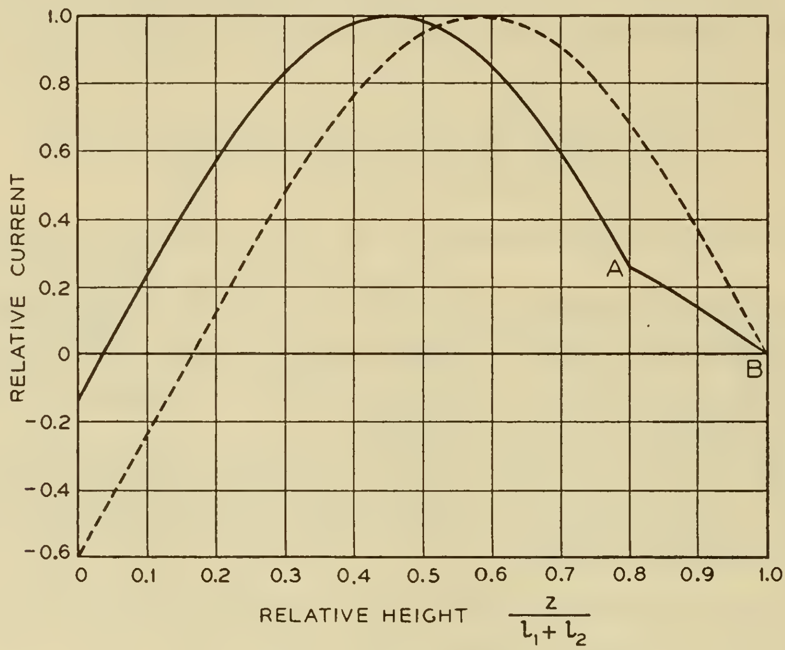

FIG. 8.19 The solid curve represents the main part of the current in the nonuniform antenna of the type shown in Fig. 8.18. The dotted curve is the corresponding current in a uniform antenna.

If we do not neglect the capacitance of the junction at $z = l_1$, the current will appear to be slightly discontinuous at this point; but the general shape will be the same.

8.29 Some of the same conclusions from another point of view

So far the entire analysis of current and potential distributions in this chapter has been based on the expressions for the field on the surface of the antenna in terms of the current and charge. We have dealt with the forces exerted by the charge and the current in one section of the antenna on the charge in another section. From this point of view the oscillations of charge and current are propagated essentially by electro-

* For experimental confirmation, see G. H. Brown, A critical study of the characteristics of broadcast antennas as affected by the antenna current distribution, *IRE Proc.*, **24**, January 1936, pp. 48–81.

static and electromagnetic induction in much the same way as in ordinary electric circuits. The conceptions of wave propagation in space, as propounded in Chapters 3 and 4, have not entered our analysis except indirectly through the expression for the field of an electric current element. We could have accepted this expression as the fundamental hypothesis of electromagnetic theory directly related to experience, instead of deriving it from Maxwell's equations which, after all, we also had to accept as postulates based on experience. This point of view is logically self-sufficient. In this chapter we used it primarily to obtain some useful approximations; but it can be employed theoretically for the exact analysis of all electromagnetic phenomena. Its reduction to practice, however, is limited by mathematical difficulties.

Another point of view is based more directly on Maxwell's equations. It is also self-sufficient and theoretically it might be used to solve exactly all electromagnetic phenomena. In practice, however, it is also limited by mathematical difficulties.* Fortunately the two points of view are supplementary in that some questions may be answered more easily from the first point of view and others more easily from the second. Here we shall merely discuss qualitatively the factors affecting current distribution from the point of view of wave propagation in three-dimensional media.

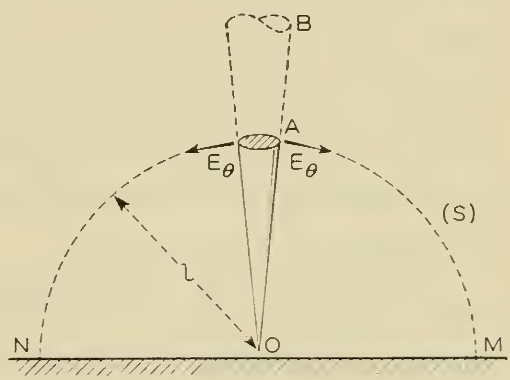

FIG. 8.20 A conical tower above a perfect ground.

Let us consider a conical tower above a perfectly conducting plane (Fig. 8.20). In the region bounded by a sphere concentric with the apex of the tower and passing through the end surface, there exist TEM waves. As shown in Section 4.6, these waves obey ordinary transmission line equations with constant L and C. If r is the distance from the generator along the surface of the tower and if we had no other waves but TEM waves, our solution would be

$$\begin{aligned} I(r) &= I_0 \sin \beta(l + \delta - r), \qquad 0 \le r \le l, \\ V_\theta(r) &= -jKI_0 \cos \beta(l + \delta - r), \end{aligned} \tag{156}$$

* Either method may be used to develop an iterative process which ultimately should give an exact solution; but the numerical complexity makes it impractical to carry out more than two or three initial steps, at best. See S. A. Schelkunoff, *Advanced Antenna Theory*.

where $V_\theta(r)$ is the voltage between the cone and the plane along a typical meridian, and should not be confused with the difference of the quasi-static potentials. The voltage along a meridian will include also the integral of the θ component of the dynamic electric intensity. For a small angle cone,

$$K = 60 \log \frac{2l}{a}, \tag{157}$$

and δ is determined by the current flowing into the " cap " on the top of the cone. Thus,

$$\frac{I(l)}{V_\theta(l)} = -\frac{\tan \beta\delta}{jK} = \frac{j\beta\delta}{K} = j\omega C\delta, \tag{158}$$

where C is the capacitance per unit length associated with the principal waves; hence, δ is known as soon as $I(l)/V(l)$ is determined. We have already seen that this ratio is proportional to a/λ.

As we have said, these equations are based on the assumption that inside the sphere S (Fig. 8.20) we have only TEM waves, or, at least, that these waves are the only ones of importance. On the spherical surface S we have a distribution of the meridian electric intensity E_θ, varying inversely as $\sin\theta$, as shown by equations 4–25. This intensity excites a wave outside S. This free-space wave will carry away a certain amount of power. It will also have a reactive effect which will tend to increase δ in equations 156.

The intensity of the free-space wave decreases as the angle of the cone decreases. This occurs because E_θ is strong in the region near the cone, precisely in the region where the action of E_θ on one side of the cone is opposed by the action on the other side, once the conductor separating the two sides is no longer present. For a given total meridian voltage, a greater part of it is thus effectively neutralized as the angle of the cone is decreased. The free-space wave has a nonvanishing E_r. Since the field must be continuous across S, we must have a wave inside S in addition to the TEM wave. This " complementary wave " represents the reaction of the free-space wave on the antenna region, and its nature is rather complicated; but the only thing that concerns us here is that this wave will modify the current distribution (equation 156), and that this modification is diminished as the cone angle is diminished.

Next let us consider a conical tower surmounted by a slender conical pole (Fig. 8.21). We assume that, if the top pole were continued down to the base, its apex would coincide with the apex of the tower. The primary effect of the discontinuity in the structure will be on the principal waves. Evidently our equations for the current and the meridian

voltage will be the same as those in Section 8.28 with

$$K_1 = 60 \log \frac{2l_1}{a_1}, \qquad K_2 = 60 \log \frac{2(l_1 + l_2)}{a_2} \tag{159}$$

substituted for Z_1' and Z_2''. If $K_2 = 3.5K_1$, we shall thus have the current distribution given by the solid curve in Fig. 8.19: that is, if we neglect the capacitance of the top of the tower. This capacitance may be included in the same way as the end capacitance was included in equations 156.

As explained in Chapter 4, a tower of arbitrary shape may be considered as a continuously deformed conical tower. The equations for

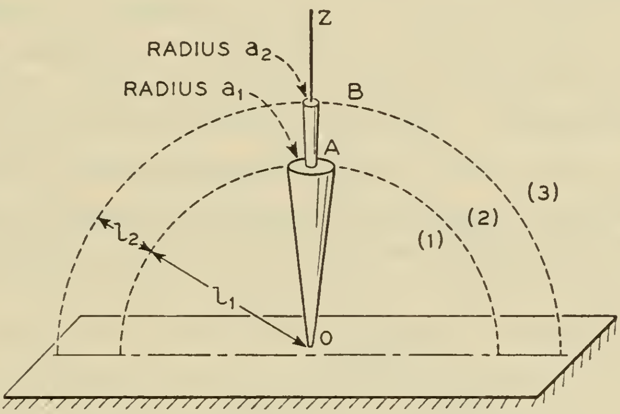

Fig. 8.21 A conical tower surmounted by a slender conical pole.

the principal waves will now depend on variable L and C. As the transverse dimensions of the tower are decreased, the relative variations in L and C become smaller. Hence, the current distribution will be nearly sinusoidal.

Our new point of view leads to the same conclusions, although by different routes. In the main, the current distribution in conductors of small cross section is sinusoidal; but there are effects due to radiation, sudden discontinuities in the dimensions of the conductor, and the " differential discontinuities " involved in gradual changes in the dimensions.

In a quantitative comparison of the equations in this section with those in the preceding sections, we should bear in mind that the voltage taken along the meridians includes a contribution from the dynamic component of electric intensity in addition to the difference of quasi-static potentials. However, as the radius of the antenna decreases, the dynamic component, determined solely by the current, remains constant if we keep the current constant; at the same time the potentials of the various points on the conductor will increase. Hence, the ratio of the meridian voltage to the potential difference will approach unity. Thus

we should expect that, while there will be a difference between K and Z_0, the ratio K/Z_0 will approach unity as a approaches zero. This is the case. This means that the first-order approximations obtained by the two methods may be different. These differences will be absorbed in the succeeding approximations.

8.30 Guided and radiated power

If the distance between two parallel wires is very small compared with λ, the characteristic impedance of the pair is much smaller when it is operated in the push-pull mode than when it is operated in the push-push mode. Hence, the currents in the wires are very nearly equal and opposite, particularly when the pair is energized by a nearly balanced generator. The distant fields produced by two closely spaced equal and opposite current elements nearly cancel each other, and the radiated power should be small. In fact, calculations show that, when the distance s between the wires is small compared with λ, the power radiated by long parallel pairs operated in the push-pull mode is independent of their length and is proportional to $(s/\lambda)^2$. The radiation from such a pair is strictly an end effect: at the generator and at the other end of the pair, spherical waves are formed, and these waves carry away a small fraction of power; there is no radiation from the parallel pair as such.

The field is very strong in the immediate vicinity of the wires; most of the power leaving the generator flows close to the wires, and it may easily be absorbed by a proper resistance at the far end; thus, the parallel pair is an effective "transmission line" for conveying power from one place to another. If the load resistance does not equal the characteristic impedance of the line, it will not accept all the power carried by the incident wave; but this power is returned to the generator and, in the steady state, the generator will supply less power to the line, just the amount accepted by the load and a small quantity for radiation.

Although the greatest fraction of the total power guided by the parallel pair is flowing close to the wires, some is far away. That small fraction is not diverted to the load at the end of the line but is entirely consumed in generating a free-space wave. Similarly, at each bend of the line (Fig. 8.22) one system of parallel wavefronts must be transformed into another, and the power flowing at large distances from the line will not follow the bends in the line but will be lost into free space.

When our object is not to convey power from one point to another but to radiate it into free space, we should create conditions favoring the greater flow of power where it is free from the guiding effect of conductors. If the parallel pair is open at the far end (Fig. 8.6b), most of the power is reflected back to the generator simply because most of it flows

close to the wires. If we start spreading the wires as in Fig. 8.6a, a greater fraction of power will flow at larger distances from the wires and will thus be free to escape. If the radius of the antenna is made smaller, more power will lie closer to the wire, and a lesser proportion will be available for radiation. In the steady state, of course, the generator supplies the radiated power and no more (except for a small heat loss in the antenna). Consequently, the foregoing statement should be interpreted as meaning that in the case of thin antennas the energy radiated per period will constitute a smaller fraction of the energy stored around the antenna than in the case of thick antennas. The Q of a thin antenna is larger than the Q of a thick antenna.

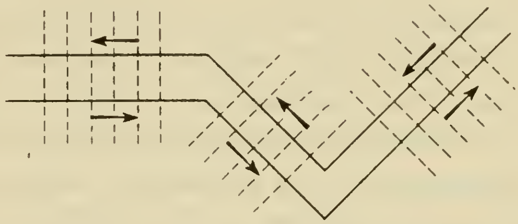

FIG. 8.22 Some radiation takes place at each bend of a parallel pair.

The quality factors of practical antennas are never large. It is physically impossible to make a long wire sufficiently thin and of material whose conductivity is sufficiently high to realize a Q larger than 20. Nevertheless, in theory the Q of a perfectly conducting antenna increases indefinitely as the radius decreases so that in the limit the antenna radiates only an infinitesimal fraction of its stored energy.

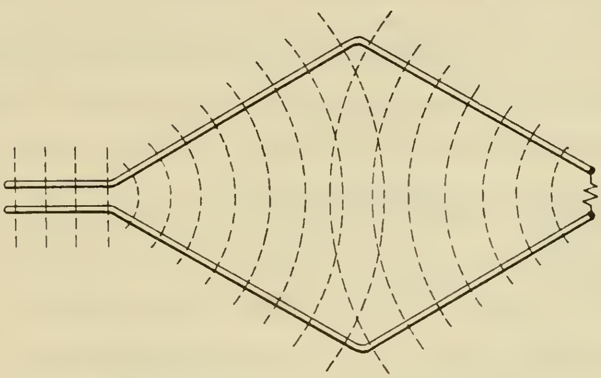

FIG. 8.23 A rhombic antenna terminated into a resistance.

In Chapter 14 we shall consider rhombic antennas (Fig. 8.23). For several reasons such antennas are terminated at the far end into a resistance such that the current waves along the wires are largely progressive. In accordance with equation 90, the asymptotic current in the rhombic is

$$I(s) = I_0(0)e^{-j\beta s}, \tag{160}$$

where s is the distance from the generator. This implies that all power

leaving the generator is absorbed by the resistance at the far end. Asymptotically this is true for, as the radius of the wire approaches zero, an increasing fraction of the total power will flow in the immediate vicinity of the wire. To state it differently: As the radius decreases less current flows in the wire for the same amount of power; hence, the distant field which carries some power away becomes smaller. Nevertheless, when the rhombic is long, it is physically impossible to make wires thin enough to convey much more than one half of the power leaving the generator to the load at the other end, even if this were desirable. On account of radiation, the amplitude of the current will gradually diminish with increasing distance from the generator. From the point of view of principal waves on rhombics, we have two discontinuities; one at the junction of the two-wire line with the rhombic where the plane wavefronts do not match the diverging spherical wavefronts, and the other at the bend of the rhombic where the diverging wavefronts do not match the converging wavefronts in the second part of the rhombic. These discontinuities require complementary fields to make the total fields continuous. Radiation is associated with these complementary fields; they also add terms to the principal current on the wires.

Linear antennas may be several wavelengths long but never as long as transmission lines except those used for laboratory purposes. This restriction on length has been implicit in the present discussion. Since the field is much stronger in the vicinity of a wire than at some distance from it, the wire acts as a transmission line. However, the guiding action of a single wire is never so complete as the guiding action of a pair of parallel wires, or of a single dielectric wire,* or of a conducting wire with a dielectric sheath.† In a true nondissipative waveguide the amplitude of a guided progressive wave remains constant along the guide. But, along a semi-infinite perfectly conducting wire, excited at its accessible end, the amplitude varies approximately inversely as the square root of $2 \log (2r/a) - 1$, provided the distance r from the end is greater than $\lambda/2$. The rate of decrease is small but nevertheless the guided power gradually breaks away from the wire. The power flow within a cylinder of some given radius b, coaxial with the wire, decreases inversely as $2 \log(2r/a) - 1$. These quantitative results can be readily obtained from Manneback's solution of Maxwell's equations for waves on infinitely thin perfectly conducting wires.‡ And, in a practical case,

* *Electromagnetic Waves*, pp. 425–431.

† G. Goubau, Surface waves and their application to transmission lines, *Jour. Appl. Phys.*, **21**, November 1950, pp. 1119–1128.

‡ C. Manneback, Radiation from transmission lines, *AIEE Jour.*, **42**, February 1923, pp. 95–105. S. A. Schelkunoff, *Advanced Antenna Theory*, John Wiley, New York, Section 2.25, to be published.

when the wire is not a perfect conductor, there is a further tendency for the power to break away from the wire as well as to be absorbed by it.

8.31 Resonance and antiresonance

By definition *at resonance* the reactive component of the input impedance vanishes, and, for a given impressed voltage, the input current is large. Hence, for a given current the potential difference across the generator is small. *At antiresonance* the reactive component also vanishes; but for a given impressed voltage the input current is small or for a given input current the voltage is large. In the nondissipative cases the input voltage vanishes at resonance and the input current at antiresonance. These are the asymptotic conditions for resonance and antiresonance in antennas; they are also approximate conditions for actual antennas, since it is known that the dissipation of power produces only a second-order effect on the resonant and antiresonant frequencies.

When the end effect in dipole antennas is included, the current is given by equation 127 where δ is defined by equation 123. Hence, at antiresonance,

$$\sin \beta(l + \delta) = 0, \qquad \beta(l + \delta) = n\pi, \qquad n = 1, 2, 3, \cdots, \tag{161}$$

or, for the antiresonant wavelengths, we find

$$\lambda = \frac{2(l + \delta)}{n}. \tag{162}$$

At resonance the input potential difference vanishes,

$$\begin{aligned} \cos \beta(l + \delta) &= 0, & \beta(l + \delta) &= \frac{(2n + 1)\pi}{2}, \\ \lambda &= \frac{4(l + \delta)}{2n + 1}, & n &= 0, 1, 2, \cdots. \end{aligned} \tag{163}$$

As the radius approaches zero, δ slowly approaches zero, and in the limit the resonant wavelengths are the odd submultiples of $4l$ whereas the antiresonant wavelengths are the even submultiples. This rule may be used for practical antennas, provided we use the effective length $l + \delta$ instead of the actual length l. For this reason we assume in what follows that the end effects have been included in the effective values of the various lengths.

In the case of the shunt-excited antenna in Fig. 8.13*b*, the resonant condition is obtained by differentiating $I(x)$ and equating to zero the discontinuity in $I'(x)$ at $x = 0$ (that is, if this point is considered as the input point). Thus, from equations 139, we find

$$\sin \beta\xi \cos \beta(l - \xi) \cos \beta l_1 + \cos \beta l \sin \beta l_1 = 0. \tag{164}$$

For the antiresonant condition,

$$\sin \beta\xi \cos \beta(l - \xi) \sin \beta l_1 - \cos \beta l \cos \beta l_1 = 0. \tag{165}$$

For the antenna network in Fig. 8.13*a*, we have, from equation 134 and other equations in Section 8.25,

$$\begin{aligned} I(x) &= -I_1(2 \sin \beta l \cos \beta x + \cos \beta l \sin \beta x), && x \leq \xi; \\ &= -\frac{2 \sin \beta l \cos \beta\xi + \cos \beta l \sin \beta\xi}{\sin \beta(l_1 - \xi)} I_1 \sin \beta(l_1 - x), \\ & && x \geq \xi. \end{aligned} \tag{166}$$

For an antiresonant condition, $I(\xi) = 0$, and

$$2 \sin \beta l \cos \beta\xi + \cos \beta l \sin \beta\xi = 0. \tag{167}$$

If neither $\cos \beta l$ nor $\cos \beta\xi$ is equal to zero, we may divide equation 167 by the product of these quantities and obtain

$$\tan \beta\xi = -2 \tan \beta l. \tag{168}$$

If $\cos \beta l = 0$, then, for the antiresonant condition, we must have $\cos \beta\xi = 0$; and, if $\cos \beta\xi = 0$, then $\cos \beta l = 0$.

Such equations as 168 may be solved graphically by plotting both sides of the equation as functions of their arguments and pairing those values of $\beta\xi$ and βl that correspond to equal ordinates. In this way $\beta\xi$ may be plotted against* βl.

Differentiating equation 166 with respect to x, we find

$$\begin{aligned} I'(x) &= \beta I_1(2 \sin \beta l \sin \beta x - \cos \beta l \cos \beta x), && x \leq \xi; \\ &= \frac{2 \sin \beta l \cos \beta\xi + \cos \beta l \sin \beta\xi}{\sin \beta(l_1 - \xi)} \beta I_1 \cos \beta(l_1 - x), && x \geq \xi. \end{aligned} \tag{169}$$

Hence, for a resonant condition, we have

$$I'(\xi - 0) = I'(\xi + 0), \tag{170}$$

or

$$\begin{aligned} &2 \sin \beta l \sin \beta\xi - \cos \beta l \cos \beta\xi \\ &\qquad = (2 \sin \beta l \cos \beta\xi + \cos \beta l \sin \beta\xi) \cot \beta(l_1 - \xi). \end{aligned} \tag{171}$$

If there is any substantial difference between the characteristic impedances of the various sections of the antenna network, we should take the actual impedances into consideration. Let the generator be at A, B of the network in Fig. 8.24. Let $AC = l_1$, $DF = l$, $DH = l_2$. Let the characteristic impedance of the "Lecher wires," AG and BH,

* *Applied Mathematics*, Chapter 3.

be K_1 and let the average characteristic impedance of the antenna, DF and CE, be K. The resonant condition corresponds to natural oscillations of the network when the input terminals A, B are short-

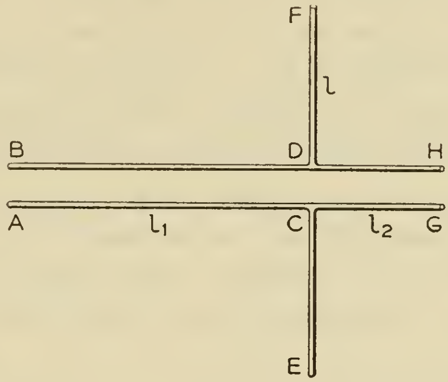

FIG. 8.24 A pair of Lecher wires and a dipole antenna.

circuited. Since no current flows between C and D, the total admittance across C, D must vanish; therefore, the resonant condition is

$$-jK_1^{-1} \cot \beta l_1 + jK^{-1} \tan \beta l + jK_1^{-1} \tan \beta l_2 = 0,$$

or

$$\cot \beta l_1 - \tan \beta l_2 = \frac{K_1}{K} \tan \beta l. \tag{172}$$

Similarly, for an antiresonant condition, we find

$$jK_1^{-1} \tan \beta l_1 + jK^{-1} \tan \beta l + jK_1^{-1} \tan \beta l_2 = 0,$$

or

$$\tan \beta l_1 + \tan \beta l_2 = -\frac{K_1}{K} \tan \beta l. \tag{173}$$

This is also the condition for the natural frequencies of the wire network.

8.32 Effects of resistance on the antenna current

Throughout the entire chapter we have assumed that the conductors are perfect. The conductivity of copper is so large that its effect on the current distribution is negligible. This is to be expected, since the attenuation due to ohmic losses is appreciable only in long transmission lines. No serious complications arise from the inclusion of the internal impedance Z_i of the conductors in equation 48; but the most significant effect is a small increase in the input resistance. This increase is easy to evaluate as follows: (1) we neglect the ohmic loss and evaluate the current; (2) we evaluate the ohmic loss corresponding to this current; (3) we divide the result by one half of the square of the input current, as

suggested in Section 8.3. In this method we neglect the square of the small correction term as compared to its first power.

8.33 Effects of proximity between antenna terminals

There is always some local capacitance between the terminals of any physical circuit (see Section 2.4). At low frequencies its effect on the performance of the circuit is negligible; but at high frequencies and particularly in the microwave region the effect may be substantial. In any practical case we are not concerned with antenna terminals as such. When the antenna is in operation, its terminals are connected either to a transmission line or to local circuits. Depending on the frequency, this transition region may, or may not, have much effect on the antenna performance. It is only in theory, when we wish to consider an antenna separately from the circuits connected to it, that it is essential to include the distance between the terminals in our calculations. Generally this distance cannot be made equal to zero without making the impedance equal to zero. But, if the wires are tapered to mere points, the impedance remains finite as the distance between the terminals vanishes. The admittance of the region in the vicinity of such terminals is calculable and can be taken out of the antenna admittance and replaced by the admittance appropriate to a different input configuration. This enables us to assign a unique meaning to the " antenna impedance " for both theoretical and practical purposes. Antenna input regions will be considered in more detail in Section 12.10.

It should be added that the distance between the terminals does not affect appreciably the current at distances comparable to or larger than the antenna radius, and a unit voltage between infinitely close terminals always produces a finite current everywhere except at the terminals.

PROBLEMS

8.5–1 Explain why the distant electric intensity of any current distribution which may be enclosed by a finite surface is substantially perpendicular to the radius drawn from a typical point within the current distribution. Show that the quasi-static potential contributes to the electric intensity perpendicular to such a radius only terms that diminish with increasing distance r at least as rapidly as $1/r^2$.

8.6–1 Show that equation 13 may be written as follows:

$$\oint E_s \frac{ds}{\lambda} = -2\pi j \iint \eta H_n \frac{dS}{\lambda^2} \cdot$$

This equation shows that, in a region of a weak magnetic field (weak in the sense that $\eta H \ll E$) enclosed within a cube whose edges are very small compared with λ,

we have approximately

$$\oint E_s\, ds = 0.$$

Hence, in such a region we may assign static potentials to various points. This explains why it is permissible to speak of a "potential drop" across the terminals of a circuit element and use the essentially static conceptions in network theory, even though the frequency may be as high as one million cycles per second. It should be noted, however, that if the distance d between two networks is such that $2\pi d/\lambda$ is not small compared with unity, the *local potentials* in the two places are unrelated. That is, two points in such different places can be at the same potential with reference to ground and yet may not be said to have "equal" potentials.

8.8–1 Derive the low-frequency expressions for the potential and dynamic component of E for a thin filament of charge and current.

Ans.

$$V = \int_{s_1}^{s_2}\int_0^{2\pi} \frac{q(s)\, ds\, d\varphi'}{8\pi^2\varepsilon r}, \qquad F_z = -j\omega\mu \int_{s_1}^{s_2}\int_0^{2\pi} \frac{I(s)\cos(s, z)}{8\pi^2 r}\, ds\, d\varphi',$$

where F_z is a typical Cartesian component of F.

8.8–2 Assume that an infinitely thin current filament of strength $I(z) = I_0 \exp j(\omega t - \beta z)$ extends along the z axis from $z = z_1$ to $z = z_2$. Find the scalar potential and the dynamic component of E.

Ans.

$$V = \frac{I_0 e^{j(\omega t - \beta z_2 - \beta r_2)}}{4\pi j\omega\varepsilon r_2} - \frac{I_0 e^{j(\omega t - \beta z_1 - \beta r_1)}}{4\pi j\omega\varepsilon r_1} + \frac{\eta I_0}{4\pi}(u - jv)e^{j(\omega t - \beta z)};$$

$$F_\rho = F_\varphi = 0;$$

$$F_z = -\frac{j\omega\mu}{4\pi} I_0 e^{j(\omega t - \beta z)}(u - jv),$$

where

$$u = \operatorname{Ci}\beta(r_2 + z_2 - z) - \operatorname{Ci}\beta(r_1 + z_1 - z),$$

$$v = \operatorname{Si}\beta(r_2 + z_2 - z) - \operatorname{Si}\beta(r_1 + z_1 - z),$$

$$r_1 = [(z_1 - z)^2 + \rho^2]^{1/2}, \qquad r_2 = [(z_2 - z)^2 + \rho^2]^{1/2}.$$

8.8–3 Solve the preceding problem for $I(z) = I_0 \exp j(\omega t + \beta z)$.

Ans.

$$V = \frac{I_0 e^{j(\omega t + \beta z_2 - \beta r_2)}}{4\pi j\omega\varepsilon r_2} - \frac{I_0 e^{j(\omega t + \beta z_1 - \beta r_1)}}{4\pi j\omega\varepsilon r_1} - \frac{\eta I_0}{4\pi}(u_1 - jv_1)e^{j(\omega t + \beta z)},$$

$$F_z = -\frac{j\omega\mu}{4\pi} I_0 e^{j(\omega t + \beta z)}(u_1 - jv_1),$$

where

$$u_1 = \operatorname{Ci}\beta(r_1 - z_1 + z) - \operatorname{Ci}\beta(r_2 - z_2 + z),$$

$$v_1 = \operatorname{Si}\beta(r_1 - z_1 + z) - \operatorname{Si}\beta(r_2 - z_2 + z).$$

8.15–1 Evaluate the characteristic impedance of a wire of radius a extending from $z = z_1$ to $z = z_2$ at vanishingly small frequencies.

Ans.

$$Z_0 = 30 \log \frac{\{z_2 - z + [(z_2 - z)^2 + a^2]^{1/2}\}\{z - z_1 + [(z - z_1)^2 + a^2]^{1/2}\}}{a^2};$$

except in the immediate neighborhood of the ends,

$$Z_0 \backsimeq 30 \log \frac{4(z_2 - z)(z - z_1)}{a^2}.$$

8.15–2 Calculate the average value of the characteristic impedance in the preceding problem.

Ans. $$\mathrm{av}(Z_0) = 60 \log \frac{2(z_2 - z_1)}{a} - 60.$$

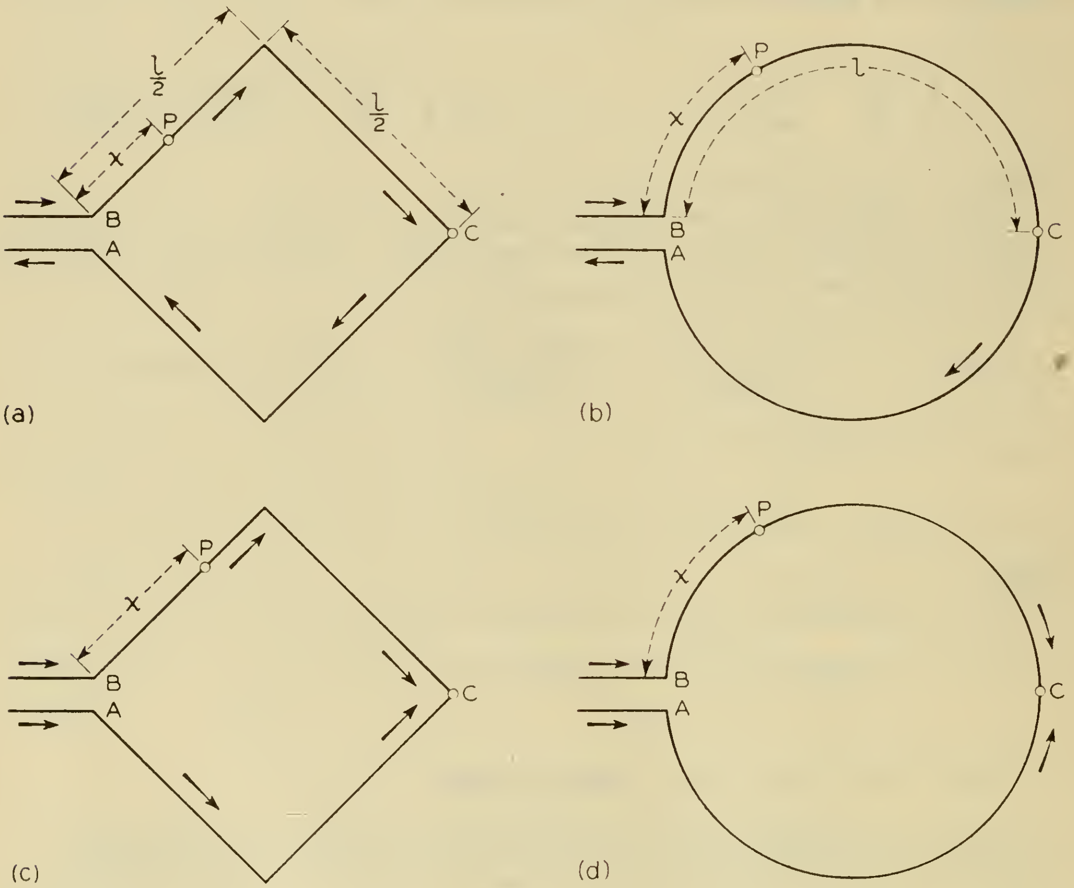

FIG. 8.25 Square and circular loop antennas fed by two-wire lines: (*a*) and (*b*), in push-pull; (*c*) and (*d*), in push-push.

8.18–1 Derive equations 109.

8.18–2 Derive equations 110.

8.18–3 Obtain the asymptotic expression for the current in loop antennas of length $2l$, fed in push-pull as indicated in Figs. 8.25*a*, *b*.

Ans. $$I(x) = A \cos \beta(l - x).$$

8.18–4 Obtain the asymptotic expression for the current in loop antennas of length $2l$, fed in push-push as indicated in Figs. 8.25*c*, *d*.

Ans. $$I(x) = A \sin \beta(l - x),$$

provided the positive direction of the current is chosen counterclockwise (or clockwise) at all points.

8.19–1 Obtain the asymptotic form of current distribution in a reflecting antenna making an angle ψ with the direction of propagation of the incident wave.

Ans. $$I(z) = A\,[\cos \beta l \cos(\beta z \cos \psi) - \cos(\beta l \cos \psi) \cos \beta z].$$

8.20–1 Let the impressed field be parallel to the antenna extending from $z = -l$ to $z = l$. Find the asymptotic current distribution when the terminals of the antenna at $z = 0$ are open.

Ans. $$I(z) = I_0 \left[\sin \beta(l - |z|) - \frac{\sin \beta l}{1 - \cos \beta l} (\cos \beta z - \cos \beta l) \right]$$
$$= I_0 \,[\sin \beta(l - |z|) - \cot \tfrac{1}{2}\beta l(\cos \beta z - \cos \beta l)].$$

8.20–2 Obtain the asymptotic current in a receiving antenna whose load is at $z = \xi$, assuming that the field is parallel to the antenna.

Ans. $$I(z) = A\,(\cos \beta z - \cos \beta l) + I_1(z),$$

where $I_1(z)$ is the current when the load is replaced by a generator.

8.20–3 Solve the preceding problem for the case of infinite load.

Ans. $$I(z) = I_1(z) - \frac{I_1(\xi)}{\cos \beta\xi - \cos \beta l} (\cos \beta z - \cos \beta l).$$

8.21–1 Obtain the asymptotic potential distribution in loop antennas (Figs. 8.25*a*, *b*; see Problem 8.18–3).

Ans. $$V(x) = jZ_0 A \sin \beta(l - x),$$

where Z_0 is the impedance of the loop.

8.21–2 Obtain the asymptotic potential in an asymmetrically fed antenna (see equation 106).

Ans. $$\begin{aligned} V(z) &= -jZ_0 A \sin \beta(l + \xi) \cos \beta(l - z), && z > \xi; \\ &= jZ_0 A \sin \beta(l - \xi) \cos \beta(l + z), && z < \xi. \end{aligned}$$

8.21–3 Obtain the asymptotic potential distribution in the symmetric mode (Fig. 8.9*a*), corresponding to the current distribution given by equation 109.

Ans. $$\begin{aligned} V(z) &= -jZ_0 A \cos \beta\xi \cos \beta(l - z), && \xi < z \leq l; \\ &= -jZ_0 A \sin \beta(l - \xi) \sin \beta z, && -\xi < z < \xi; \\ &= jZ_0 A \cos \beta\xi \cos \beta(l + z), && -l \leq z < -\xi. \end{aligned}$$

8.21–4 Obtain the asymptotic potential in the antisymmetric mode (Fig. 8.9*b*).

Ans. $$\begin{aligned} V(z) &= -jZ_0A \sin \beta\xi \cos \beta(l - z), && \xi < z \le l; \\ &= jZ_0A \sin \beta(l - \xi) \cos \beta z, && -\xi < z < \xi; \\ &= -jZ_0A \sin \beta\xi \cos \beta(l + z), && -l \le z < -\xi. \end{aligned}$$

8.25–1 Derive equations 139.

8.26–1 Discuss the advantages of introducing new variables in equation 144:

$$I' = \tfrac{1}{2}(I_1 - I_2), \qquad V' = V_1 - V_2, \qquad I'' = I_1 + I_2, \qquad V'' = \tfrac{1}{2}(V_1 + V_2),$$

in case L_{11} is not very different from L_{22}.

8.28–1 Derive the asymptotic current distribution in the antenna shown in Fig. 8.18, assuming that the impedance ratio Z_0''/Z_0' is maintained constant.

Ans.

$$\begin{aligned} I(z) &= A \sin \beta(l_1 + l_2 - z), && l_1 \le z \le l_1 + l_2; \\ &= A\left[\sin \beta l_2 \cos \beta(l_1 - z) + \frac{Z_0''}{Z_0'} \cos \beta l_2 \sin \beta(l_1 - z)\right], && 0 \le z \le l_1; \\ &= A\left[\sin \beta(l_1 + l_2 - z) + \frac{Z_0'' - Z_0'}{Z_0'} \cos \beta l_2 \sin \beta(l_1 - z)\right], && 0 \le z \le l_1. \end{aligned}$$

8.30–1 By direct substitution verify that the following expressions (C. Manneback, Radiation from transmission lines, *AIEE Jour.*, **42**, February 1923, pp. 95–105),

$$H_\varphi = \frac{I_0 e^{-j\beta r}(1 + \cos \theta)}{4\pi r \sin \theta}, \qquad E_\theta = \eta H_\varphi, \qquad E_r = -\frac{I_0 e^{-j\beta r}}{4\pi j\omega\varepsilon r^2},$$

satisfy Maxwell's equations. Show that this field implies an electric current along the axis $\theta = 0$ and that its strength is $I(z) = I_0 e^{-j\beta z}$.

8.30–2 Using the expressions in the preceding problem, show that the intensity of the magnetic field generated by a progressive current filament, $I(z) = I_0 e^{-j\beta z}$, extending along the axis $\theta = 0$ from $z = 0$ to $z = z_1$ is given by

$$4\pi\rho H_\varphi = I_0 e^{-j\beta r}(1 + \cos \theta) - I_0 e^{-j\beta(z_1 + r_1)}(1 + \cos \theta_1),$$

where ρ is the distance from the axis, r_1 the distance from the end of the filament at $z = z_1$, and θ_1 is the polar angle if this end is taken as the origin of the second spherical coordinate system.

8.30–3 Using the expression in the preceding problem, show that the magnetic intensity of the field generated by a sinusoidal current, $I(r) = A \sin \beta r$, extending along $\theta = 0$ from $r = 0$ to $r = \lambda/2$, is given by

$$4\pi\rho H_\varphi = jA(e^{-j\beta r} + e^{-j\beta r_1}).$$

8.30–4 Using the result of the preceding problem and Maxwell's equations in cylindrical coordinates, obtain the intensity of the electric field.

Ans. $$4\pi\rho E_\rho = jA\eta(e^{-j\beta r} \cos \theta + e^{-j\beta r_1} \cos \theta_1),$$

$$E_z = -\frac{jA\eta}{4\pi}\left(\frac{e^{-j\beta r}}{r} + \frac{e^{-j\beta r_1}}{r_1}\right).$$

8.31–1 Obtain approximate resonant wavelengths of loop antennas (Figs. 8.25*a*, *b*).

Ans. $$\lambda_n = \frac{2l}{n}, \qquad n = 1, 2, 3, \cdots.$$

8.31–2 Obtain approximate antiresonant wavelengths of loop antennas.

Ans. $$\lambda_n = \frac{4l}{2n+1}, \qquad n = 0, 1, 2, 3, \cdots.$$

9

IMPEDANCE, RECIPROCITY, EQUIVALENCE

In a radio transmission system the transmitting and receiving antennas are only two of the many elements serving various purposes. The transmission of signals from one antenna to the other depends on their current distribution, on the fields produced by these distributions, and on the manner in which these fields are affected by the earth and by the atmosphere. But, as they pass from their source to the transmitting antenna, and from the receiving antenna to their destination, these signals are affected by the circuital properties of the terminal equipment and by the antenna impedances. To calculate the impedance of a specific antenna we have to solve the corresponding field problem — that is, we have to solve Maxwell's equations subject to special boundary conditions at the surface of the antenna. However, some of the most important general properties of the impedance may be obtained from much more basic considerations. These properties are common to all dynamical systems, mechanical and acoustical systems as well as electrical, and they are independent of the particular form of the dynamical equations as long as these equations are linear. These properties are basic in the sense that, if we were to find an inconsistency between them and Maxwell's equations, we should have to reject Maxwell's equations.

Such properties were considered by Otto Brune with special reference to electric networks;* but his results are easily extended to all linear dynamical systems and to systems with an infinite number of degrees of freedom. In this chapter we shall consider some of these general properties of impedances.

* Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency, *Jour. Math. Phys.*, **10**, August 1931, pp. 191–236.

9.1 Impedances as functions of a complex variable

Let us assume that the instantaneous values of the electric and magnetic intensity are the real parts of complex quantities $\mathring{E} \exp(pt)$, $\mathring{H} \exp(pt)$, where p is the *oscillation constant*,

$$p = \xi + j\omega. \tag{1}$$

The imaginary part ω is the frequency of oscillations, and the real part ξ is the relative rate of growth of the amplitude. In practice, we are interested chiefly in sinusoidal oscillations for which $\xi = 0$; but the more general form (equation 1) enables us to obtain a better insight into the dependence of antenna impedances on frequency by using the theory of functions of a complex variable. Performing the differentiation with respect to t in Maxwell's equations 2–18 and 2–19 and canceling the exponential time factor, we have

$$\int E_s\, ds = -p\mu \iint H_n\, dS, \qquad \int H_s\, ds = (g + p\varepsilon) \iint E_n\, dS. \tag{2}$$

Irrespective of a particular field problem, the solutions of these equations $E(x, y, z; p)$, $H(x, y, z; p)$ are functions of the coordinates of a typical point and of the complex variable p. These solutions are analytic functions of p, except perhaps for some isolated values of p.

The voltage and current at the input terminals of a transmitting antenna, or any other physical circuit, are determined by the integrated values of E and H, and thus are functions of p alone; $V(p)$ and $I(p)$, let us say. The ratio of these functions is called the input impedance $Z(p)$ of the antenna, and its reciprocal the input admittance $Y(p)$,

$$Z(p) = \frac{V(p)}{I(p)}, \qquad Y(p) = \frac{I(p)}{V(p)}; \tag{3}$$

hence, the *generalized Ohm's law*,

$$V(p) = Z(p)\, I(p), \qquad I(p) = Y(p)\, V(p). \tag{4}$$

The actual form of Maxwell's equations has little bearing on these equations; the only requirement is that the field equations be *linear* so that the exponential time factor can be canceled from them. The so-called "electric circuit elements" are merely physical circuits whose impedances are particularly simple functions of p. The circuit is a *resistor* if its impedance is independent of p,

$$Z(p) = R; \tag{5}$$

it is an *inductor* if its impedance is proportional to p,

$$Z(p) = Lp; \tag{6}$$

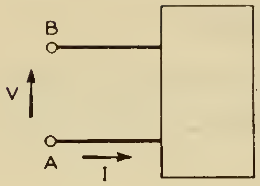

FIG. 9.1 A schematic representation of an impedor.

and it is a *capacitor* if its impedance is inversely proportional to p,

$$Z(p) = \frac{1}{Cp}. \tag{7}$$

Figure 9.1 shows the conventional schematic representation of an *impedor:* that is, any concealed passive network (either discrete or continuous) with two accessible terminals. In the case of an antenna the interior of the impedor includes the earth and the space around it. We use this representation when we wish to stress our interest in the voltage and current at the antenna terminals and our lack of interest in the interior structure.

9.2 Zeros and infinities of impedance functions

The *zeros* of the input impedance $Z(p)$ are the roots of

$$Z(p) = 0. \tag{8}$$

They give those values of the oscillation constant for which the voltage across the input terminals vanishes while the current does not. Hence, they represent the natural oscillations of the antenna when its terminals are short-circuited.

The *infinities* of the impedance are the zeros of the admittance,

$$Y(p) = 0. \tag{9}$$

They give the oscillation constants for which the current through the input terminals vanishes while the voltage does not; they represent the natural oscillations of the antenna with its terminals floating (or " open ").

The *natural oscillation constants* of any *passive* physical circuit, that is, a circuit without concealed sources of power, *must lie either in the left half of the complex* p *plane or on the imaginary axis*; otherwise, the real part of p would be positive, and the oscillations would grow in amplitude without any contribution of power to the circuit. An antenna in free space loses power by radiation, whether its terminals are short-circuited or left floating; hence, the zeros and infinities of its impedance are in the left half of the p plane. *The only exception is the point at the origin,* $p = 0$. This point corresponds to a static field, since at this point $\exp(pt) = 1$ at all times. If the terminals of an antenna consisting of two separate conductors are floating, we can place opposite charges on these conductors and create a voltage across the input terminals while the current will be zero; hence, $p = 0$ is an infinity of the impedance of

such an antenna. All such antennas may be called *dipole antennas* (Fig. 9.2*a*).

Similarly, we may consider a loop antenna (Fig. 9.2*b*): that is, an antenna consisting of a single conductor so bent that its ends are brought close together. Then, if the terminals are short-circuited and if the conductor is perfect, a steady current can flow in the loop since there is no

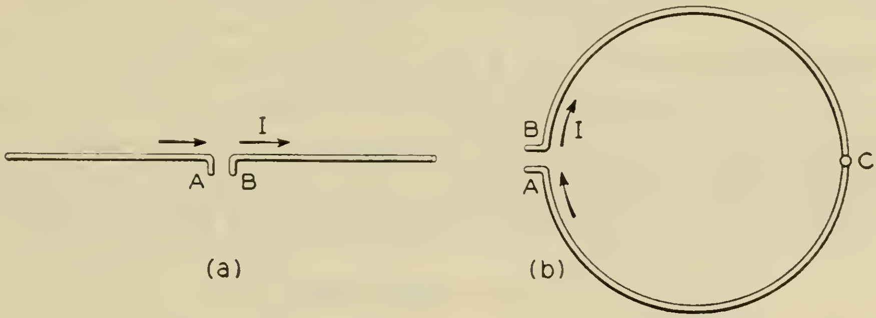

FIG. 9.2 (*a*) A dipole antenna and (*b*) a loop antenna.

loss by radiation. Thus $p = 0$ *is a zero of the impedance of a perfectly conducting loop antenna* and, of course, *an infinity of its admittance.*

If $p = p_1$ is a zero of $Z(p)$, then,

$$Z(p) = (p - p_1)^n f(p), \tag{10}$$

where $f(p)$ reduces to a nonzero finite value at $p = p_1$ and n is positive. In the theory of functions the term " zero " is usually reserved for the case in which n is an integer; otherwise, the point is a *branch point.* In the vicinity of a branch point $Z(p)$ is multiple-valued; the various values coalesce at the branch point itself. The exponent n is called the *order* of the zero. The zero is said to be *simple* if $n = 1$.

In the case of infinities n is negative. If n is an integer, the infinity is said to be a *pole of order* $|n|$. If $n = -1$, the pole is said to be simple.

There is an important theorem concerning the distribution and character of zeros and poles on the imaginary axis: *those zeros and poles of a passive impedance which lie on the imaginary axis are simple, and they separate each other.* For the proof the reader is referred to Brune's paper already cited. From this theorem it follows at once that no passive impedance could be proportional to the nth integral power of p unless $n = \pm 1$ for *all* values of p; otherwise, $p = 0$ would not be a simple zero or pole. This means, for instance, that the voltage across a passive physical circuit could not be proportional to the second time derivative of the current,

$$V = A \frac{d^2 I}{dt^2}, \tag{11}$$

as it can be proportional to the first time derivative,

$$V = L\frac{dI}{dt}\cdot \tag{12}$$

If equation 11 were true, we should have, for an exponentially varying current,

$$V = Ap^2I, \qquad Z(p) = Ap^2, \tag{13}$$

and the impedance would have a zero of the second order at the origin. There is no such fundamental objection to equation 12 for an inductor. A physical inductor would always have some resistance, and its impedance must contain a constant term. A more exhaustive analysis would show, in fact, that its impedance must contain the higher powers of p,

$$Z(p) = R + pL + a_2p^2 + a_3p^3 + \cdots; \tag{14}$$

but it can also be shown that, if we were not limited by available materials and practicable dimensions of the structure, all coefficients except L could be made arbitrarily small. All that we would have to do is to select increasingly better conductors, to decrease the length of the coil and simultaneously decrease the radius of the wire sufficiently to keep L constant. This illustrates the difference between the *physically unattainable* ideal inductor defined by equation 12 and the *physically impossible* passive element described by equation 11. For convenience, much of our thinking is done in terms of idealized elements and systems; the results thus obtained are approximately true for actual systems, to the extent to which the ideal systems approximate the actual; but the results which might be obtained for physically impossible systems could be of no value to us.

In the preceding chapter we have seen that, as the radius approaches zero, the radiation from the antenna also approaches zero; hence, the damping constants of natural oscillations will approach zero, and the zeros and poles of the antenna impedance will approach the imaginary axis. The positions of the zeros may be found from the condition that the input voltage should vanish, which is the condition we used to determine the limits approached by the resonant frequencies in Section 8.31. For a dipole antenna the latter are given by equation 8–163 where $\beta = \omega\sqrt{\mu\varepsilon} = p\sqrt{\mu\varepsilon}/j$; since the end effect represented by δ approaches zero in the limit, the asymptotic positions of the zeros of the dipole antenna are given by

$$p_m = \frac{jm\pi}{2l\sqrt{\mu\varepsilon}}, \qquad m = 1, 3, 5, 7, \cdots. \tag{15}$$

The asymptotic positions of the poles are obtained from equation 8–161,

$$p_m = \frac{jm\pi}{2l\sqrt{\mu\varepsilon}}, \qquad m = 0, 2, 4, 6, \cdots. \tag{16}$$

To these we must also add the zeros and poles corresponding to the negative values of m for the following reason. The coefficients in Maxwell's equations 2 are real; therefore, the coefficients in $Z(p)$ are real. Taking the conjugate of equation 8, we obtain

$$[Z(p)]^* = Z(p^*) = 0. \tag{17}$$

Hence, if $p = p_n$ is a zero of $Z(p)$, then $p = p_n^*$ is also a zero. The same is true of the zeros of $Y(p)$ and, hence, of the poles of $Z(p)$. Thus, *the zeros and poles of $Z(p)$ occur in conjugate pairs*, or else they must lie on the real axis. The conjugates of equations 15 and 16 are given by the negative values of m.

Figure 9.3 shows the asymptotic distribution of zeros (hollow circles) and poles (solid circles) of the impedance of a dipole antenna. For a loop antenna, the positions of zeros and poles are interchanged. An antenna of nonzero radius radiates power, and its natural oscillations are damped; hence, the zeros and poles will move off the imaginary axis into the left half of the p plane (all except $p = 0$). They are close to the imaginary axis for antennas of small radius, and far away for those of large radius. Figures 9.4 and 9.5 show a spherical antenna and the location of the smallest zeros of its input impedance. The poles are not shown because their positions depend on the distance between the hemispheres.

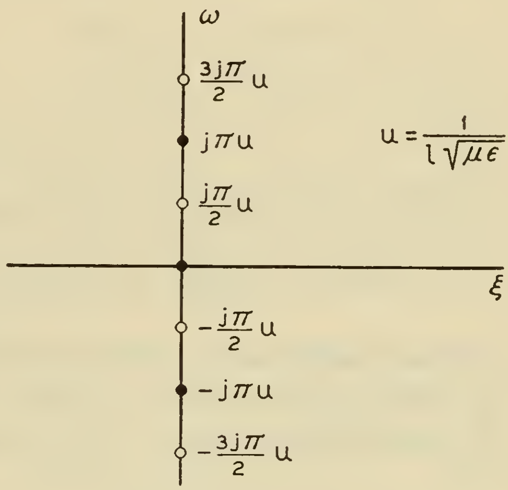

Fig. 9.3 The asymptotic distribution of zeros (hollow circles) and poles (solid circles) of the impedance of a dipole antenna. For a loop antenna, the positions of the zeros and poles are interchanged.

In the next section we shall find that the zeros and poles determine the impedance except for a constant factor; hence, they determine the behavior of the impedance as a function of frequency.

9.3 Expressions for $Z(p)$ and $Y(p)$ in terms of zeros and poles

The impedance of a network consisting of a finite number of resistors,

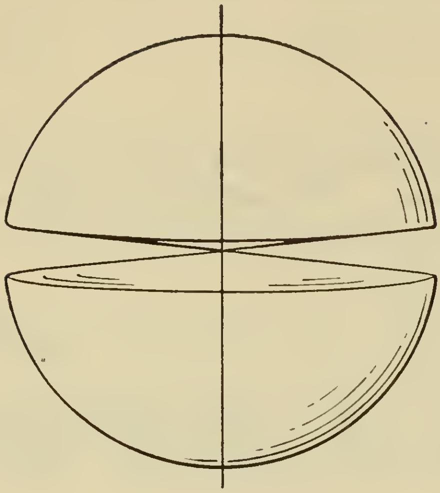

FIG. 9.4 A spherical antenna.

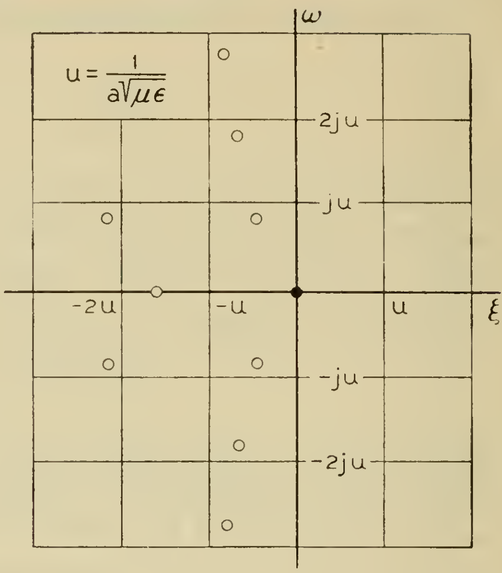

FIG. 9.5 Location of the smallest zeros of the input impedance of the spherical antenna.

inductors, and capacitors is a rational fraction,

$$Z(p) = \frac{N(p)}{D(p)} = \frac{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0}, \tag{18}$$

with *real* coefficients; so is the admittance. This conclusion follows from the differential equations of the network. For example, the equation for the instantaneous values of the current in a series circuit (Fig. 9.6) is

$$L \frac{d\tilde{I}_1}{dt} + R\tilde{I}_1 + \frac{\int \tilde{I}_1 \, dt}{C} = \tilde{V}_1, \tag{19}$$

where $\tilde{V}_1$ is the input voltage. Substituting

$$\tilde{V}_1 = V_1 e^{pt}, \qquad \tilde{I}_1 = I_1 e^{pt}, \tag{20}$$

and solving, we find the impedance and admittance,

$$\begin{aligned} Z_1(p) &= \frac{V_1}{I_1} = R + pL + \frac{1}{pC} = \frac{p^2LC + pRC + 1}{pC}, \\ Y_1(p) &= \frac{pC}{p^2LC + pRC + 1}. \end{aligned} \tag{21}$$

Similarly, for a parallel circuit (Fig. 9.7) we find

$$Z_2(p) = \frac{pL}{p^2LC + pGL + 1}, \qquad Y_2(p) = G + pC + \frac{1}{pL}, \tag{22}$$

where $G = 1/R$ is the conductance.

If the network contains more meshes, there will be more differential equations. On substituting typical exponential variables $V \exp(pt)$ and $I \exp(pt)$, each differentiation with respect to t becomes equivalent to a multiplication by p. After canceling the exponential factor, we obtain linear algebraic equations in typical variables V, I whose coefficients are polynomials or rational fractions (when there are terms such as $1/pC$ or

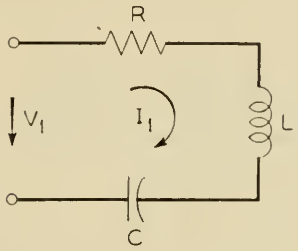

FIG. 9.6 A series circuit.

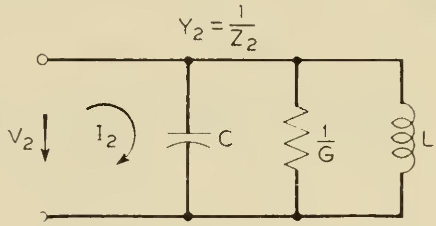

FIG. 9.7 A parallel circuit.

$1/pL$). Solving these equations for any particular ratio V/I, we find that this ratio must be of the form suggested in equation 18. If V and I belong to the same pair of terminals, the ratio represents the input impedance across that pair of terminals. Since a zero (or a pole) at infinity must be simple, the *difference between the degrees of the numerator and the denominator cannot exceed unity.*

By the fundamental theorem of algebra, the polynomials $N(p)$ and $D(p)$ in equation 18 can be factored; thus,

$$Z(p) = \frac{a_n(p - p_1')(p - p_2')(p - p_3')\cdots}{b_m(p - p_1)(p - p_2)(p - p_3)\cdots}, \tag{23}$$

where $p_1, p_2, p_3, \cdots$ are the poles of $Z(p)$ and $p_1', p_2', p_3', \cdots$ are the zeros. Since

$$p - p_1 = -p_1\left(1 - \frac{p}{p_1}\right), \tag{24}$$

equation 23 may be written as

$$Z(p) = \frac{(-)^n a_n p_1' p_2' \cdots p_n' \left(1 - \frac{p}{p_1'}\right)\left(1 - \frac{p}{p_2'}\right)\cdots}{(-)^m b_m p_1 p_2 \cdots p_m \left(1 - \frac{p}{p_1}\right)\left(1 - \frac{p}{p_2}\right)\cdots}$$

$$= \frac{a_0\left(1 - \frac{p}{p_1'}\right)\left(1 - \frac{p}{p_2'}\right)\cdots\left(1 - \frac{p}{p_n'}\right)}{b_0\left(1 - \frac{p}{p_1}\right)\left(1 - \frac{p}{p_2}\right)\cdots\left(1 - \frac{p}{p_m}\right)}. \tag{25}$$

For example, the zeros of the impedance of the series circuit may be found from equations 21,

$$Z_1(p) = 0, \qquad p^2LC + pRC + 1 = 0,$$

$$p_{1,2} = -\frac{R}{2L} \pm \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]^{1/2}. \tag{26}$$

Then,

$$Z_1(p) = \frac{L(p - p_1)(p - p_2)}{p} = \frac{\left(1 - \dfrac{p}{p_1}\right)\left(1 - \dfrac{p}{p_2}\right)}{pC}, \tag{27}$$

$$Y_1(p) = \frac{p}{L(p - p_1)(p - p_2)}.$$

Evidently p_1 and p_2 are negative real and distinct when

$$\frac{R}{2L} > (LC)^{-1/2}, \quad \text{or} \quad R > 2K, \tag{28}$$

where

$$K = \left(\frac{L}{C}\right)^{1/2}. \tag{29}$$

In the complex p plane these values are represented by two points A and B, on the negative real axis (Fig. 9.8*a*). When $R = 2K$, the two zeros coincide and become a *double* zero.

When $R < 2K$, the zeros are conjugate imaginaries,

$$p_1 = \xi_1 + j\omega_1, \qquad p_2 = \xi_1 - j\omega_1,$$

where

$$\xi_1 = -\frac{R}{2L}, \qquad \omega_1 = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} = \left[\frac{1 - \frac{1}{4}\Delta^2}{LC}\right]^{1/2}, \tag{30}$$

and

$$\Delta = \frac{R}{\sqrt{L/C}} = \frac{R}{K}. \tag{31}$$

These zeros are shown in Fig. 9.8*c*.

Similarly, for the parallel circuit (Fig. 9.7) the infinities of the impedance are

$$p_{1,2} = -\frac{G}{2C} \pm \left[\left(\frac{G}{2C}\right)^2 - \frac{1}{LC}\right]^{1/2}, \tag{32}$$

and the impedance itself may be expressed as

$$Z_2(p) = \frac{p}{C(p - p_1)(p - p_2)}. \tag{33}$$

The preceding equations for the impedances and admittances exemplify all the important properties of the more general functions. The zeros and poles are either in the left half of the p plane or on the

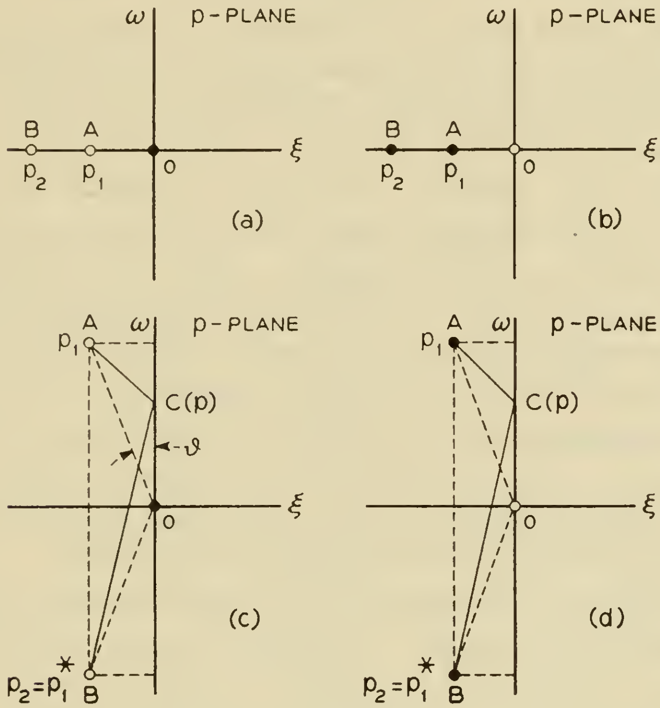

FIG. 9.8 Zeros (hollow circles) and poles (solid circles) of the impedances of simple series and parallel circuits: (*a*) for the series circuit when $R > 2K$; (*b*) for the parallel circuit when $R > 2K$; (*c*) for the series circuit when $R < 2K$; (*d*) for the parallel circuit when $R < 2K$.

imaginary axis. They are all on the imaginary axis only when the circuits are nondissipative ($R = 0$ or $G = 0$). In slightly dissipative circuits the zeros and poles are close to the imaginary axis, and the angle ϑ shown in Fig. 9.8*c*, and defined by

$$\tan \vartheta = -\frac{\xi_1}{\omega_1}, \tag{34}$$

is small compared with unity. As the dissipation increases, they move away from the imaginary axis. In practice, we are interested in the impedances at real frequencies, when $p = j\omega$ is on the imaginary axis. The absolute value of a product of several factors equals the product of the absolute values of the factors. The absolute value of a typical

linear factor $p - p_1$ in equation 23 equals the length of the segment joining the points p and p_1 (Fig. 9.8c). Hence, if a particular zero is close to the imaginary axis, the impedance at real frequencies in the vicinity of this zero is small, and we have the phenomenon known as *resonance.* Similarly, the impedance is large for frequencies in the vicinity of a pole, located near the imaginary axis. If several zeros and poles are near the imaginary axis, the impedance will fluctuate between small and large values as the frequency passes these points. As the zeros and poles recede from the imaginary axis, the fluctuations become less pronounced, and the "resonance curves" become flatter.

Normally the zeros and poles are simple; but, for some special combinations of circuit constants (when $R = 2K$ in the above examples, for instance), they may coalesce and become multiple zeros and poles. When they are on the imaginary axis, they are simple and separate each other. When they are off the imaginary axis, they are either conjugate imaginaries or negative real. In the case of more complicated networks, there are more zeros and poles; but their essential properties are the same. If we count the zero (or pole) at infinity ($p = \infty$), we find that the number of zeros equals the number of poles, that is, if we count the multiple zeros and poles according to the degree of their multiplicity. Series and parallel circuits are *dual* in the sense that the impedance of one has the same form as the admittance of the other.

As the number of elements in the circuit increases, the number of zeros and poles also increases. In Chapter 2 we have seen that continuous structures, including all free space, are limits of networks with an increasingly larger number of increasingly smaller meshes. The number of their zeros and poles will be infinite. In fact, this must be true of any physical circuit since all physical circuits are continuous and cannot be wholly dissociated from the surrounding space. As we deliberately localize regions of concentration of electric and magnetic fields (capacitors and inductors, respectively) and regions in which the electromagnetic energy is transformed into heat (resistors), we find that some zeros and poles move closer to the origin, $p = 0$, and form a cluster more or less clearly separated from the distant zeros and poles. The latter do not affect the impedance at frequencies close to the cluster, and we have a "network of lumped elements."

Transmission lines are simple examples of circuits with infinitely many zeros and poles. Thus, the impedance of a nondissipative line of length l, closed through a resistance R at the far end, is

$$Z(p) = K \frac{R \cosh(pl\sqrt{LC}) + K \sinh(pl\sqrt{LC})}{K \cosh(pl\sqrt{LC}) + R \sinh(pl\sqrt{LC})}, \tag{35}$$

where $K = \sqrt{L/C}$ is the characteristic impedance. If $R < K$, the zeros are given by

$$p_n = \frac{-\tanh^{-1}(R/K) \pm jn\pi}{l\sqrt{LC}}, \tag{36}$$

and the poles by

$$p_n = \frac{-\tanh^{-1}(R/K) \pm j(n + \frac{1}{2})\pi}{l\sqrt{LC}}. \tag{37}$$

They are equispaced on a straight line parallel to the imaginary axis (Fig. 9.9). If $R > K$, the positions of the zeros and poles are inter-

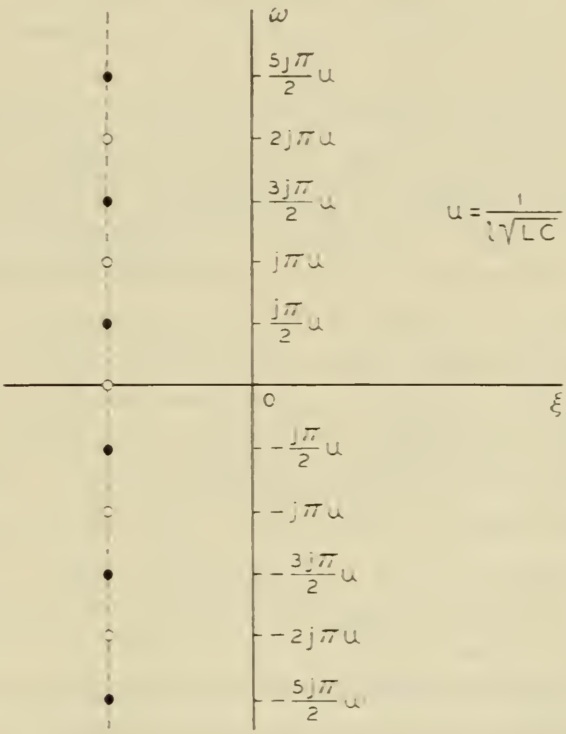

FIG. 9.9 Zeros and poles of a nondissipative uniform transmission line terminated into a resistance different from its characteristic impedance.

changed. The inverse hyperbolic tangent of unity is infinite; hence, the line of zeros and poles recedes leftward to infinity as R approaches K. As R approaches zero or infinity, the line of zeros and poles approaches the imaginary axis. If we keep R fixed and let the characteristic impedance K increase indefinitely, the zeros and poles will also approach the imaginary axis (since R/K will approach zero), and the resonances will tend to become sharper.

To illustrate the infinite product forms of the impedance, we shall assume that $R = 0$ in equation 35; then $Z(p)$ is proportional to the

hyperbolic tangent of $pl\sqrt{LC}$, and*

$$Z(p) = K \tanh pl\sqrt{LC} = \frac{pLl \prod_{n=1}^{\infty} \left(1 - \frac{p}{p_n'}\right)\left(1 - \frac{p}{p_n'^*}\right)}{\prod_{n=0}^{\infty} \left(1 - \frac{p}{p_n}\right)\left(1 - \frac{p}{p_n^*}\right)}, \tag{38}$$

$$p_n' = \frac{jn\pi}{l\sqrt{LC}}, \qquad p_n = \frac{j(2n+1)\pi}{2l\sqrt{LC}}.$$

Similar equations may be obtained for a line open at the far end ($R = \infty$) and for a line closed through a given impedance.

In the case of two transmission lines, so separated that the interaction between them is negligible, we have two strings of zeros and poles. As the two lines approach each other, the interaction increases, and the positions of the zeros and poles are altered; but there are still two strings. For n coupled transmission lines there are n strings of zeros and poles. When the coupling is strong, we may even find it difficult to distinguish between separate strings "belonging" to the various transmission lines; we shall merely observe that the zeros and poles are scattered on an infinitely long ribbon of a certain width. In general, the zeros and poles may be scattered throughout the entire left half of the p plane without any definite pattern — as they are in the case of a spherical antenna (Fig. 9.4). Such a distribution is typical of any physical circuit whose physical dimensions are of comparable magnitudes. When one dimension is much larger than the other two, those zeros and poles that are closest to the origin will form a recognizable string. Thus, we find a similarity between the patterns of zeros and poles belonging to the impedance functions of thin antennas and transmission lines and a corresponding similarity in the behavior of the impedance functions.

Irrespective of the number of zeros and poles, the impedance functions may be expressed as ratios of products of linear factors. This theorem is a special case of the Weierstrass theorem in the theory of functions of a complex variable. When the number of factors is infinite, questions of convergence arise. For instance, the two product forms (equations 23 and 25) are equally valid when the number of factors is finite; but, when the number of factors is infinite, the form 23 is divergent. This is due to the fact that the first term $a_n p^n$ of the product approaches infinity as n approaches infinity except when p is zero. On the other hand, the form 25 remains valid as long as we arrange the zeros

* Edwin P. Adams, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Washington, 1922, p. 130.

and poles according to the order of their magnitudes, as, for instance, in equation 38. The factors in the product may be reshuffled only if certain " Weierstrass convergence factors " are included; but this discussion would take us beyond the scope of this book. The main fact, as far as we are concerned, is that *the impedance of any physical circuit may be expressed as the ratio of two products of linear factors exhibiting the natural oscillation constants of the circuit with its terminals at first floating and then short-circuited.* Thus, for a dipole antenna (Fig. 9.2*a*),

$$Z(p) = \frac{\left(1 - \frac{p}{p_1}\right)\left(1 - \frac{p}{p_1^*}\right)\left(1 - \frac{p}{p_3}\right)\left(1 - \frac{p}{p_3^*}\right)\cdots}{pC\left(1 - \frac{p}{p_2}\right)\left(1 - \frac{p}{p_2^*}\right)\left(1 - \frac{p}{p_4}\right)\left(1 - \frac{p}{p_4^*}\right)\cdots}, \tag{39}$$

where $p_1, p_1^*, p_3, p_3^*, \cdots$ are the oscillation constants with the antenna terminals short-circuited, and $p_2, p_2^*, p_4, p_4^*, \cdots$ are the corresponding constants with the terminals floating. The factor p in the denominator is characteristic of doublet antennas which, by definition, have a pole at $p = 0$. For real frequencies,

$$Z(j\omega) = \frac{\left(1 - \frac{j\omega}{p_1}\right)\left(1 - \frac{j\omega}{p_1^*}\right)\left(1 - \frac{j\omega}{p_3}\right)\cdots}{j\omega C\left(1 - \frac{j\omega}{p_2}\right)\left(1 - \frac{j\omega}{p_2^*}\right)\left(1 - \frac{j\omega}{p_4}\right)\cdots}. \tag{40}$$

As $\omega \to 0$,

$$Z(j\omega) \to \frac{1}{j\omega C}, \tag{41}$$

where C is the d-c capacitance; and it should be noted that C in equation 40 is a constant. Thus, the *d-c capacitance of an antenna together with its natural oscillation constants determines the antenna impedance at all frequencies.* When the damping constants of the natural oscillations are small, we may neglect their squares; hence,

$$\begin{aligned}\left(1 - \frac{j\omega}{p_1}\right)\left(1 - \frac{j\omega}{p_1^*}\right) &= 1 - j\omega\,\frac{p_1 + p_1^*}{p_1 p_1^*} - \frac{\omega^2}{p_1 p_1^*} \\ &= 1 - \frac{2j\omega\xi_1}{\omega_1^2 + \xi_1^2} - \frac{\omega^2}{\omega_1^2 + \xi_1^2} \\ &\simeq 1 - \frac{\omega^2}{\omega_1^2} - \frac{2j\omega\xi_1}{\omega_1^2}.\end{aligned} \tag{42}$$

Therefore,

$$Z(j\omega) \simeq \frac{\left(1 - \dfrac{\omega^2}{{\omega_1}^2} - \dfrac{2j\xi_1\omega}{{\omega_1}^2}\right)\left(1 - \dfrac{\omega^2}{{\omega_3}^2} - \dfrac{2j\xi_3\omega}{{\omega_3}^2}\right)\cdots}{j\omega C\left(1 - \dfrac{\omega^2}{{\omega_2}^2} - \dfrac{2j\xi_2\omega}{{\omega_2}^2}\right)\left(1 - \dfrac{\omega^2}{{\omega_4}^2} - \dfrac{2j\xi_4\omega}{{\omega_4}^2}\right)\cdots} \cdot \tag{43}$$

For a perfectly conducting loop antenna (Fig. 9.2*b*), we find, similarly,

$$Z(p) = \frac{pL\left(1 - \dfrac{p}{p_1}\right)\left(1 - \dfrac{p}{{p_1}^*}\right)\left(1 - \dfrac{p}{p_3}\right)\left(1 - \dfrac{p}{{p_3}^*}\right)\cdots}{\left(1 - \dfrac{p}{p_2}\right)\left(1 - \dfrac{p}{{p_2}^*}\right)\left(1 - \dfrac{p}{p_4}\right)\left(1 - \dfrac{p}{{p_4}^*}\right)\cdots} \cdot \tag{44}$$

The factor p in the numerator represents the zero at $p = 0$ which is characteristic of the impedance of a closed circuit without loss when the current is steady. The coefficient L is the d-c inductance of the loop, and, as $p = j\omega \to 0$,

$$Z(j\omega) \to j\omega L. \tag{45}$$

9.4 Resonance and antiresonance in simple circuits

Under certain conditions the response of an antenna to an impressed voltage is similar to the response of either a series or a parallel circuit. To show this we shall express the impedances of these simple circuits in terms of parameters having more general meaning than the resistance, inductance, and capacitance. First we shall consider the series circuit (Fig. 9.6). Letting $p = j\omega$ in equation 21, we have

$$Z_1(j\omega) = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \cdot \tag{46}$$

The reactive component vanishes at a certain frequency $\omega = \omega_r$ given by

$$\omega_r L - \frac{1}{\omega_r C} = 0, \qquad \omega_r = \frac{1}{\sqrt{LC}} \cdot \tag{47}$$

This frequency is called the *resonant frequency* of the circuit. Since the magnitude of the impedance,

$$|Z_1(j\omega)| = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}, \tag{48}$$

is minimum at $\omega = \omega_r$, the current for a fixed impressed voltage is maximum. In accordance with equation 47, the impedances of the

inductor and capacitor become equal at resonance, and, in view of equation 29,

$$\omega_r L = \frac{1}{\omega_r C} = K. \tag{49}$$

This equation assigns a physical meaning to the parameter K which originally was defined simply as the square root of the ratio of L to C. We may now express the impedance of the series circuit at any frequency in terms of the resonant frequency ω_r defined by equation 47, the characteristic impedance K of the circuit, and the parameter Δ defined by equation 31; thus,

$$\begin{aligned} Z_1(j\omega) &= K\left[\Delta + j\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right], \\ |Z_1(j\omega)| &= K\left[\Delta^2 + \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)^2\right]^{1/2}. \end{aligned} \tag{50}$$

These forms are convenient because Δ and ω/ω_r are dimensionless parameters.

The magnitude of the current in response to a unit voltage is, then,

$$|I_1| = \frac{1}{|Z_1(j\omega)|}, \qquad |I_1|_{\max} = \frac{1}{K\Delta}. \tag{51}$$

Figure 9.10*a* represents the current relative to the maximum current as a function of f/f_r; the lower curve represents the reciprocal of this response, or the relative magnitude of the voltage needed to produce a given current at different frequencies.

The response drops to $1/\sqrt{2}$ fraction of its maximum value when

$$\left|\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right| = \left|\frac{f}{f_r} - \frac{f_r}{f}\right| = \Delta. \tag{52}$$

There is one such frequency on either side of the resonant frequency,

$$\frac{f_1}{f_r} = -\tfrac{1}{2}\Delta + (1 + \tfrac{1}{4}\Delta^2)^{1/2}, \qquad \frac{f_2}{f_r} = \tfrac{1}{2}\Delta + (1 + \tfrac{1}{4}\Delta^2)^{1/2}. \tag{53}$$

The difference $f_2 - f_1$ is called the " width " of the resonant curve; the parameter Δ is seen to be *equal to the relative width of the curve*,

$$\Delta = \frac{f_2 - f_1}{f_r}. \tag{54}$$

The inphase and the quadrature components of the response are shown, respectively, by the solid and the dotted curves in Fig. 9.11. These curves also represent the conductance and the susceptance of the

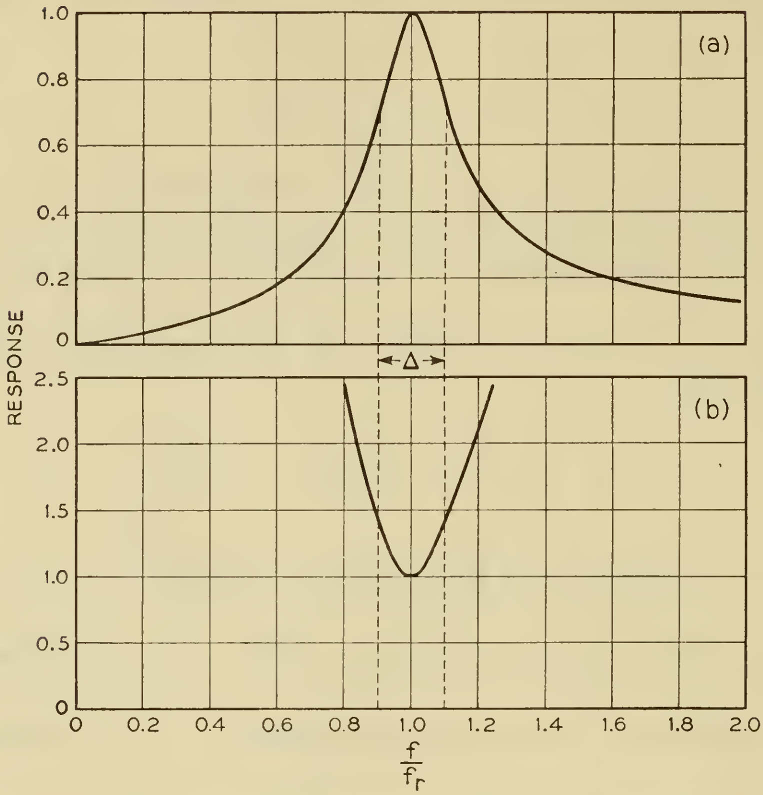

FIG. 9.10 The magnitude of a relative response and its reciprocal (resonance curves).

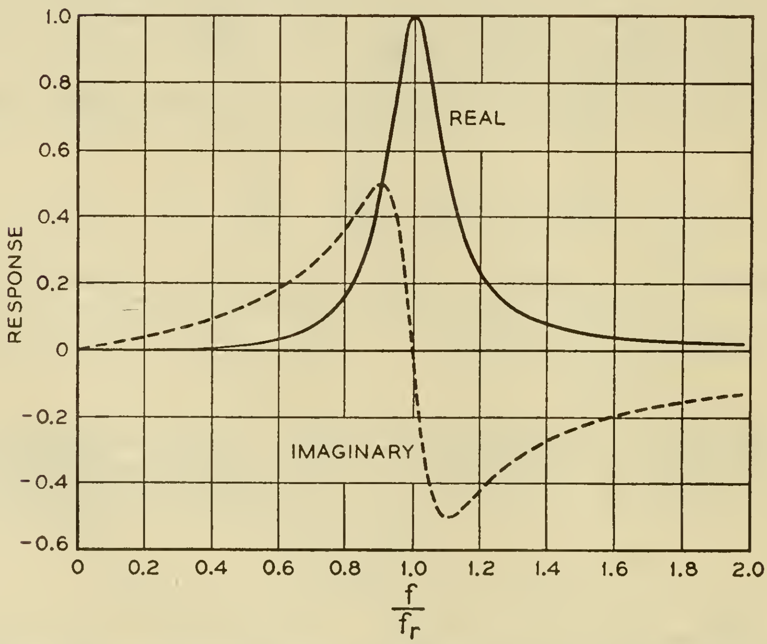

FIG. 9.11 Real and imaginary parts of a relative response (resonance curves).

circuit in terms of the admittance at resonance. The *difference between the maximum and minimum susceptance is equal to the maximum conductance.*

The reciprocal of the relative width of the resonant curve is called the *quality factor:*

$$Q = \frac{f_r}{f_2 - f_1} = \frac{1}{\Delta} = \frac{K}{R} = \frac{\omega_r L}{R}. \tag{55}$$

At resonance the energy $\mathcal{E}$ stored in the circuit and the average power P_{av} dissipated in R are

$$\mathcal{E} = \tfrac{1}{2}LI_a^2, \qquad P_{av} = \tfrac{1}{2}RI_a^2, \tag{56}$$

where I_a is the current amplitude. Therefore,

$$Q = \frac{\omega_r \mathcal{E}}{P_{av}}, \quad \Delta = \frac{P_{av}}{\omega_r \mathcal{E}}. \tag{57}$$

At resonance the impedance of the circuit is real, and the generator supplies merely the power dissipated in the resistance; there is no other exchange of energy between the generator and the circuit. If Δ is small and Q large, the energy contributed to the circuit per cycle is small. We may look on this energy as the energy needed to sustain natural oscillations at a constant level, for the difference between the resonant frequency (equation 47) and the natural frequency ω_1 given by equation 30 is small. Hence, we conclude that

$$Q \simeq \frac{\omega_1 \mathcal{E}}{P_{av}}, \qquad \Delta \simeq \frac{P_{av}}{\omega_1 \mathcal{E}}, \tag{58}$$

where $\mathcal{E}$ is the total energy of a freely oscillating circuit and P_{av} is the power dissipated in the resistance. Thus, we have a connection between resonance and natural oscillations. From equation 30 we also find

$$\xi_1 = -\frac{R}{2L} = -\frac{\omega_1 R}{2\omega_1 L} \simeq \frac{\omega_1 R}{2\omega_r L} = -\frac{\omega_1 R}{2K} = -\tfrac{1}{2}\omega_1 \Delta. \tag{59}$$

From this equation and from equation 34, we have

$$\Delta = -\frac{2\xi_1}{\omega_1} = 2 \tan \vartheta. \tag{60}$$

Also,

$$p_1 = \xi_1 + j\omega_1 = -\tfrac{1}{2}\omega_1 \Delta + j\omega_1 = (-\tfrac{1}{2}\Delta + j)\omega_1. \tag{61}$$

The response of the parallel circuit is reciprocal to that of the series circuit. The curve in Fig. 9.10a represents the absolute value of the input impedance in terms of the maximum impedance, and, hence, the

relative voltage required to produce a given current; the curve in Fig. 9.10*b* represents the relative response of the circuit to a given voltage; and the curves in Fig. 9.11 represent the resistive and reactive components of the input impedance in terms of the maximum impedance. The circuit is said to be *antiresonant* at $f = f_r$. The input admittance of the circuit may be obtained by substituting $p = j\omega$ in equation 22; then it may be expressed in terms of the "characteristic impedance" K, the width of the resonant curve Δ, and the antiresonant frequency f_r or ω_r; thus,

$$Y_2(j\omega) = K^{-1}\left[\Delta + j\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right],$$
$$K = \sqrt{\frac{L}{C}}, \qquad \Delta = GK = \frac{G}{\omega_r C}, \qquad \omega_r = \frac{1}{\sqrt{LC}}. \tag{62}$$

The difference between the maximum and minimum reactance is equal to the maximum resistance.

9.5 Resonance and antiresonance in complicated circuits

Let us assume that a particular zero $p = p_1$ of any circuit whatsoever is close to the imaginary axis, and let us suppose that there are no other zeros or poles in its vicinity In the neighborhood of this zero,

$$Z(p) \simeq (p - p_1)(p - p_1^*) f(p_1), \tag{63}$$

since all other factors $p - p_2$, $p - p_3, \cdots$ are substantially constant for small variations in p. We might have replaced $p - p_1^*$ by $p_1 - p_1^*$ and included this factor in $f(p_1)$; but the above form is more convenient for comparing $Z(p)$ with the impedance of a series circuit. On the imaginary axis,

$$\begin{aligned}(p - p_1)(p - p_1^*) &= (j\omega - \xi_1 - j\omega_1)(j\omega - \xi_1 + j\omega_1)\\ &= -\omega^2 - 2j\xi_1\omega + \omega_1^2 + \xi_1^2.\end{aligned} \tag{64}$$

Neglecting the square of the small quantity ξ_1, we find

$$(p - p_1)(p - p_1^*) = j\omega_1\omega\left[-\frac{2\xi_1}{\omega_1} + j\left(\frac{\omega}{\omega_1} - \frac{\omega_1}{\omega}\right)\right]. \tag{65}$$

Therefore, equation 63 becomes

$$Z(p) \simeq A\left[-\frac{2\xi_1}{\omega_1} + j\left(\frac{\omega}{\omega_1} - \frac{\omega_1}{\omega}\right)\right], \tag{66}$$

where A is a constant depending on the other zeros and poles of $Z(p)$. In view of equation 60, this expression is of the same form as equation 50 for the impedance of the series circuit. The only possible difference is

that A may be complex. The absolute value of A plays the part of the characteristic impedance K of the series circuit; the phase of A affects the position of the resonant frequency, but its value is not important, since the phase of the bracketed factor in equation 66 changes very rapidly from almost $-90°$ when ω is smaller than ω_1 to almost $+90°$ when ω is larger than ω_1. Hence, there will be some value of ω, not far from ω_1, for which the phase of $Z(p)$ will vanish, and there will be resonance.

Similarly, in the vicinity of a pole, close to the imaginary axis and well separated from other zeros and poles, any circuit behaves as a parallel circuit. These considerations simplify the treatment of self-resonant antennas (see Chapter 11).

9.6 Small dipole and loop antennas

The impedance of a dipole antenna is given by equation 39, and it may be expanded in a power series

$$Z(p) = \frac{1}{pC} + R + pL + Ap^2 + \cdots. \tag{67}$$

To show this, we note that, if we multiply out the various factors in the numerator, we obtain a power series in p, beginning with a constant term equal to unity. The remaining factors except $1/pC$ may also be expanded in power series,

$$\left(1 - \frac{p}{p_2}\right)^{-1} = 1 + \frac{p}{p_2} + \frac{p^2}{{p_2}^2} + \frac{p^3}{{p_2}^3} + \cdots . \tag{68}$$

The product of such factors and the preceding power series is a power series beginning with unity. When we divide the result by pC, we obtain a series of the form shown in equation 67. The series 68 is convergent as long as the absolute value of p is less than the absolute value of p_2; therefore the series 67 is convergent as long as the absolute value of p is less than the absolute value of the pole nearest to the origin (excepting $p = 0$).

On the imaginary axis, equation 67 becomes

$$Z(j\omega) = \frac{1}{j\omega C} + R + j\omega L - A\omega^2 + \cdots. \tag{69}$$

In Chapter 5 we have seen that the radiated power approaches zero as the frequency approaches zero; hence, if there is no dissipation in the antenna and the surrounding medium, $R = 0$, and equation 69 becomes

$$Z(j\omega) = \frac{1}{j\omega C} + j\omega L - A\omega^2 + \cdots. \tag{70}$$

The coefficients C, L, A are independent of ω, of course. The coefficient

C is the d-c capacitance, L the d-c inductance, and $-A\omega^2$ the radiation resistance at low frequencies. In the next chapter we shall show that these terms may be calculated without solving the more difficult problem of obtaining $Z(p)$ at all frequencies.

The admittance of the dipole antenna is the reciprocal of equation 70,

$$Y(j\omega) = j\omega C + j\omega^3 LC^2 - AC^2\omega^4 + \cdots. \tag{71}$$

Thus, the radiation conductance at low frequencies varies as the fourth power of the frequency.

Similarly, the admittance of a perfectly conducting loop antenna for frequencies below the first resonant frequency may be expressed as a power series,

$$Y(j\omega) = \frac{1}{j\omega L} + j\omega C - B\omega^2 + \cdots, \tag{72}$$

where L is the d-c inductance of the loop, C the d-c capacitance, and $-B\omega^2$ the radiation conductance at low frequencies. Its reciprocal,

$$Z(j\omega) = j\omega L + j\omega^3 CL^2 - BL^2\omega^4 + \cdots, \tag{73}$$

shows that the radiation resistance of the loop varies as the fourth power of the frequency when the frequencies are sufficiently low.

The preceding expressions are useful when the dimensions of the antenna are small compared with the wavelength.

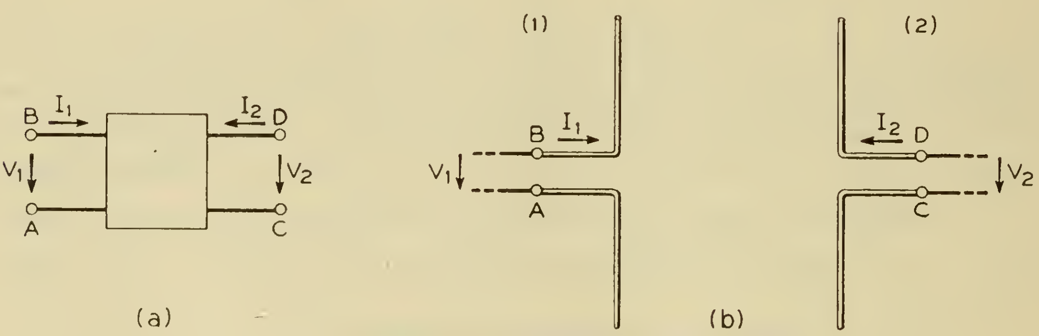

FIG. 9.12 Schematic diagrams of (*a*) a transducer and (*b*) a pair of antennas regarded as a transducer.

9.7 Linear transducers

A structure with two or more *pairs* of accessible terminals is called a *transducer*. The difference between a *four-terminal transducer* and a four-terminal network or a *four-pole* is that in the transducer the terminals are paired and in the four-pole they are not. In the transducer (Fig. 9.12) the input is always either across A, B or across C, D; never

across A, C or A, D. In a four-pole the input may be across any two terminals. Two antennas do not form a four-pole, except when they are very close to each other;* but they form a transducer as long as the gaps between their respective input terminals are small.

A transducer is linear if the voltages across each pair of terminals are linear functions of the currents:

$$V_1 = Z_{11}I_1 + Z_{12}I_2,$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2. \tag{74}$$

This implies, of course, that the currents are also linear functions of the voltages. Thus, solving equations 74 for I_1 and I_2, we obtain

$$I_1 = Y_{11}V_1 + Y_{12}V_2,$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2, \tag{75}$$

where

$$Y_{11} = \frac{Z_{22}}{D}, \qquad Y_{12} = -\frac{Z_{12}}{D}, \qquad Y_{21} = -\frac{Z_{21}}{D}, \qquad Y_{22} = \frac{Z_{11}}{D},$$
$$D = Z_{11}Z_{22} - Z_{12}Z_{21}. \tag{76}$$

The directions of the voltages shown in Fig. 9.12 are for the field voltages; they are opposite to the voltages impressed by the generators.

If $I_2 = 0$, then,

$$Z_{11} = \frac{V_1}{I_1}, \qquad Z_{21} = \frac{V_2}{I_1}; \tag{77}$$

that is, if a unit current is passing through A, B when C, D are floating, the voltage across A, B is Z_{11} and that across C, D is Z_{21}. Similarly, if $I_1 = 0$, then,

$$Z_{12} = \frac{V_1}{I_2}, \qquad Z_{22} = \frac{V_2}{I_2}. \tag{78}$$

The quantities Z_{12} and Z_{21} are called the *mutual* or *transfer impedances.*

If the transducer is short-circuited at the second pair of terminals, $V_2 = 0$, and, from equations 75, we find

$$Y_{11} = \frac{I_1}{V_1}, \qquad Y_{21} = \frac{I_2}{V_1}; \tag{79}$$

that is, one volt across A, B when C and D are connected will produce

*For the simple reason that we can attach no meaning to the impedance across a pair of distant terminals.

Y_{11} amperes at A, B and Y_{21} amperes across C, D. Similarly, if A and B are connected, $V_1 = 0$, and

$$Y_{12} = \frac{I_1}{V_2}, \qquad Y_{22} = \frac{I_2}{V_2}. \tag{80}$$

Solving equations 75 for V_1, V_2 and comparing with equations 74, we find

$$Z_{11} = \frac{Y_{22}}{\Delta}, \qquad Z_{12} = -\frac{Y_{12}}{\Delta}, \qquad Z_{21} = -\frac{Y_{21}}{\Delta}, \qquad Z_{22} = \frac{Y_{11}}{\Delta},$$
$$\Delta = Y_{11}Y_{22} - Y_{12}Y_{21}. \tag{81}$$

By definition, the above equations are characteristic of all linear transducers: electric, mechanical, electromechanical, acoustic, etc. Whether a particular physical system is linear depends on the dynamical equations governing its behavior. Maxwell's equations, for example, are linear if the conductivity, permeability, and dielectric constant are independent of the field intensity; hence, in this case the associated electric transducers will be linear. Ferromagnetic substances are nonlinear, especially near saturation; there are resistive substances, such as Thirite, which are also nonlinear. Any transducer which contains any nonlinear substance will be nonlinear. Although the earth's crust contains some nonlinear substances, the amount is negligible in its effect on wave propagation, and antenna pairs form substantially linear transducers. The impedance coefficients may be either measured* or calculated.†

9.8 Reciprocity theorems

In circuit theory, it is shown that, for any transducer consisting of a network of linear resistances, inductances, and capacitances, the matrices of the coefficients in equations 74 and 75 are symmetric; that is:

$$Z_{21} = Z_{12}, \qquad Y_{21} = Y_{12}. \tag{82}$$

We have already pointed out (Section 2.4) that a continuous medium may be considered as the limit of a network. Hence, the reciprocity theorem must apply to transducers whose parts are continuous media.‡

* C. R. Englund and A. B. Crawford, The mutual impedance between adjacent antennas, *IRE Proc.*, 17, August 1929, pp. 1277–1295.

† P. S. Carter, Circuit relations in radiating systems and applications to antenna problems, *IRE Proc.*, **20**, June 1932, pp. 1004–1041.

‡ For a direct proof based on Maxwell's equations see, for instance, *Electromagnetic Waves*, pp. 476–479. It should be noted, however, that reciprocity theorems are properties of linear dynamical systems and as such are independent of specific

Since Z_{21} is the voltage produced across C, D by one ampere through A, B, and Z_{12} is the voltage across A, B due to one ampere through C, D (Fig. 9.12), the above "reciprocity theorem" may be stated as follows: *In any physical linear network, the positions of a generator having infinite*

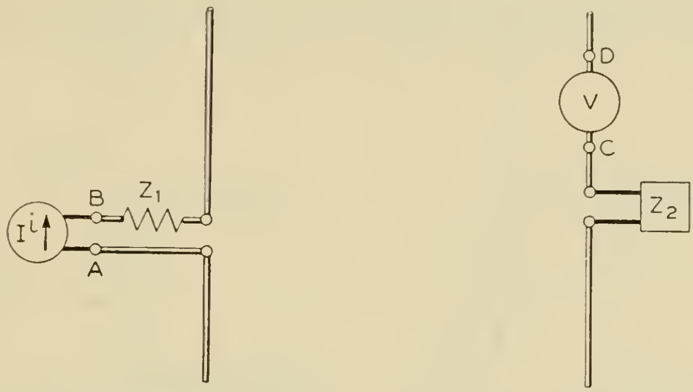

FIG. 9.13 The positions of a generator with infinite internal impedance at A, B and a voltmeter with infinite internal impedance at C, D may be interchanged without affecting the voltmeter's reading.

internal impedance and a voltmeter may be interchanged without affecting the voltmeter's reading (Fig. 9.13).

Likewise, *in any physical linear network the positions of a generator having zero internal impedance and an ammeter may be interchanged without affecting the ammeter's reading* (Fig. 9.14).

The voltmeter and ammeter in these theorems are assumed to be ideal; that is, the impedance of the voltmeter must be infinite and that of the ammeter must be zero. A generator of finite impedance may always be replaced by a generator whose impedance is zero in series with its internal impedance. It may also be replaced by a generator with infinite impedance in parallel with its internal impedance. Similarly, a voltmeter whose impedance is finite may be replaced by an ideal volt-

forms of the dynamical equations. Thus, equations 82 may be obtained from Lagrange's equations for either mechanical or electric linear systems. Also, solely from the principle of conservation of energy, we can prove that $Z_{12} = -Z_{21}^*$ for any nondissipative linear transducer (including electromechanical transducers). Hence, if Z_{12} is real — as it is in electromechanical systems — then $R_{12} = -R_{21}$ (gyroscopic coupling). It should also be noted that the variables in the transducer equations must correspond to generalized forces and velocities, such that the element of work is given by $p\,\delta q$, where p is the generalized force and δq the generalized virtual displacement. In the case of cylindrical electromagnetic waves, for example, E_z and H_φ are linearly related; but $Z_{12} \neq Z_{21}$. If, however, we rewrite the same equations in terms of new variables E_z and ρH_φ, where ρ is the distance from the axis of the wave, then the reciprocity holds.

meter in parallel with its internal impedance; and an ammeter with a finite impedance is equivalent to this impedance in series with an ideal ammeter. The ideal elements may then be interchanged as stated in the reciprocity theorems.

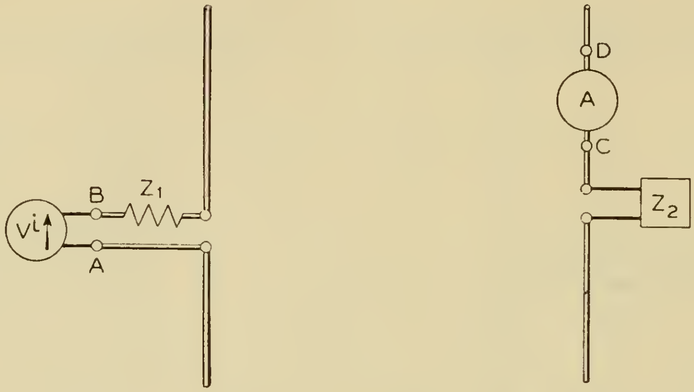

FIG. 9.14 The positions of a generator with zero internal impedance at A, B and an ammeter with zero internal impedance at C, D may be interchanged without affecting the ammeter's reading.

9.9 Circuit equivalence theorems

In addition to the reciprocity theorem, there are two important circuit equivalence theorems. These may be stated as follows:

The first circuit equivalence theorem: *At a pair of terminals, any linear network containing one or more generators of zero impedance acts as a generator whose electromotive force equals the voltage appearing at the terminals when no load impedance is connected, and whose internal impedance is the impedance measured at these terminals when all generators are short-circuited.*

Thus, an antenna in an arbitrary impressed field (Fig. 9.15*a*) is equivalent to the circuit in Fig. 9.15*b*, in which V is the voltage appearing at the antenna terminals A, B when Z is disconnected, and Z_A is the antenna impedance. For finite networks, this theorem was first enunciated by Helmholtz, and then independently by Thévenin.

The second circuit equivalence theorem: *At a pair of terminals, any linear network containing one or more generators of infinite impedance acts as a generator whose current equals the current appearing at the terminals when these are short-circuited, and whose internal admittance is the admittance measured at these terminals when all generators are open-circuited.*

Thus, an antenna in an arbitrary impressed field (Fig. 9.16*a*) is equivalent to the circuit in Fig. 9.16*b*, in which I is the current at the antenna terminals A, B when these are short-circuited, and Y_A is the antenna admittance. For finite networks this theorem was first enunciated by E. L. Norton.

To prove the first theorem,*

$$I = \frac{V}{Z_A + Z}, \tag{83}$$

we need only to obtain the current through an impedance Z across A, B (Fig. 9.12) when a given voltage is applied across the terminals C, D of

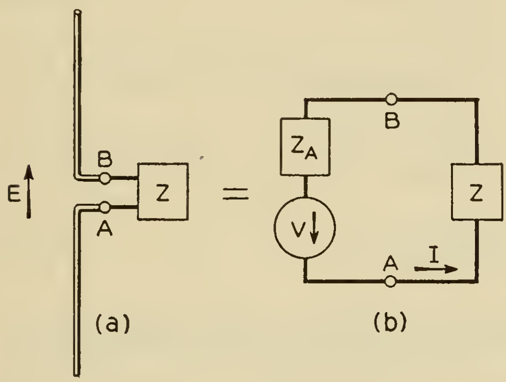

FIG. 9.15 Application of the first circuit equivalence theorem to a receiving antenna.

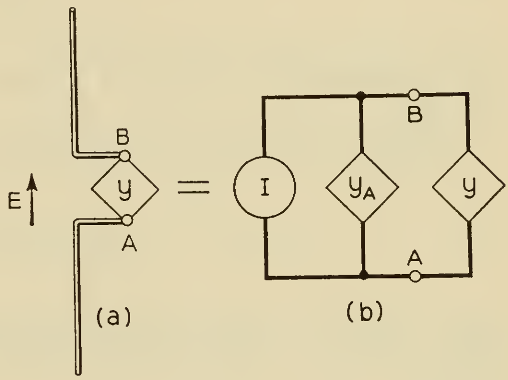

FIG. 9.16 Application of the second circuit equivalence theorem to a receiving antenna.

the transmitting antenna producing the field, and to interpret the result. If I is the current from B to A in the direction of V_1, then,

$$I = -I_1. \tag{84}$$

If Z is the impedance between A, B,

$$V_1 = ZI. \tag{85}$$

Substituting in equations 74,

$$\begin{aligned} ZI &= -Z_{11}I + Z_{12}I_2, \\ V_2 &= -Z_{12}I + Z_{22}I_2, \end{aligned} \tag{86}$$

and solving for I in terms of V_2, we find

$$I = \frac{Z_{12}V_2}{Z_{11}Z_{22} - Z_{12}^2 + ZZ_{22}} = \frac{Z_{12}V_2/Z_{22}}{Z_{11} - (Z_{12}^2/Z_{22}) + Z}. \tag{87}$$

This is the actual current through Z across A, B.

Now if the circuit is open at A, B, the voltage $V_1 = V$ across these

* There exist logical rather than analytic proofs of equivalence theorems which are more general because they are independent of the specific equations of motion of a given dynamical system. For an example of such a proof see Sections 16.2 and 16.3 which are devoted to field equivalence theorems.

terminals may be obtained by letting $I_1 = 0$ in equations 74; thus,

$$V = Z_{12}I_2, \qquad V_2 = Z_{22}I_2, \qquad V = \frac{Z_{12}V_2}{Z_{22}}. \tag{88}$$

To find the impedance Z_A seen from A, B when C, D are short-circuited, we let $V_2 = 0$ in equations 74 and solve for the ratio V_1/I_1; thus,

$$Z_A = Z_{11} - \frac{Z_{12}^2}{Z_{22}}. \tag{89}$$

Substituting from equations 88 and 89 in equation 87, we obtain equation 83.

Similarly, we can prove that the voltage across A, B in Fig. 9.16 is

$$V = \frac{I}{Y_A + Y}, \tag{90}$$

where I and Y_A are obtained as stated in the second circuit equivalence theorem.

Unless the two antennas are very close together, the antenna impedance Z_A of one antenna is not affected by the presence of the other. It is only in antenna arrays that we should be careful to consider the reaction between the antennas, and then we must remember that Z_A and Y_A *in equations* 83 *and* 90 *are not reciprocals;* for Z_A is obtained on the assumption that the other antenna is short-circuited, whereas Y_A is determined on the assumption that the other antenna is open.

9.10 Reciprocity of current distributions

The *transfer admittance** $Y(z_1; z_2)$ between two points on a wire (Fig. 9.17*a*) is the current at $z = z_2$ due to a unit voltage at $z = z_1$. By the reciprocity theorem, the transfer admittance is a symmetric function,

$$Y(z_1; z_2) = Y(z_2; z_1). \tag{91}$$

The points need not be on the same wire; thus (Fig. 9.17*b*),

$$Y(z_1; z_2') = Y(z_2'; z_1). \tag{92}$$

Suppose now that the unit voltage is not concentrated at $z = z_1$ but is uniformly distributed in the interval $(z_1 - \frac{1}{2}s_1, z_1 + \frac{1}{2}s_1)$; then the current at some point $z = z_2$ is

$$Y(z_1 - \tfrac{1}{2}s_1, z_1 + \tfrac{1}{2}s_1; z_2) = \frac{1}{s_1}\int_{z_1-\frac{1}{2}s_1}^{z_1+\frac{1}{2}s_1} Y(z_1; z_2)\, dz_1. \tag{93}$$

* This function exists except possibly at $z_2 = z_1$ where it often has a logarithmic singularity.

Since $Y(z_1;\, z_2)$ is symmetric,

$$Y(z_1 - \tfrac{1}{2}s_1, z_1 + \tfrac{1}{2}s_1; z_2) = \frac{1}{s_1}\int_{z_1-\frac{1}{2}s_1}^{z_1+\frac{1}{2}s_1} Y(z_2; z_1)\, dz_1$$
$$= Y(z_2; z_1 - \tfrac{1}{2}s_1, z_1 + \tfrac{1}{2}s_1); \qquad (94)$$

that is, *the current at* $z = z_2$ *due to a unit voltage uniformly distributed in the interval* $z_1 - \frac{1}{2}s_1 < z < z_1 + \frac{1}{2}s_1$ *equals the average current in this interval due to a unit voltage at* $z = z_2$.

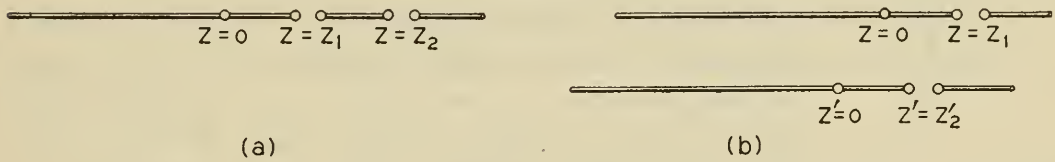

FIG. 9.17 Illustrations for the reciprocity equations.

This theorem applies to nonuniformly distributed voltages, provided the same weight factor is used in averaging the current. The weight factor expresses interconnections between different parts of the generator and the load.

Finally, if we integrate equations 93 and 94 between $z = z_2 - \frac{1}{2}s_2$ and $z = z_2 + \frac{1}{2}s_2$, we find

$$Y(z_1 - \tfrac{1}{2}s_1, z_1 + \tfrac{1}{2}s_1; z_2 - \tfrac{1}{2}s_2, z_2 + \tfrac{1}{2}s_2)$$
$$= Y(z_2 - \tfrac{1}{2}s_2, z_2 + \tfrac{1}{2}s_2; z_1 - \tfrac{1}{2}s_1, z_1 + \tfrac{1}{2}s_1); \quad (95)$$

that is, *the average current in the interval* $(z_2 - \frac{1}{2}s_2, z_2 + \frac{1}{2}s_2)$ *due to a unit voltage distributed uniformly in the interval* $(z_1 - \frac{1}{2}s_1, z_1 + \frac{1}{2}s_1)$ *equals the average current in the latter interval due to a unit voltage spread uniformly over the former.*

9.11 Reflecting antennas

Consider a wire (Fig. 9.18), and let $E_z(z)$ be the electric intensity parallel to the wire. A typical impressed elementary voltage is $E_z(z)\, dz$; hence, the current at $z = \xi$ will be

$$I(\xi) = \int_{-l}^{l} E_z(z)\; Y(z; \xi)\, dz. \qquad (96)$$

Since Y is a symmetric function,

$$I(\xi) = \int_{-l}^{l} E_z(z)\; Y(\xi; z)\, dz. \qquad (97)$$

Hence, if we know the current $Y(\xi; z)$ in a transmitting antenna in which a unit voltage is impressed at $z = \xi$, we can determine the current at this

point when the antenna is in a given field; but *we can say nothing about the current at other points.* We would have this complete information only if we knew the current distribution in the transmitting antenna corresponding to a generator in every position $z = \xi$.

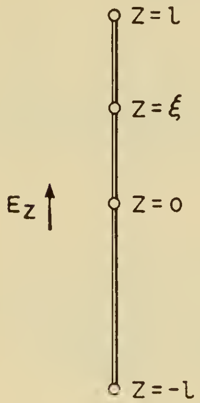

FIG. 9.18 A reflecting antenna.

9.12 Receiving antennas

In a receiving antenna, there is an impedance Z, the "load," inserted at $z = \xi$. By the first circuit equivalence theorem we can find the current through the load if we know the impedance Z_A of the transmitting antenna as seen from the terminals of the load, and the voltage V across these terminals when the load is disconnected. Now, if the terminals are short-circuited, the current is given by equation 97. To make the terminals effectively disconnected, we may impress a voltage between these terminals that just cancels $I(\xi)$. This voltage is $-Z_A I(\xi)$, and its negative is the field voltage which appears between the open terminals. Thus,

$$V = Z_A \int_{-l}^{l} E_z(z)\, Y(\xi; z)\, dz. \tag{98}$$

The current through the load is, then,

$$I = \frac{Z_A \int_{-l}^{l} E_z(z)\, Y(\xi; z)\, dz}{Z_A + Z}. \tag{99}$$

The solution of the transmitting-antenna problem gives all the information needed for the solution of the receiving-antenna problem, *provided the load and the generator are in the same relative position.*

Since $Y(\xi; z)$ is the antenna current which would be produced by a unit voltage at $z = \xi$, and $1/Z_A$ is the corresponding input current, we may write equation 98 as follows:

$$V = \frac{\int_{-l}^{l} E_z(z)\, I(\xi; z)\, dz}{I_i}; \tag{100}$$

where $I(\xi; z)$ is the current distribution produced by a generator at $z = \xi$ and I_i is the input current.

9.13 Reciprocity of transmission and reception

Consider two antennas whose impedances when used as radiators of power are Z_1 and Z_2. Let the first antenna be used for radiation and

the second for reception. Let the generator impedance connected to the first antenna be the conjugate of Z_1, so that one half of the power goes into the antenna, where it is partly dissipated and partly radiated. If V is the voltage developed by the generator, the power given up to the antenna is

$$P_1 = \tfrac{1}{2}R_1 \frac{VV^*}{4R_1^2} = \frac{VV^*}{8R_1}. \tag{101}$$

If the load impedance of the receiving antenna is the conjugate of Z_2, the received power P_2 is maximum. If the transfer admittance* is Y_{12}, the current through the load is $Y_{12}V$ and the power absorbed by the load is

$$P_2 = \tfrac{1}{2}R_2|Y_{12}V|^2 = \tfrac{1}{2}R_2Y_{12}Y_{12}^*VV^*. \tag{102}$$

The ratio of the powers defined in equations 102 and 101 is

$$\frac{P_2}{P_1} = 4R_1R_2Y_{12}Y_{12}^*. \tag{103}$$

The right-hand side is a symmetric function and we have proved that, under the conditions stated, the power transfer is the same regardless of the direction of transfer.

9.14 Reciprocity of radiation patterns

The radiation patterns of a given antenna are the same whether the antenna is used for radiation or reception. This is an immediate consequence of the theorem in the preceding section. Besides the given antenna, we imagine a current element as a test antenna. If the current element is on the surface of a large sphere centered on the given antenna and is always oriented to receive the maximum power, the received power is proportional to the radiation intensity Φ. If, now, the current element radiates the power previously radiated by the given antenna, the latter will receive from each direction the power that was previously received by the element and, therefore, in proportion to Φ.

With so many reciprocal relationships between transmitting and receiving antennas existing *at their terminals*, it is important to remember the fact, mentioned earlier, that the current distributions in these antennas are usually very different. Consequently, the pattern of power reradiated by a receiving antenna is different from its radiation pattern as a transmitter (or from its receiving pattern).

* Of the transducer consisting of the antennas, the generator impedance together with the matching networks, and the load.

PROBLEMS

9.3–1 Obtain the zeros and poles of a dissipative transmission line of length l short-circuited at the far end.

Ans. There is one zero at $p_0 = -R/L$; the remaining zeros are given by

$$p_n = -\frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right) \pm \left[\frac{1}{4}\left(\frac{R}{L} - \frac{G}{C}\right)^2 - \frac{k_n{}^2}{LCl^2}\right]^{1/2},$$

where $k_n = n\pi$, $n = 1,2,3,\cdots$. The poles are given by the same formula with $k_n = (n - \frac{1}{2})\pi$. (*Note.* If the line is distortionless, $RC = GL$, and the zeros and poles are equidistributed on a straight line parallel to the imaginary axis.)

9.3–2 Obtain the zeros and poles of a dissipative transmission line of length l open at the far end.

Ans. One pole is at $p_0 = -G/C$; the remaining poles are given by the formula in the preceding problem with $k_n = n\pi$. The zeros are given by the same formula with $k_n = (n - \frac{1}{2})\pi$.

9.7–1 Obtain the relative voltages that should be impressed on an end-fire couplet. Assume that the distance between the antennas is $\lambda/4$.

Ans. The voltage impressed on the forward antenna (in the direction of main radiation) should be $-j(Z_{11} + jZ_{12})/(Z_{11} - jZ_{12})$ times as large as the voltage impressed on the other antenna.

9.7–2 Obtain the relative voltages that should be impressed on the various antennas in an end-fire array of three elements a quarter wavelength apart.

Ans. In the direction of the array the voltages should be proportional to $(Z_{11} - Z_{13}) - jZ_{12}$, $-jZ_{11}$, $-(Z_{11} - Z_{13}) - jZ_{12}$.

9.7–3 Obtain the relative voltages that should be impressed on the various antennas in a three-element broadside array.

Ans. The voltages impressed on the antennas at the ends should be $(Z_{11} + Z_{12} + Z_{13})/(Z_{11} + 2Z_{12})$ times as large as the voltage impressed on the middle antenna.

9.7–4 Derive the following formula:

$$|Z_{12}| = (g_1 g_2 R_{i,1} R_{i,2})^{1/2} \frac{\lambda}{2\pi r},$$

for the magnitude of the mutual impedance of two antennas, distance r apart. Assume that the receiving antenna is terminated into its image impedance in order to receive the maximum power.

9.12–1 Derive the following transmission formula for two parallel dipole antennas in a dissipative medium, assuming that the antennas have a common equatorial plane:

$$\frac{P_2}{P_1} = \frac{\omega^2\mu^2 e^{-2\alpha r}}{64\pi^2 R_{i,1} R_{i,2} r^2} \left| \frac{\int_{-l_1}^{l_1} I_1(z)\,dz}{I_{i,1}} \right|^2 \left| \frac{\int_{-l_2}^{l_2} I_2(z)\,dz}{I_{i,2}} \right|^2$$

$$= \frac{900\pi^2 \mu_r{}^2 e^{-2\alpha r}}{R_{i,1} R_{i,2} \lambda_{\mathrm{air}}{}^2 r^2} \left| \frac{\int_{-l_1}^{l_1} I_1(z)\,dz \int_{-l_2}^{l_2} I_2(z)\,dz}{I_{i,1} I_{i,2}} \right|^2,$$

where

P_1 = power supplied to the terminals of the first antenna,
P_2 = maximum power received by the second antenna,
$R_{i,1}$, $R_{i,2}$ = input resistances of the antennas,
$I_1(z)$, $I_2(z)$ = currents in the antennas when they are used as transmitting antennas,
$2l_1$, $2l_2$ = the lengths of the antennas,
α = the intrinsic attenuation constant of the medium,
r = the distance between the antennas.

9.12–2 Show that the formula in the preceding problem is equivalent to

$$\frac{P_2}{P_1} = \frac{225\pi^2\mu r^2 e^{-2\alpha r}}{\lambda_{\text{air}}{}^2 r^2 \hat{P}_1 \hat{P}_2} \left| \int_{-l_1}^{l_1} I_1(z)\,dz \int_{-l_2}^{l_2} I_2(z)\,dz \right|^2,$$

where $\hat{P}_1$ and $\hat{P}_2$ are the powers radiated by $I_1(z)$ and $I_2(z)$.

9.12–3 Derive the following transmission formula for two small coplanar loops:

$$\frac{P_2}{P_1} = \frac{\omega^2\mu^2|\sigma|^4 S_1{}^2 S_2{}^2 e^{-2\alpha r}}{64\pi^2 r^2 R_1 R_2} = \frac{900\pi^2\mu_r{}^2 S_1{}^2 S_2{}^2 |\sigma|^4 e^{-2\alpha r}}{R_1 R_2 \lambda_{\text{air}}{}^2 r^2},$$

where S_1 and S_2 are the areas of the loops.

9.13–1 The *effective length of a straight linear transmitting antenna* is defined as the moment of its current distribution divided by the input current, where the *moment of the current distribution* is defined as the sum of the moments of its current elements. The *effective length of a straight linear receiving antenna* is defined as the voltage induced between open terminals of the antenna divided by the incident electric intensity when this intensity is parallel to the antenna. Prove that these effective lengths are equal.

9.13–2 The concepts of the preceding problem may be generalized as follows. Consider *any* antenna and a point A within it or in its vicinity. Consider another point B so distant that the choice of A has a negligible effect on the direction AB. And finally choose some direction BC perpendicular to AB. The *effective length of the antenna acting in the transmitting capacity*, with respect to the pair of directions AB and BC, may be defined as the length of a uniform current filament parallel to BC and carrying a current equal to the input current of the given antenna when the current filament produces the same electric intensity along BC as the given antenna. The *effective length of the antenna acting in the receiving capacity*, with respect to waves traveling in the direction BA and such that their electric intensity is parallel to BC, may be defined as the voltage induced across open terminals of the antenna divided by the incident intensity. Prove that these effective lengths are equal.

9.13–3 Show that, if an antenna emits waves that are linearly polarized at large distances, and if s is its effective length, then,

$$A_{\text{eff}} = \frac{\eta|s|^2}{4R_i}, \qquad g = \frac{\pi\eta|s|^2}{R_i\lambda^2},$$

where R_i is the input resistance of the antenna.

10

SMALL ANTENNAS

10.1 Small antennas

The electrical performance of an antenna depends on its dimensions relative to the wavelength at which it is operated; the absolute dimensions are not important. For this reason, an *antenna is said to be small* if its largest dimension, measured from its input terminals, does not exceed one eighth of the wavelength. "Small antennas" may thus be large by ordinary standards. If the frequency is 60,000 cps, $\lambda = 5000$ meters; and any practicable antenna must inevitably be small by our definition. In most cases one dimension of the antenna is so much larger than the rest that small antennas may properly be called *short antennas.* In the domestic broadcast frequency range, antennas can be made comparable in length to $\lambda/4$ and even $\lambda/2$; but, because of space limitations and the great expense involved in the construction of such antennas, only the transmitting antennas are made large.* Even in the case of still shorter waves, antennas may have to be made short owing to space limitations, as, for example, ship-borne or plane-borne antennas.

Small antennas are treated in a separate chapter for two directly opposite reasons. In some respects small antennas are much simpler than the larger antennas, and their theory is a good introduction to the general theory; in other respects small antennas are more complicated, and this is another reason for the separate treatment.

10.2 Antenna impedance

In the limiting case of vanishingly small frequency the dipole antenna is just a capacitor and the loop antenna is an inductor. The remaining

* Even broadcast antennas are sometimes short. For an experimental study of such antennas, see Carl E. Smith and E. M. Johnson, Performance of short antennas, *IRE Proc.*, 35, October 1947, pp. 1026–1038.

terms in the expressions for their impedances (equations 9–69 and 9–73) are negligible. As the frequency increases, these terms become more significant. Thus, for a dipole antenna, we have

$$Z_{\mathrm{an}} = \frac{1}{j\omega C_{\mathrm{an}}} + R + j\omega L_{\mathrm{an}} - A\omega^2 + \cdots. \tag{1}$$

In Section 9.6 we have seen that $R = 0$ for a perfectly conducting antenna in a nondissipative medium, and the first resistive term $-A\omega^2$ represents the radiation resistance. The first three coefficients, C_{an}, L_{an}, A, may be calculated, at least in certain cases, without deriving the complete expression for Z_{an}. Since they are independent of the frequency, we can obtain them by assuming that the frequency is vanishingly small. As illustrated in the following section, the antenna capacitance C_{an} is obtained by solving an electrostatic problem. The antenna inductance L_{an} is then obtained from this solution by making the voltage proportional to time, so that a constant charging current flows; then the stored magnetic energy $\mathcal{E}_m$ can be calculated, and from it the inductance

$$L_{\mathrm{an}} = \frac{2\mathcal{E}_m}{I_{\mathrm{in}}^2}, \tag{2}$$

where I_{in} is the input current. Finally, the third coefficient is obtained by calculating the radiated power on the assumption that the current varies infinitely slowly.

Thus, from the point of view of electrical behavior, short antennas are essentially simple electric circuits with a resistance depending on the frequency of operation.

The series expansion (equation 1) is also valid for imperfectly conducting antennas, but its interval of convergence is smaller and we may no longer assert that $-A\omega^2$ represents the radiation resistance. There will be other terms proportional to ω^2 due to the skin effect in the antenna. In this case it is preferable to segregate those terms of the series that depend on the internal impedance of the wire and incorporate them into R, thereby increasing the radius of convergence of the remainder of the series. Similarly, we can incorporate into R the ground resistance, dielectric losses, etc., thereby making it a complicated function of frequency but retaining the essential simplicity of the rest of the series.

10.3 Antenna capacitance

At distances from the input terminals large compared with the transverse dimensions of the antenna the capacitance per unit length for principal waves is given by equation 4–36. In this case r is very nearly equal to z

(Fig. 10.1), and we shall write

$$C(z) = \frac{\pi\varepsilon}{\log(2z/\rho)}, \tag{3}$$

where the antenna radius ρ may be a function of z. For antennas constructed of separate wires forming a "cage structure," an "effective radius" may be determined from the dimensions of the cage and the

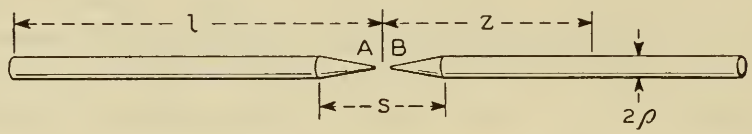

Fig. 10.1 A dipole antenna.

wires (equation 4–44). Similar formulas may be derived for flat strips, for rods of square cross section, etc. In all such cases the capacitance per unit length may be expressed in the form of equation 3, where ρ is an appropriate effective radius.

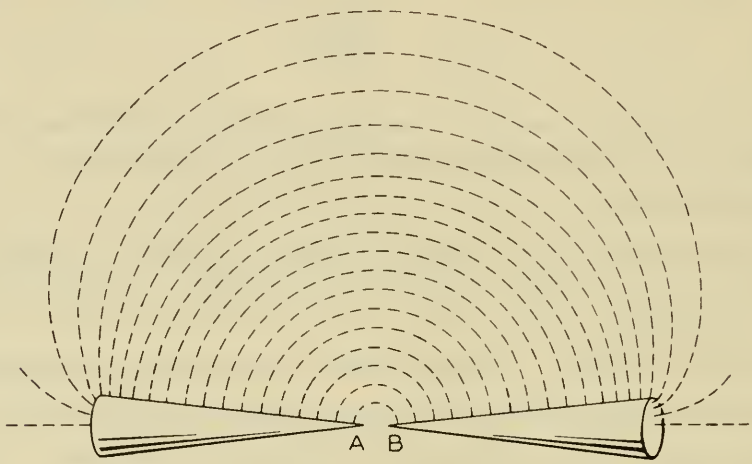

Fig. 10.2 A biconical antenna.

The capacitance given by equation 3 is that computed for principal waves in which the electric lines run in circles (or very nearly in circles) between the arms of the antenna. This condition would prevail if the arms were infinitely long; it prevails in a finite antenna only far enough from the outer ends for the "end effect" to be negligible. Near the ends the electric lines will bulge out, as illustrated in Fig. 10.2. To understand this effect, let us imagine the antenna as a part of an infinitely long antenna (Fig. 10.3). The antenna arms are oppositely charged. The positive charge on FH repels the positive charge at D. If FH is removed, this force of repulsion is removed; and the forces of repulsion

from the left of D will force some charge to move nearer to the end D. Hence, for the same potential difference, there is more charge concentrated per unit length near the ends than there would have been if the conductors were continued. The end effect must represent an increase in the capacitance.

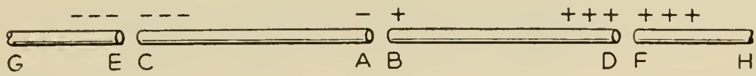

FIG. 10.3 A dipole antenna cut out of an infinitely long wire.

The end effect is not particularly large. Thus, for a biconical antenna (Fig. 10.2), the total antenna capacitance obtained from equation 3 is

$$C_{\mathrm{an}} = Cl = \frac{\pi\varepsilon l}{\log(2l/a)}, \tag{4}$$

where l is the length of one arm and a is the maximum radius of the arm. The end effect may be obtained from the following expression for the total* charge per unit length of a thin cone which we give without derivation:

$$q(z) = \frac{\pi\varepsilon V}{\log(2l/a)} + \frac{\pi\varepsilon V}{2[\log(2l/a)]^2}\log\frac{l+z}{l-z}, \tag{5}$$

where V is the voltage between the arms. Integrating this with respect to z, we find

$$C_{\mathrm{an}} = \frac{\pi\varepsilon l}{\log(2l/a)} + \frac{\pi\varepsilon l \log 2}{[\log(2l/a)]^2} \simeq \frac{\pi\varepsilon l}{\log(2l/a) - \log 2}. \tag{6}$$

In obtaining this expression we have integrated only the charge on the lateral surface of the cone. If we include the charge on the end face, we obtain the complete expression for the total capacitance of the biconical antenna,

$$C_{\mathrm{an}} = \frac{\pi\varepsilon l}{\log(2l/a) - \log 2} + 2\varepsilon a. \tag{7}$$

It should be observed that, while the charge density is infinite at the ends $z = \pm l$, it is large only in their immediate neighborhood. For example, the two terms in equation 5 are equal at a distance from the end given approximately by $l - z = a^2/2l = (a/2l)a$, which is an exceedingly small fraction of the radius.

* That is, the charge associated with the principal and the higher modes of distribution.

In Sections 10.7 and 10.8 we shall develop two methods of calculating the antenna capacitance. By either of these methods we shall obtain the following formula for the capacitance of a thin antenna whose shape beyond conical input tips extending from $z = -\frac{1}{2}s$ to $z = \frac{1}{2}s$ is arbitrary,*

$$C_{\mathrm{an}} = \frac{\pi\varepsilon(l - \frac{1}{2}s)}{\log(2l/a_{\mathrm{lm}}) - 1 - \log 2 + f(s, l)} + \frac{\pi\varepsilon s}{2\log[2s/\rho(\frac{1}{2}s)]} + 2\varepsilon\rho(l), \tag{8}$$

where $\rho(s/2)$ and $\rho(l)$ are the radii at $z = s/2$ and $z = l$, a_{lm} is the *logarithmic mean radius* defined by

$$\log a_{\mathrm{lm}} = (l - \tfrac{1}{2}s)^{-1}\int_{s/2}^{l} \log \rho(z)\, dz. \tag{9}$$

The function $f(s, l)$ represents a small correction term, which by one method (see equation 38) is found to be

$$f(s, l) \simeq \frac{s}{2l - s} \log \frac{2l}{s} \cdot \tag{10}$$

By the other method (see equations 49 and 54), we find

$$f(s, l) \simeq \frac{s}{2l - s}\left[\log \frac{2l}{s} - 2\log 2\right]. \tag{10'}$$

The important factor giving the order of magnitude of $f(s, l)$ is $s/2l$.

For a cylindrical antenna the above mean radius equals the actual radius. The second term in equation 8 represents the approximate capacitance of the *conical* input tips; for other shapes it should be replaced by the appropriate capacitance of the input region. The last term is the capacitance between the flat outer ends of the antenna; if the ends are hemispheres, the coefficient 2 should be replaced by π.

If the logarithmic mean radius of the antenna approaches zero while the ratio s/a_{lm} remains constant, the antenna capacitance approaches the following asymptotic limit:

$$C_{\mathrm{an}} = \frac{\pi\varepsilon l}{\log(2l/a_{\mathrm{lm}}) - 1 - \log 2} \cdot \tag{11}$$

The length of the input region does not enter this expression. For

* Assuming that s is at least equal to the diameter at $z = \frac{1}{2}s$ and preferably somewhat larger.

practical dimensions, the effect of s is usually small, although we could *imagine* dimensions for which the effect would be enormous.*

The capacitance of a vertical antenna above a perfect ground (Fig. 10.4) is twice that given by the above equations.

In the following sections we shall see that the capacitance plays a part in determining the current distribution in small antennas and the radiation of power from them. Two methods of obtaining the capacitance will be given in Sections 10.7 and 10.8. Often, however, experiments with scale models may be required to supplement calculations. Large collections of pertinent formulas may be found in two publications of the Bureau of Standards: *Radio Instruments and Measurements*, C74, second edition (1924, reprinted 1937), Superintendent of Documents, Government Printing Office, Washington 25, D.C.; *Methods, Formulas, and Tables for Calculation of Antenna Capacity*, Frederick W. Grover, Sci. Pap. of the Bureau of Standards, No. 568 (out of print, but available in libraries).

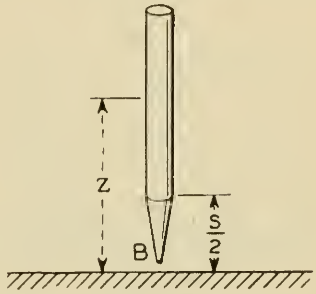

FIG. 10.4 A vertical antenna.

10.4 Antenna current

To calculate the inductance and the radiation resistance we need to know the antenna current. If $q(z)$ is the charge per unit length, the charge on an element Δz is $q(z)\,\Delta z$. The time rate of increase of this charge must equal the difference between the current $I(z)$ entering the element and the current $I(z + \Delta z)$ leaving it; thus,

$$\frac{\partial q}{\partial t}\,\Delta z = I(z) - I(z + \Delta z).$$

Dividing by Δz and passing to the limit, we find

$$\frac{\partial I}{\partial z} = -\frac{\partial q}{\partial t}, \qquad \frac{dI}{dz} = -j\omega q. \tag{12}$$

Integrating from $z = z$ to $z = l$,

$$I(l) - I(z) = -j\omega \int_z^l q(z)\,dz,$$

or

$$I(z) = I(l) + j\omega \int_z^l q(z)\,dz. \tag{13}$$

* See Section 12.10.

It has already been explained that the third and fourth coefficients in equation 1 may be obtained on the assumption that ω tends to zero; hence, for these purposes we may assume that $q(z)$ in the above equation represents the static distribution of charge.

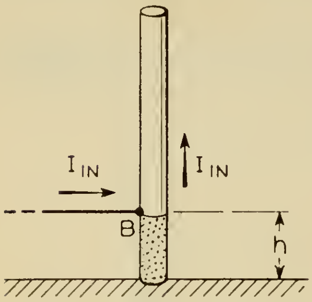

Fig. 10.5 An antenna supported by an insulating base.

The input current is

$$I_{\text{in}} = I(l) + j\omega \int_h^l q(z)\, dz, \tag{14}$$

where we have assumed that the antenna begins at $z = h$ (Fig. 10.5). The insulating base of the antenna may be treated more conveniently as part of the feed system rather than as part of the antenna. The impedance between point B and the ground will then be the antenna impedance in parallel with the impedance of the insulating base. For purposes of mathematical simplification, we may sometimes wish to make $h = 0$; then we must assume a conical taper as shown in Fig. 10.4; otherwise, we shall complicate our problem instead of simplifying it.

If the shape of the antenna is such that $I(l) = 0$ and the charge per unit length is constant, $q(z) = q_0$, we have

$$I(z) = j\omega q_0 \int_z^l dz = j\omega q_0 (l - z). \tag{15}$$

Later in this chapter we shall calculate such a shape* for the case in which the input terminals are infinitely close. Then,

$$I(z) = I(0)\left(1 - \frac{z}{l}\right), \qquad I(0) = j\omega q_0 l = j\omega C_{\text{an}} V, \tag{16}$$

and the current distribution is exactly linear.

For other shapes equation 16 gives the asymptotic current distribution as the radius of the antenna approaches zero. For any practical antenna, no matter how thin, there is fairly substantial deviation from this simple form. For a biconical antenna, for instance, we find, from equations 5 and 13,

$$I(z) = \frac{j\omega\pi\varepsilon V l}{\log(2l/a)}\left(1 - \frac{z}{l}\right) +$$
$$\frac{j\omega\pi\varepsilon V l}{2[\log(2l/a)]^2}\left[2 \log \frac{2l}{l+z} + \left(1 - \frac{z}{l}\right)\log\frac{l+z}{l-z}\right] + I(l). \tag{17}$$

* Shown in Fig. 10.9.

10.5 Radiation resistance and effective length of a small antenna

In a small antenna all current elements are so close that the influence coefficient K_{12} in equation 5–84 is substantially constant,

$$K_{12} = 80\pi^2. \tag{18}$$

Since all the current elements have the same phase, $\cos \vartheta_{mn} = 0$, and equation 5–84 becomes

$$P = \frac{40\pi^2}{\lambda^2} \int_{-l}^{l} \int_{-l}^{l} |I(z_1)|\,|I(z_2)|\,dz_1\,dz_2 = \frac{40\pi^2}{\lambda^2} \left[\int_{-l}^{l} |I(z)|\,dz \right]^2 \cdot \tag{19}$$

The quantity in brackets is the magnitude of the moment of the current distribution. Let us define the effective length, $2l_{\text{eff}}$, of the antenna as the length of a current element carrying a current equal to the input current and having the same moment as the actual current; that is,

$$2l_{\text{eff}} I_{\text{in}} = \int_{-l}^{l} I(z)\,dz, \qquad 2l_{\text{eff}} = \frac{1}{I_{\text{in}}} \int_{-l}^{l} I(z)\,dz,$$
$$l_{\text{eff}} = \frac{1}{I_{\text{in}}} \int_{0}^{l} I(z)\,dz. \tag{20}$$

Substituting this effective length in equation 19, we have

$$P = 40\pi^2 \left(\frac{2l_{\text{eff}}}{\lambda} \right)^2 |I_{\text{in}}|^2. \tag{21}$$

On the other hand, if R_{in} is the input resistance,

$$P = \tfrac{1}{2} R_{\text{in}} |I_{\text{in}}|^2. \tag{22}$$

From these two equations, we have

$$R_{\text{in}} = 80\pi^2 \left(\frac{2l_{\text{eff}}}{\lambda} \right)^2 \tag{23}$$

for a short antenna in free space.

For a vertical antenna of length l above a perfect ground (Fig. 10.4), the input resistance is half the above value (see Section 4.18),

$$R_{\text{in}} = 160\pi^2 \left(\frac{l_{\text{eff}}}{\lambda} \right)^2 \cdot \tag{24}$$

When the current distribution is linear (equation 16), $l_{\text{eff}} = \frac{1}{2}l$, and the above equations become

$$R_{\text{in}} = 80\pi^2 \left(\frac{l}{\lambda} \right)^2 \quad \text{in free space,}$$
$$R_{\text{in}} = 40\pi^2 \left(\frac{l}{\lambda} \right)^2 \quad \text{above perfect ground.} \tag{25}$$

It should be remembered that in the first equation l is the length of one antenna arm, whereas in the second equation it is the entire length. *For equal total lengths* the input resistance of an antenna just above the ground is *twice* that of the same antenna in free space.

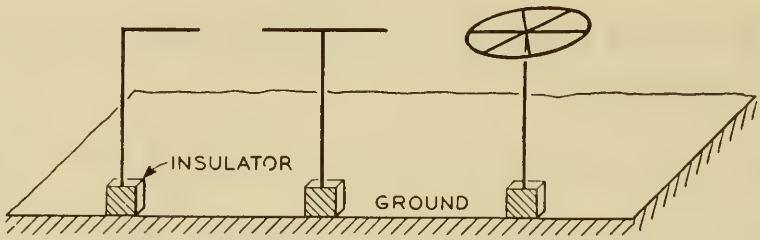

FIG. 10.6 Types of capacitive loading at the top of an antenna.

A convenient expression for the effective length may be obtained in terms of the distribution of antenna capacitance. First we integrate by parts the integral for l_{eff} (equation 20),

$$l_{\text{eff}} = \left.\frac{z\,I(z)}{I_{\text{in}}}\right|_0^l - \frac{1}{I_{\text{in}}}\int_0^l z\,\frac{dI}{dz}\,dz$$

$$= \frac{l\,I(l)}{I_{\text{in}}} - \frac{1}{I_{\text{in}}}\int_0^l z\,\frac{dI}{dz}\,dz. \tag{26}$$

Since

$$I_{\text{in}} = j\omega C_{\text{an}}V, \qquad I(l) = j\omega C_{\text{cap}}V, \qquad \frac{dI}{dz} = -j\omega q = -j\omega\,C(z)V, \tag{27}$$

equation 26 becomes

$$l_{\text{eff}} = \frac{lC_{\text{cap}} + \int_0^l z\,C(z)\,dz}{C_{\text{an}}} = \frac{lC_{\text{cap}} + \int_0^l z\,C(z)\,dz}{C_{\text{cap}} + \int_0^l C(z)\,dz}. \tag{28}$$

By analogy with the usual definition of the center of gravity we might describe l_{eff} as the height of the "center of capacitance." Instead of mass distribution we have capacitance distribution. Equation 28 is almost self-evident if, instead of subdividing the current in the antenna into elements by horizontal sections, we subdivide it into vertical filaments of unequal length but carrying uniform currents. During the charging period, the charge per unit voltage $C(z)\,dz$ on a typical element is raised to the height z; hence, we have a uniform current filament of moment proportional to $z\,C(z)\,dz$. The numerator in equation 28 is thus proportional to the total moment and the denominator to the input current.

The conception of "effective length" or "effective height" is very useful in long-wave radio communication; it enables us to appraise rapidly the effectiveness of the antenna in producing distant fields. It shows the importance of capacitive loading at the top (Fig. 10.6) since

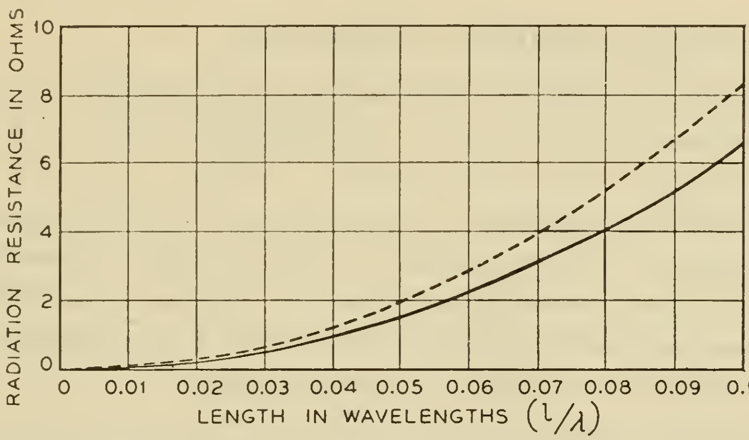

FIG. 10.7 Radiation resistance of a short antenna. The dotted curve represents the radiation resistance of an infinitely thin antenna; the solid curve is for an antenna of a small but finite radius. The difference depends on $\log(2l/a)$.

this raises the center of capacitance and thus increases the effective height of the antenna structure.

Figure 10.7 shows the radiation resistance of short cylindrical antennas. The dotted curve is for the limiting case of an infinitely thin antenna, or for antennas of finite radius but so shaped (see Fig. 10.9) that the antenna current is linearly distributed.

10.6 Antenna inductance

For principal waves the inductance per unit length is

$$L = \frac{\mu}{\pi} \log \frac{2z}{a} \cdot \tag{29}$$

The stored magnetic energy is

$$\mathcal{E}_m = \tfrac{1}{2} L_{\text{an}} \, |I_{\text{in}}|^2 = \tfrac{1}{2} \int_0^l L(z) \, |I(z)|^2 \, dz; \tag{30}$$

therefore,

$$L_{\text{an}} = \frac{1}{|I_{\text{in}}|^2} \int_0^l L(z) \, |I(z)|^2 \, dz. \tag{31}$$

The end effect may be neglected since the current is small near the ends.

Using the linear current distribution given by equation 16, we find

the following approximate expression for the inductance of short cylindrical antennas in free space:

$$\mathrm{L}_{\mathrm{an}} = \frac{\mu}{\pi}\int_0^l \left(1 - \frac{z}{l}\right)^2 \log\frac{2z}{a}\,dz = \frac{\mu l}{3\pi}\left(\log\frac{2l}{a} - \frac{11}{6}\right). \quad (32)$$

10.7 Calculation of antenna capacitance

One method of obtaining the capacitance of an antenna is based upon the field representation in terms of principal and higher order (or complementary) waves. By this method it is easy to evaluate the major part of the antenna capacitance; for this part is obtained from the principal wave alone. The evaluation of the end effect, depending on the higher-order waves, is not difficult, but it requires some knowledge of Legendre functions of fractional orders; since it is in general small, we shall not consider the details of its calculation.

The capacitance per unit length associated with the principal wave is given by equation 3, provided $2z$ is fairly large compared with ρ. The principal part of the antenna capacitance is, then,

$$C_{\mathrm{pr}} = \int_{s/2}^{l} C(z)\,dz. \quad (33)$$

To evaluate this integral, we write

$$\frac{2z}{\rho} = \frac{2l}{a_{\mathrm{lm}}}\,\frac{z}{l}\,\frac{a_{\mathrm{lm}}}{\rho}, \quad (34)$$

where, at the present stage of our analysis, a_{lm} is merely some constant, as yet undetermined. We now write

$$\frac{1}{\log(2z/\rho)} = \frac{1}{u+v+w},$$
$$u = \log\frac{2l}{a_{\mathrm{lm}}}, \qquad v = \log\frac{z}{l}, \qquad w = \log\frac{a_{\mathrm{lm}}}{\rho}. \quad (35)$$

As a_{lm} approaches zero, u tends to infinity while v and w remain generally finite. Let us expand, therefore, in inverse powers of u,

$$\frac{1}{\log(2z/\rho)} = \frac{1}{u}\left(1 + \frac{v+w}{u}\right)^{-1} = \frac{1}{u} - \frac{v+w}{u^2} + \frac{(v+w)^2}{u^3} + \cdots, \quad (36)$$

and use only the first two terms in the integral for C_{pr} (equation 33). Thus,

$$C_{\mathrm{pr}} = \pi\varepsilon\left[(l - \tfrac{1}{2}s)u^{-1} - u^{-2}\int_{s/2}^{l}\log\frac{z}{l}\,dz - u^{-2}\int_{s/2}^{l}\log\frac{a_{\mathrm{lm}}}{\rho}\,dz\right]. \quad (37)$$

We now *define* a_{lm} in such a way that the last term is zero; and we see that a_{lm} is the logarithmic mean radius as defined in equation 9. Evaluating the remaining terms, we find

$$C_{pr} = \frac{\pi\varepsilon(l - \frac{1}{2}s)}{u}\left[1 + \frac{1}{u} - \frac{s\log(2l/s)}{u(2l - s)}\right]$$
$$\simeq \frac{\pi\varepsilon(l - \frac{1}{2}s)}{u\left[1 - \dfrac{1}{u} + \dfrac{s\log(2l/s)}{u(2l - s)}\right]} = \frac{\pi\varepsilon(l - \frac{1}{2}s)}{u - 1 + \dfrac{s}{2l - s}\log\dfrac{2l}{s}}. \tag{38}$$

This gives the first term of equation 8 except for $(-\log 2)$ in the denominator which represents the end effect. In the first approximation, the end effect is independent of the shape of the antenna and may therefore be deduced from the second term of equation 5 for the conical antenna as shown in equation 6.

10.8 Second method of calculating antenna capacitance

The method explained in the preceding section is simple, but it is limited to thin antennas of any shape and thick antennas of particular shapes. An older and more generally applicable method for calculating the antenna capacitance is based on the equation for the potential at a point P,

$$V = \iint \frac{Q\,dS}{4\pi\varepsilon r_{12}}, \tag{39}$$

where r_{12} is the distance between P and a typical point on the antenna at which the surface density of charge is Q. Since each antenna arm is an equipotential, we have to solve the following pair of integral equations of the first kind,

$$\iint_{(S_1)} \frac{Q\,dS}{4\pi\varepsilon r_{12}} = V_1, \qquad \iint_{(S_2)} \frac{Q\,dS}{4\pi\varepsilon r_{12}} = V_2, \tag{40}$$

where V_1 and V_2 are the potentials of the antenna arms. There is an added condition that the total charges on these arms are equal and opposite,

$$\iint_{(S_2)} Q\,dS = -\iint_{(S_1)} Q\,dS. \tag{41}$$

When these equations are solved, the capacitance is found from the definition

$$C_{an} = \frac{\iint_{(S_1)} Q\,dS}{V_1 - V_2}. \tag{42}$$

This equation gives the total capacitance of the antenna without separating it into parts associated with the principal and higher-order waves (or, rather, modes of charge distribution, since we are considering an electrostatic case). The concept of modes does not enter this method of analysis.

Equation 39 leads at once to a fairly simple method of calculating the antenna capacitance. We begin with the assumption that the charge density is constant so that Q may be taken outside the integral sign; then we compute the potential over the surface of each conductor; and, finally, we calculate the average potential of each conductor. We then assume that this average potential equals approximately the actual potential assumed when the uniformly distributed charge is allowed to redistribute itself so that the original differences in potential between various points on each conductor are reduced to zero. Thus, in equation 42 we assume that V_1 and V_2 are the average potentials of the antenna arms. This method of calculating the antenna capacitance was used as far back as 1914 and eliminated the necessity of model experiments.* The value of the capacitance so obtained is always somewhat smaller than the exact value, as shown by Maxwell.† Maxwell also gave a method for finding an upper limit to the capacitance and a method of obtaining the higher-order approximations. Even the first-order approximations obtained by these methods are good, because it can be shown that the first-order errors in the assumed charge distribution lead only to second-order errors in the capacitance.

If the antenna is a surface of revolution, r_{12} in equation 39 may be expressed conveniently in cylindrical coordinates; thus,

$$r_{12} = [\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z - z')^2]^{1/2}, \qquad (43)$$

where (ρ, φ, z) and (ρ', φ', z') are two typical points.

Simpler equations than 39 may be obtained if each antenna arm is either a solid conductor or a hollow conductor with no holes in it; for then the interior of each arm is at the same potential as the surface.

* G. W. O. Howe, The calculation of aerial capacitance, *Wireless Eng.*, **20**, April 1943, pp. 157–158. The capacity of radio telegraphic antennae, *Electrician*, **73**, August and September 1914, pp. 829–832, 859–864, 906–909. The capacity of aerials of the umbrella type, *Electrician*, **75**, September 1915, pp. 870–872. The calculation of capacity of radio telegraph antennae, including the effects of masts and buildings, *Wireless World*, **4**, October and November 1916, pp. 549–556, 633–638. The capacity of an inverted cone and the distribution of its charge, *Proc. Phys. Soc.* (London), **29**, August 15, 1917, pp. 339–344.

† J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1, Oxford Press, 1904, p. 149.

Hence, for a surface of revolution, we may write

$$\int_{s/2}^{l} \frac{q(z')\,dz'}{4\pi\varepsilon\sqrt{[\rho(z')]^2 + (z'-z)^2}} = V_1,$$
$$\int_{-l}^{-s/2} \frac{q(z')\,dz'}{4\pi\varepsilon\sqrt{[\rho(z')]^2 + (z'-z)^2}} = V_2, \tag{44}$$

where $q(z')$ is the charge per unit length in the direction of the axis, and z refers to a typical point on the axis in the interior of the antenna.

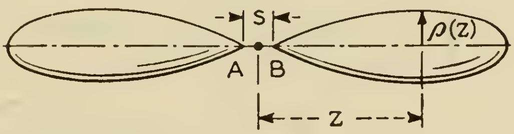

FIG. 10.8 An antenna bounded by a surface of revolution.

If the antenna is symmetric (Fig. 10.8),

$$V_1 = -V_2 = \tfrac{1}{2}V_0, \qquad q(-z') = -q(z'),$$

where V_0 is the voltage between the arms. In this case equations 44 reduce to a single integral equation,

$$-\int_{-l}^{-s/2} \frac{q(z')\,dz'}{\sqrt{[\rho(z')]^2 + (z'-z)^2}} +$$
$$\int_{s/2}^{l} \frac{q(z')\,dz'}{\sqrt{[\rho(z')]^2 + (z'-z)^2}} = 2\pi\varepsilon V_0, \qquad \tfrac{1}{2}s < z < l. \tag{45}$$

If the radius of the antenna is small, the integrand in the second term is large in the vicinity of $z' = z$, and the value of the integral is determined primarily by the charge density at this point. Hence, in the first approximation we may assume that $q(z') = q(z)$ and rewrite the above equation as follows:

$$-q(z)\int_{-l}^{-s/2} \frac{dz'}{\sqrt{[\rho(z')]^2 + (z'-z)^2}} +$$
$$q(z)\int_{s/2}^{l} \frac{dz'}{\sqrt{[\rho(z')]^2 + (z'-z)^2}} = 2\pi\varepsilon V_0. \tag{46}$$

From this, we obtain a better approximation,

$$q(z) = \frac{2\pi\varepsilon V_0}{F(z)}, \tag{47}$$

where

$$F(z) = \int_{s/2}^{l} \frac{dz'}{\sqrt{[\rho(z')]^2 + (z' - z)^2}} + \int_{-s/2}^{-l} \frac{dz'}{\sqrt{[\rho(z')]^2 + (z' - z)^2}} \cdot \tag{48}$$

The corresponding value of the capacitance per unit length is

$$C(z) = \frac{q(z)}{V_0} = \frac{2\pi\varepsilon}{F(z)} \cdot \tag{49}$$

To obtain higher-order approximations, we might rewrite equation 45 as follows:

$$q(z)\,F(z) = 2\pi\varepsilon V_0 + \int_{s/2}^{l} \frac{q(z) - q(z')}{\sqrt{\rho^2 + (z' - z)^2}}\,dz' + \int_{-l}^{-s/2} \frac{q(z') - q(z)}{\sqrt{\rho^2 + (z' - z)^2}}\,dz'. \tag{50}$$

In obtaining the earlier approximation we assumed that $q(z)$ is independent of z so that the last terms in this equation disappear. Having determined the first approximation, we may substitute it in these terms and recalculate $q(z)$. Repeating this process, we obtain a sequence of successive approximations. To simplify the integration, we introduce the average value of $F(z)$ in the interval $(\frac{1}{2}s, l)$,

$$\bar{F} = \frac{1}{l - \frac{1}{2}s} \int_{s/2}^{l} F(z)\,dz, \tag{51}$$

and write equation 50 as follows:

$$q(z) = \frac{2\pi\varepsilon V_0}{\bar{F}} + \frac{1}{\bar{F}} \int_{s/2}^{l} \frac{q(z) - q(z')}{\sqrt{\rho^2 + (z' - z)^2}}\,dz' + \frac{1}{\bar{F}} \int_{-l}^{-s/2} \frac{q(z') - q(z)}{\sqrt{\rho^2 + (z' - z)^2}}\,dz' + \left[1 - \frac{F(z)}{\bar{F}}\right] q(z). \tag{52}$$

Even so, the successive integrations become more complicated and would soon require numerical integration.* Fortunately the first approximation is usually sufficient for practical purposes.

If the antenna arms are cylindrical, we assume that ρ is constant in equation 48 and evaluate the integrals by making the substitution $t = z' - z$. Thus,

$$F(z) = \int_{s/2-z}^{l-z} \frac{dt}{\sqrt{\rho^2 + t^2}} - \int_{-l-z}^{-s/2-z} \frac{dt}{\sqrt{\rho^2 + t^2}} \cdot \tag{53}$$

* Even in the simple case of a circular cylinder, the integration is lengthy; see Erik Hallén, Lösung zweier Potentialprobleme der Elektrostatik, *Arkiv för Matematik, Astronomi, och Fysik*, **21A**, No. 22, Stockholm, 1929.

We have evaluated integrals of this type in Section 8.15. Thus, we find

$$F(z) = \log\frac{[l-z+\sqrt{(l-z)^2+\rho^2}][z-\tfrac{1}{2}s+\sqrt{(z-\tfrac{1}{2}s)^2+\rho^2}][z+\tfrac{1}{2}s+\sqrt{(z+\tfrac{1}{2}s)^2+\rho^2}]}{\rho^2[l+z+\sqrt{(l+z)^2+\rho^2}]} \tag{54}$$

From this and equation 49 we obtain an approximate capacitance per unit length of a cylindrical antenna.

In conclusion, it should be noted that it is easier to solve the inverse problem, in which we assign a certain distribution of charge, evaluate the potential and then the shape of the equipotential surfaces corresponding to the assumed distribution of charge. Since any equipotential surface can be replaced by a conducting sheet without disturbing the field, we thus obtain antenna shapes for which the charge distribution and the capacitance are known exactly. Assume, for example, that a negative charge is distributed on the axis uniformly from $z = -l$ to $z = 0$ while an equal positive charge is distributed uniformly from $z = 0$ to $z = l$. The potential of these two linear filaments is

$$V = -\frac{q_0}{4\pi\varepsilon}\int_{-l}^{0}\frac{dz'}{\sqrt{\rho^2+(z-z')^2}} + \frac{q_0}{4\pi\varepsilon}\int_{0}^{l}\frac{dz'}{\sqrt{\rho^2+(z-z')^2}}. \tag{55}$$

These integrals are of the same type as those in equation 53. Assuming $s = 0$ in equation 54, we find

$$V = \frac{q_0}{4\pi\varepsilon}\log\frac{[l-z+\sqrt{(l-z)^2+\rho^2}](z+\sqrt{z^2+\rho^2})^2}{\rho^2[l+z+\sqrt{(l+z)^2+\rho^2}]}. \tag{56}$$

Thus, we obtain an equation for the equipotential surfaces,

$$\frac{[l-z+\sqrt{(l-z)^2+\rho^2}](z+\sqrt{z^2+\rho^2})^2}{\rho^2[l+z+\sqrt{(l+z)^2+\rho^2}]} = e^{4\pi\varepsilon V/q_0}. \tag{57}$$

Let $\rho = kz$, where k is a constant; then, for $z > 0$, we have, from equation 56,

$$V = \frac{q_0}{2\pi\varepsilon}\log\frac{1+\sqrt{1+k^2}}{k} + \frac{q_0}{4\pi\varepsilon}\log\frac{l-z+\sqrt{(l-z)^2+k^2z^2}}{l+z+\sqrt{(l+z)^2+k^2z^2}}. \tag{58}$$

As z approaches zero, the second term approaches zero and V approaches a constant limit; hence, in the vicinity of $z = 0$ the equipotential surfaces are cones given by $\rho = kz$.

When ρ is small compared with z and $l - z$, equation 57 becomes

$$\rho = kz\sqrt{\frac{l-z}{l+z}}, \qquad k \simeq 2e^{-2\pi\varepsilon V/q_0}. \tag{59}$$

This is an approximate equation for the equipotential surfaces when k is small compared with unity. Figure 10.9 shows the shape when $k = \frac{1}{4}$. Equation 59 is not quite correct in the vicinity of $z = l$. The surface given by this equation cuts the z axis at $z = l$, whereas the exact equipotential surface given by equation 57 cuts the z axis at $z = l + 0.125k^2l$. The radius of the exact surface at $z = l$ is $\rho = 0.25k^2l$. For small k, ρ is small, and the simplified expression 59 is good.

The charge per unit length of an antenna arm, assumed to coincide with an equipotential surface, may be obtained if we multiply the absolute value of grad V by $2\pi\rho\varepsilon$. For small k we shall find that the charge per unit length is very nearly equal to q_0 except in the immediate vicinity

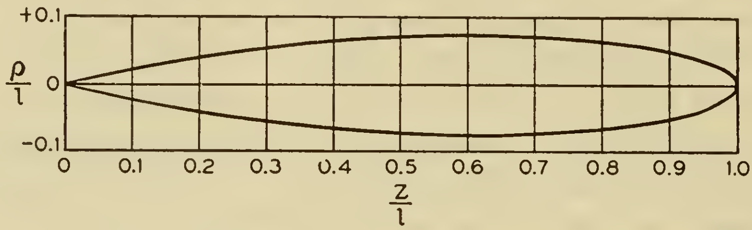

Fig 10.9 One arm of an antenna with a uniformly distributed *total* capacitance, that is, the capacitance associated with the principal *and* higher-order waves. The capacitance associated with only the principal waves is distributed uniformly on biconical antennas.

of the outer ends of the antenna. Thus, Fig. 10.9 shows an antenna with uniformly distributed capacitance. This should be contrasted with the biconical antenna for which only that part of the total capacitance is uniformly distributed which is associated with the principal wave.

We have already mentioned that the electric charge distributes itself on a conductor in such a way that the stored electric energy is minimum, and, therefore, the capacitance $C = q^2/2\mathcal{E}$ is maximum; hence, first-order errors in an *assumed* charge distribution will lead to second-order errors in the calculated capacitance. Calculations based on this principle are fairly simple. For instance, let us assume that the electric charge on a cylindrical antenna is distributed uniformly, so that the potential is given by equation 56 with $\rho = a$, where a is the radius of the antenna. Taking the average potential over one arm of the antenna,

$$\bar{V} = \frac{q_0}{2\pi\varepsilon l}\int_0^l \log\frac{2z}{a}\,dz - \frac{q_0}{4\pi\varepsilon l}\int_0^l \log\frac{l+z}{l-z}\,dz$$

$$= \frac{q_0}{2\pi\varepsilon}\left[\log\frac{2l}{a} - 1\right] - \frac{q_0}{2\pi\varepsilon}\log 2. \qquad (60)$$

From this we obtain the capacitance of the cylindrical antenna agreeing with equation 11.

In order to find a second approximation to the charge density on a cylinder we note that, for a constant $\rho = a$, the potential of a uniformly distributed axial charge is a slowly varying function of z over most of the range $0 < z < l$. This suggests that we treat q_0 in equation 56 as variable, so that

$$q(z) = 4\pi\varepsilon V \left[\log \frac{[l - z + \sqrt{(l-z)^2 + a^2}]\,(z + \sqrt{z^2 + a^2})^2}{a^2\,[l + z + \sqrt{(l+z)^2 + a^2}]} \right]^{-1} \cdot \qquad (61)$$

When $z \gg a$ and $l - z \gg a$, then,

$$q(z) = 4\pi\varepsilon V \left(2 \log \frac{2z}{a} + \log \frac{l - z}{l + z} \right)^{-1} \cdot \qquad (62)$$

The region $-2a < z < 2a$ must receive separate consideration (see Section 12.10).

10.9 Small loop antennas

The field of an electric current I flowing in a small loop (Fig. 10.10) of area S may be obtained by solving equations 4–9, 4–10, and 4–11. Since, however, we have already found the field of an electric current element, we can simplify our task considerably. Thus, the voltage induced in the loop by a current element at P_0 is

$$V_{AB} = -j\omega\mu H_z S. \qquad (63)$$

Since H_z is in the equatorial plane of the element, we find its value by substituting $\theta = \pi/2$ in equation 4–81; hence,

$$V_{AB} = -j\omega\mu \frac{\sigma I s}{4\pi r}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r} S. \qquad (64)$$

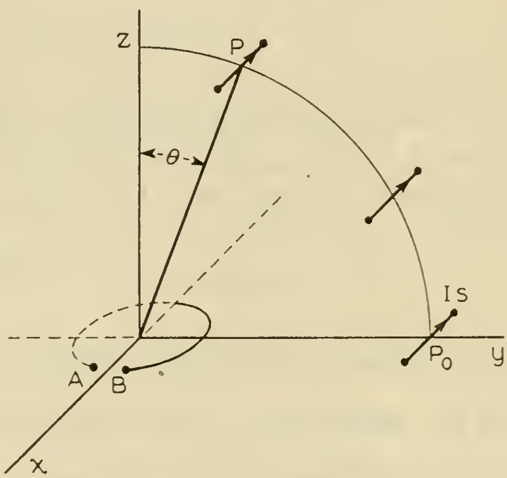

FIG. 10.10 Illustrating the calculation of the field of a loop antenna.

By the reciprocity theorem, the same voltage is induced in the element by the current I in the loop. In terms of the electric intensity E_φ of the field of the loop,

$$V_{AB} = E_\varphi s. \qquad (65)$$

Comparing this with equation 64, we find E_φ in the equatorial plane of

the loop; thus,

$$E_\varphi = -\frac{j\omega\mu\sigma IS}{4\pi r}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r}. \tag{66}$$

In any other position P of the element in the axial plane of the loop, H_z at the loop differs from equation 66 by the factor $\sin\theta$; hence, the complete expression for the electric intensity is

$$E_\varphi = -\frac{j\omega\mu\sigma IS}{4\pi r}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r}\sin\theta. \tag{67}$$

Substituting in equations 4–10 and 4–9, we find

$$\begin{aligned} H_\theta &= \frac{\sigma^2 IS}{4\pi r}\left(1 + \frac{1}{\sigma r} + \frac{1}{\sigma^2 r^2}\right) e^{-\sigma r}\sin\theta, \\ H_r &= \frac{\sigma IS}{2\pi r^2}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r}\cos\theta. \end{aligned} \tag{68}$$

In nondissipative media, $\sigma = j\beta$, and

$$\begin{aligned} E_\varphi &= \frac{\eta\beta^2 IS}{4\pi r}\left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r}\sin\theta, \\ H_\theta &= -\frac{\beta^2 IS}{4\pi r}\left(1 + \frac{1}{j\beta r} - \frac{1}{\beta^2 r^2}\right) e^{-j\beta r}\sin\theta, \\ H_r &= \frac{j\beta IS}{2\pi r^2}\left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r}\cos\theta. \end{aligned} \tag{69}$$

The receiving properties of the loop are contained in equation 63; thus, if ψ is the angle between the normal to the plane of the loop and the magnetic intensity H_0, the voltage induced in the loop is

$$|V| = \omega\mu H_0 S\cos\psi = \frac{2\pi\eta H_0 S\cos\psi}{\lambda} = \frac{2\pi E_0 S\cos\psi}{\lambda}\cdot \tag{70}$$

The voltage induced in a coil of n turns is n times as large.

10.10 Radiation resistance of a small loop

At large distances from the loop,

$$E_\varphi = \frac{\eta\beta^2 IS}{4\pi r} e^{-j\beta r}\sin\theta; \tag{71}$$

hence, the radiated power is

$$P = \frac{1}{2\eta}\int_0^{2\pi}\int_0^{\pi} r^2 E_\varphi E_\varphi^* \sin\theta\, d\theta\, d\varphi = \frac{\eta}{12\pi}(\beta^2 S)^2 II^*. \tag{72}$$

Thus, the radiation resistance of the loop is

$$R = \frac{\eta}{6\pi}(\beta^2 S)^2 = 320\pi^4\frac{S^2}{\lambda^4} \simeq 31{,}000\frac{S^2}{\lambda^4}\cdot \tag{73}$$

If the loop has n turns, the total circulating current is* nI; hence, the input resistance due to radiation is n^2 times as large as for a single turn — assuming, of course, that all the turns are connected in series. There will be no change in the radiation resistance if the turns are connected in parallel.

10.11 Inductance of a small loop

In a small, single-turn loop the current is substantially uniform and the approximate inductance may be obtained by integrating equation 29. From Fig. 10.11 we find that, for a loop of radius b,

$$z = b \sin \psi; \tag{74}$$

hence, if the wire radius is a,

$$L_{\text{loop}} = \frac{\mu b}{\pi} \int_{\log}^{\pi} \left(\frac{2b}{a} \sin \psi \right) d\psi$$

$$= \mu b \log \frac{b}{a} \cdot \tag{75}$$

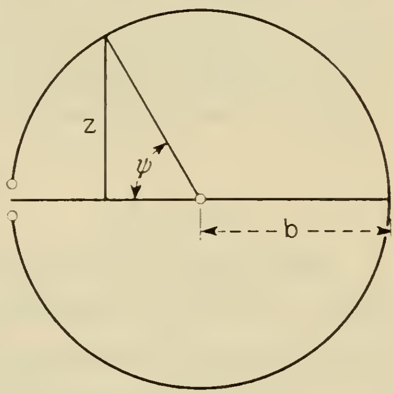

FIG. 10.11 Illustrating the calculation of the inductance of a loop antenna.

10.12 Capacitance of a small loop

The capacitance of a small, single-turn loop may be evaluated by a method analogous to that used in Section 10.6 for obtaining the inductance of a dipole antenna. Thus, the stored electric energy may be expressed as

$$\mathcal{E}_e = \tfrac{1}{2} C_{\text{loop}} |V_i|^2 = \frac{1}{2} \int_0^{\pi} C(z) \, |V(z)|^2 \, d(b\psi); \tag{76}$$

therefore,

$$C_{\text{loop}} = \frac{\int_0^{\pi} C(\psi) \, |V(\psi)|^2 \, d(b\psi)}{|V_i|^2} \cdot \tag{77}$$

For the voltage, we have

$$V(\psi) = I \int_{\psi}^{\pi} \frac{\mu}{\pi} \left(\log \frac{2b}{a} + \log \sin \psi \right) d(b\psi) \tag{78}$$

$$= \frac{\mu b I}{\pi} (\pi - \psi) \log \frac{2b}{a} + \frac{\mu b I}{\pi} \int_{\psi}^{\pi} \log \sin \psi \, d\psi. \tag{79}$$

The first term is a linear function of ψ, and, if we neglect the second

* Provided that the total length of the wire is small compared with $\lambda/4$; otherwise, the currents in the various turns will be unequal.

term, the equation reduces to

$$\frac{V(\psi)}{V(0)} = 1 - \frac{\psi}{\pi}. \tag{80}$$

Similarly, if we replace $C(\psi)$ by its average value round the loop and integrate equation 77 using equation 80, then,

$$C_{\text{loop}} = \tfrac{1}{3}C_{\text{av}}b\pi, \tag{81}$$

where C_{av} is the average value of the capacitance between the two halves of the loop per unit length along the circumference. Since $C(\psi)$ is a slowly varying function, its average value is approximately the reciprocal of the average value of the reciprocal function. The latter can be evaluated (see equation 75); thus,

$$C_{\text{av}} = \frac{\pi\varepsilon}{\log(b/a)}. \tag{82}$$

Returning to equation 79, we note that the first term increases as the radius of the wire diminishes, but the second term remains constant. Thus equation 80 represents the asymptotic form of the voltage distribution in a small loop, and equation 82 becomes more and more accurate as the ratio b/a increases.

It is not excessively difficult to evaluate the second term in equation 79. Thus, from equation 6.814 on p. 138 of *Smithsonian Mathematical Formulae*, we find

$$\log 2 \sin\psi = \log 2 + \log \sin\psi = -\sum_{n=1}^{\infty} \frac{\cos 2n\psi}{n}. \tag{83}$$

Therefore,

$$P(\psi) = \int_{\psi}^{\pi} \log \sin\psi \, d\psi = -(\pi - \psi)\log 2 + \tfrac{1}{2}\sum_{n=1}^{\infty} \frac{\sin 2n\psi}{n^2}. \tag{84}$$

Substituting in equation 79, we have

$$V(\psi) = \frac{\mu b I}{\pi}(\pi - \psi)\log\frac{b}{a} + \frac{\mu b I}{2\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\psi}{n^2}. \tag{85}$$

Hence,

$$\frac{V(\psi)}{V(0)} = 1 - \frac{\psi}{\pi} + \frac{1}{2\pi\log(b/a)}\sum_{n=1}^{\infty}\frac{\sin 2n\psi}{n^2}. \tag{86}$$

The sum of the series is zero when $\psi = 0, \pi/2, \pi$. When $\psi = \pi/4$,

$$\sum_{n=1}^{\infty}\frac{\sin 2n\psi}{n^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = 0.916. \tag{87}$$

For $\psi = 3\pi/4$, the value is -0.916.

10.13 Practical loop antennas

In practice, it is not essential to know exactly either the inductance or the capacitance of a loop antenna. The inductance is normally tuned

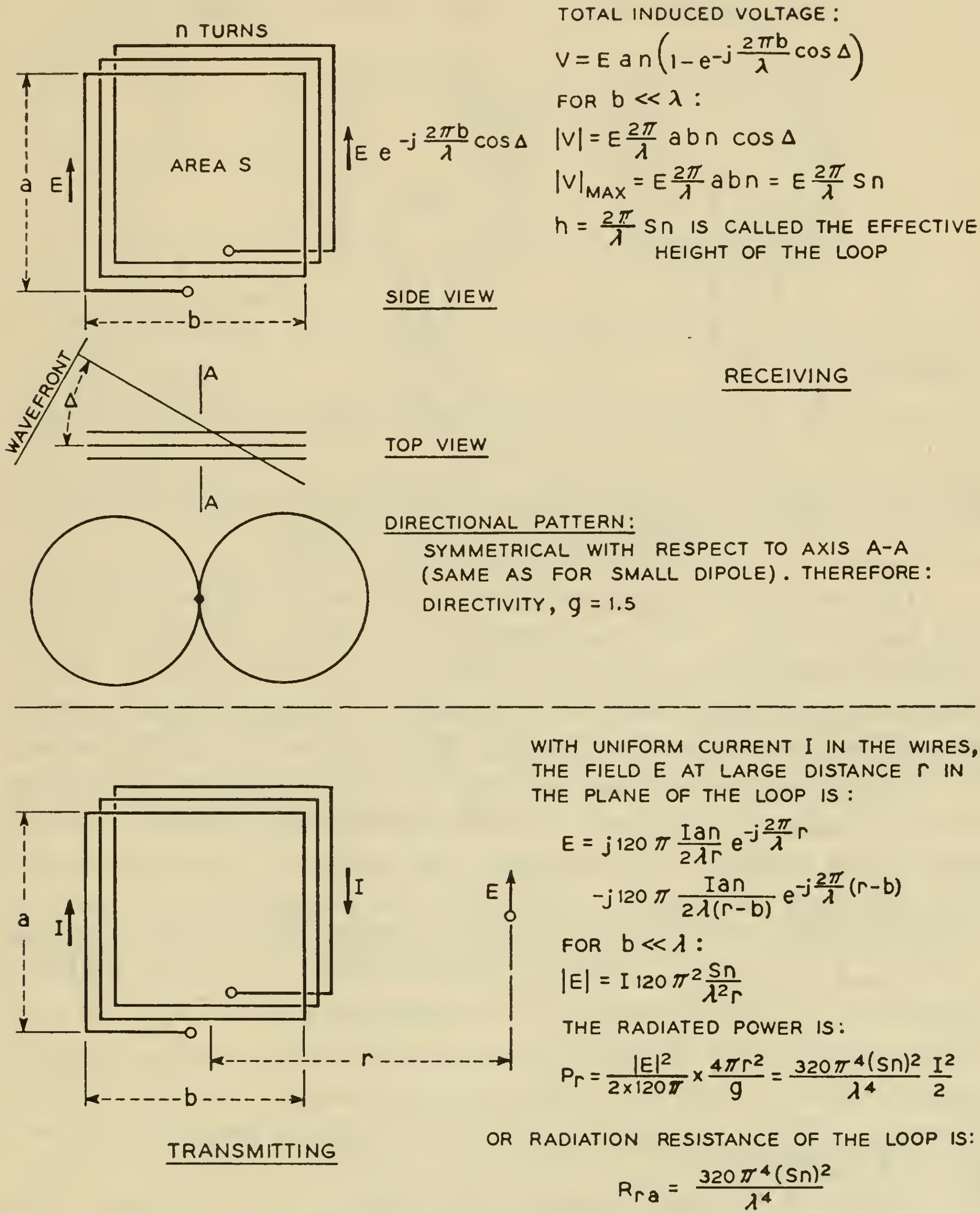

FIG. 10.12 A summary of the basic properties of a small loop antenna.

out with a variable capacitor, and only its order of magnitude is required. The capacitance of the loop merely affects the tuning capacitance. Usually it is not even necessary to know the local field. The essential

information needed for most ordinary purposes may be obtained by much simpler methods than those used in the preceding sections, particularly if we assume that the loop is rectangular. A summary of this information is presented in Fig. 10.12.

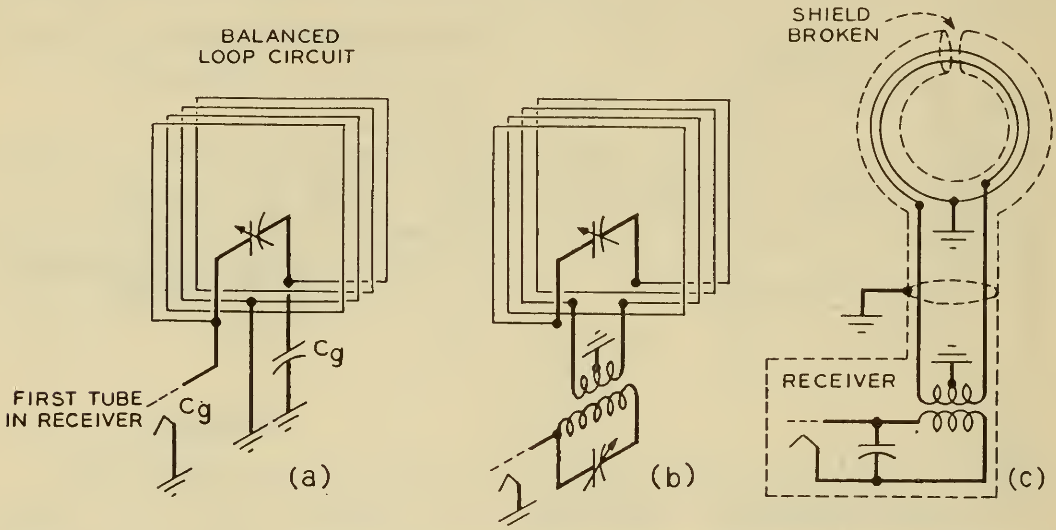

Fig. 10.13 Balanced methods of feeding loop antennas to eliminate the dipole type of radiation from unbalanced currents.

The loop antenna is generally used for direction finding at broadcast and longer waves. This use depends on its figure 8 radiation pattern which has a null along the perpendicular to the plane of the loop. Hence, as we rotate the loop round its vertical axis, we receive a signal until the perpendicular points in the direction of the source of incoming waves.

The figure 8 pattern shown in Fig. 10.12 is obtained only if the loop circuit is balanced to ground;* an unbalance causes the loop to act also as a dipole antenna. Consequently, the nulls are obscured and their directions may be changed. This effect has frequently been called the " antenna effect "; but this is no longer considered good usage. Figures 10.13*a* and *b* show typical balanced loop circuits. A better balance is obtained by enclosing the loop in an electrostatic shield (Fig. 10.13*c*).† Such a shield insures that all parts of the loop will have the same capacitance to ground, irrespective of the loop orientation or of the proximity of various objects.‡ Single-turn balanced shielded loops are discussed by L. L. Libby.§

* It is a good example of the effects one must look for in the presence of the earth.

† J. E. Browder, Design values for loop-antenna input circuits, Fig. 1, *IRE Proc.*, 35, May 1947, pp. 519–525.

‡ Frederick E. Terman, *Radio Engineering*, Third Edition, McGraw-Hill, New York, p. 821.

§ Special aspects of balanced shielded loops, *IRE Proc.*, 34, September 1946, pp. 641–646.

10.14 Magnetically and dielectrically loaded antennas

For a given current in the winding, a loop wound on a magnetic core (Fig. 10.14a) produces a stronger field than the loop alone. The current in the loop magnetizes the core, and the core becomes a magnetic doublet whose field is superimposed in phase on the field of the current. If the

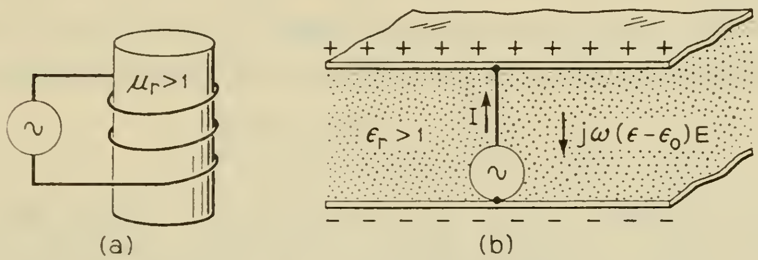

FIG. 10.14 Magnetically and dielectrically loaded antennas.

loop and its core are small, the directive pattern is still of the figure 8 shape; hence, the directivity of the loop is not affected by the core. Since for the given current the field and, hence, the radiated power are increased by the core, the radiation resistance must become larger. Thus, let H_1 and H_2 be the distant magnetic intensities of the loop alone and of the loop with the magnetic core; then,

$$H_1 = AI, \qquad H_2 = kAI, \qquad k > 1, \tag{88}$$

where A and k are proportionality factors. The radiation intensity is proportional to the square of the magnetic intensity, and, *for the same shape of the radiation pattern*, the radiated power is proportional to the radiation intensity; therefore,

$$P_2 = k^2 P_1. \tag{89}$$

If R_1 and R_2 are, respectively, the radiation resistance of the loop alone and of the loop with the magnetic core, then,

$$P_1 = \tfrac{1}{2}R_1 I^2, \qquad P_2 = \tfrac{1}{2}R_2 I^2. \tag{90}$$

Consequently,

$$R_2 = k^2 R_1. \tag{91}$$

The heat loss in the loop is unaffected by the presence of the core. Hence, if the heat loss in the core is kept negligible, the radiation efficiency and the power gain of the loop are increased.

In the case of a dipole antenna or a capacitor antenna (Fig. 10.14b), loaded with a dielectric, the effect is the opposite. The density $j\omega(\varepsilon - \varepsilon_0)E$ of the polarization current* in the dielectric is in the direc-

* Polarization current is the excess of displacement current in a given medium over the displacement current in vacuum (for equal electric intensities).

tion opposite to that of the current I in the dipole. Polarization currents produce fields just as conduction currents do; but in the present case they weaken the field of the conduction currents.

When the edge effects are negligible, the polarization current is

$$j\omega(\varepsilon - \varepsilon_0)ES = j\omega\varepsilon ES\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) = I\left(1 - \frac{\varepsilon_0}{\varepsilon}\right),$$

where S is the area of the capacitor and $I = j\omega\varepsilon ES$ is the conduction current in the wire. Hence, the effective radiating current is

$$I - I\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) = I\,\frac{\varepsilon_0}{\varepsilon}\,.$$

It may be noted that in any homogeneous medium, the radiation resistance of either a current element or a loop is proportional to the intrinsic impedance $\eta = \sqrt{\mu/\varepsilon}$ of the medium. Consequently, an increase in μ makes the radiation resistance larger, and an increase in ε makes it smaller.

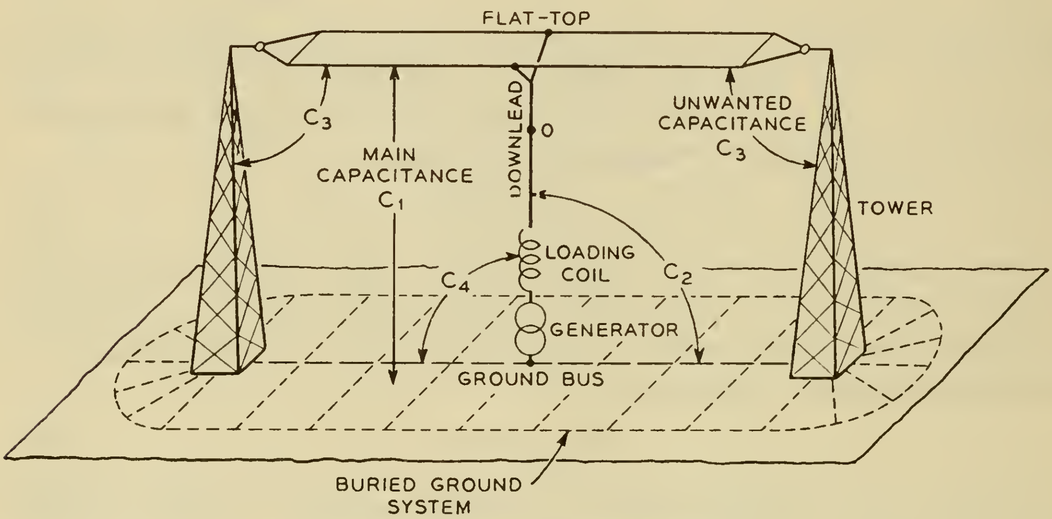

FIG. 10.15 A long-wave antenna.

10.15 Long-wave antennas

During severe magnetic storms very long waves provide the most reliable means for long-distance radio communication. This is one reason why they are still being used, even though the efficiencies of long-wave antennas are low and progress in the short-wave art has been remarkable. The simplest type of long-wave antenna is shown in Fig. 10.15. A single " downlead " is connected to a " flat-top " consisting of parallel horizontal wires supported by two towers. Cornelius J. DeGroot, in his paper on " The High Power Station at Malabar, Java,"*

* *IRE Proc.*, **12**, December 1924, pp. 693–722.

describes a long-wave station designed for direct communication with Holland on wavelengths ranging from 7 to 20 km. The station was located in a deep ravine, and the mountains on the two sides were used as towers for supporting the downlead.

Another long-wave station with a world-wide range of communication is described by E. H. Shaughnessy in his paper on "The Rugby Radio Station of the British Post Office."* In the Rugby Station designed to operate on $\lambda = 18{,}000$ meters (16.66 kc/sec), twelve towers are provided to support two flat-top capacitive loading systems. This station is situated near Rugby on a site about one and a half miles long by one mile wide. As originally designed, the capacitance of the larger antenna system was to be 0.045 μf and that of the smaller 0.033 μf. It was estimated that a minimum working current of 500 amperes in an efficient antenna supported on 250-meter (820-ft) masts would be required. To provide a safe margin and to allow for a possible 50 per cent loss in efficiency, the station was designed for 1000 kw. The total resistance of the smaller (0.033 μf) antenna was measured to be about three quarters of an ohm. The mean geometric height of the antenna was 236 meters (775 ft), and the effective height, deduced from measurements, was 185 meters. Hence, the radiation resistance is only 0.16 ohm.

With such small radiation resistances, it is essential to provide as good a ground as possible; otherwise, the efficiency will be exceedingly low. A buried ground system is very common for reasons of convenience; but an elevated counterpoise is better electrically. According to Maxwell, a grounded screen with wires 1 ft apart and 2 ft 6 in. above the ground will carry 80 per cent of the current. If the same screen is insulated, there will be less than 10^{-4} per cent earth current. These figures are based on electrostatic considerations which are permissible for such long waves. The greater the height of the screen compared with the distance between the wires, the greater the shielding.

In addition to ground losses there will be losses in the wires, in tuning coils, dielectric losses in insulators, and leakage losses. If these losses are represented by series resistances in the antenna circuit, the resistances will vary with wavelength as follows:

Radiation resistance	$\propto \lambda^{-2}$
Wire resistance	$\propto \lambda^{-1/2}$
Dielectric loss resistance	$\propto \lambda$
Leakage resistance	$\propto \lambda^2$

At frequencies considerably below the first absorption band, the con-

* *IEE Jour.*, **64**, June 1926 pp. 683–713.

ductivity g of a dielectric is proportional to the frequency. The admittance of an insulator is proportional to $g + j\omega\varepsilon$; hence, it is proportional to the frequency. The impedance will then be proportional to the wavelength. If there is a leakage across an insulator and if this leakage is represented by a large constant resistance R in parallel with the capacitance of the antenna, the power loss will be $VV^*/2R$. Since $V = I/j\omega C$, the equivalent series resistance will vary inversely as the square of the frequency or directly as the square of the wavelength. For further discussion of these problems, the reader should consult a paper by T. L. Eckersley, "An investigation of transmitting aerial resistances."*

The main antenna capacitance C_1 is provided by the flat-top (Fig. 10.15). It is deliberately made large compared with the capacitance C_2 of the downlead in order to raise as high as possible the center of capacitance O, and thus to increase the effective height of the antenna and the radiation resistance. The capacitances C_3 between the flat-top and the supporting towers are unwanted, because the charging currents in the towers are in the opposite direction to the current in the downlead and thus tend to cancel the distant field of the antenna.

The loading coil is provided to tune out the antenna capacitance. The capacitance C_4 of this coil to ground is undesirable because the charging currents do no useful work and result only in an additional loss of power.

10.16 Multiple-tuned antennas

Alexanderson's "multiple-tuning" principle may be employed to decrease the ground losses. This provides several downleads separated widely enough to insure that the associated ground currents do not appreciably overlap. At the same time the distances between the downleads are so small compared with λ that, as far as the distant field is concerned, they are equivalent to a single downlead; but the earth resistance is reduced in the ratio n, the number of the downleads, to unity. This occurs because the current in each downlead is now I/n, and the associated ground loss is $\frac{1}{2}R_g\ (I/n)^2$, where R_g is the ground contribution to the resistance of a single downlead antenna; the total ground loss is n times as large since the mutual resistances are negligible. Hence, the total ground loss associated with n downleads is $R_gI^2/2n$, compared with $R_gI^2/2$ for a single downlead.

The RCA Rocky Point antenna (Fig. 10.16) combines the multiple-tuning principle with an artificial ground. To quote†: "The antenna,

* *IEE Jour.*, **60**, May 1922, pp. 581–594.

† E. F. W. Alexanderson, A. E. Reoch, and C. H. Taylor, The electrical plant of transocean radio telegraphy, *AIEE Jour.*, **42**, June 1923, pp. 693–703.

in effect, stands on a plate of copper 2000 feet wide and 3 miles long." The combined resistance of the antenna and the ground system is 40 hundredths of an ohm; it is made up as follows:

	Ohms
Radiation resistance at 16,500 meters	0.05
Ground resistance	0.10
Tuning coil resistance	0.15
Conductor resistance	0.05
Insulator and other losses	0.05
Total resistance	0.40

The unit is operated with 200 kw in the antenna. The corresponding antenna current is 200 amperes (effective); this gives a moment of 60,000 meter-amperes.

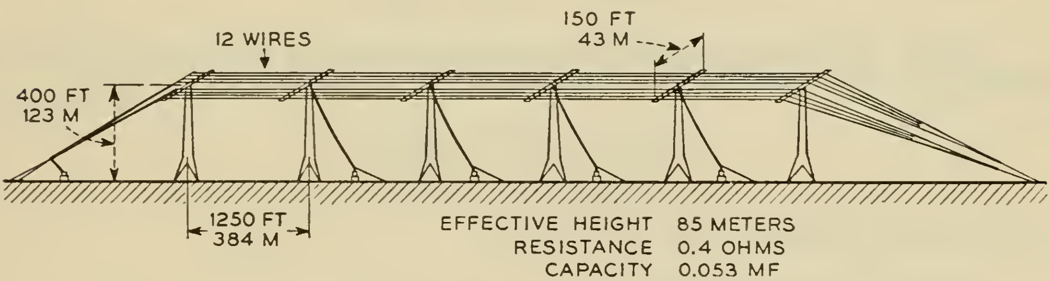

FIG. 10.16 A multiple-tuned antenna.

For further information on multiple-tuned long-wave antennas, the reader is referred to the paper just cited, and to a paper by N. Lindenblad and W. W. Brown, Main considerations in antenna design, *IRE Proc.*, **14**, June 1926, pp. 291–323.

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PROBLEMS

10.4–1 A short wire of length $2l$ is in a uniform field of intensity E_0 parallel to it. What is the form of current distribution?

Ans. $I(z) = A(l^2 - z^2)$, where z is the distance from the center and A is a constant.

10.4–2 Solve the preceding problem when the wire is broken at $z = 0$.

Ans. $I(z) = Bz(l - z)$ if $0 < z < l$ and $I(z) = -Bz(l + z)$ if $-l < z < 0$, where B is a constant.

10.5–1 Calculate the moment p of the current in the wire under the conditions of Problem 10.4–1, and the power P reradiated by the wire.

Ans. $$p = \tfrac{4}{3}Al^3, \qquad P = \frac{640\pi^2}{9\lambda^2}A^2l^6.$$

10.5–2 Solve Problem 10.5–1 under the conditions of Problem 10.4–2.

Ans. $$p = \tfrac{1}{3}Bl^3, \qquad P = \frac{40\pi^2}{9\lambda^2}B^2l^6.$$

10.6–1 Find the approximate transfer admittance of a short antenna between $z = 0$ and $z = z$.

Ans. $$I(z) = \frac{1}{Z_i}\left(1 - \frac{|z|}{l}\right),$$

where Z_i is the impedance seen at the center.

10.6–2 Find A in Problem 10.4–1.

Ans. $A = E_0/Z_il$, where Z_i is defined in Problem 10.6–1.

10.6–3 Find B in Problem 10.4–2, neglecting the interaction between the two halves of the wire.

Ans. $B = 2E_0/Z_i'l$, where Z_i' is the impedance seen from the center of a wire of length l.

10.6–4 Show that, to the extent to which we can neglect the inductance and the difference between $\log(2l/a)$ and $\log(l/a)$, the constants A and B in Problems 10.4–1 and 10.4–2 are equal.

10.6–5 What is the voltage induced between the terminals of the broken wire in Problem 10.4–2?

Ans. $$V = E_0l.$$

10.6–6 What is the current through the load of a perfectly conducting short receiving antenna, assuming that the antenna is tuned and that the load is equal to the radiation resistance?

Ans. $$I = \frac{E_0\lambda^2}{160\pi^2 l}.$$

10.6–7 What is the received power under the conditions of the preceding problem?

Ans. $$P = \frac{E_0^2\lambda^2}{640\pi^2}.$$

10.6–8 Using the result of the preceding problem, calculate the effective area of a short perfectly conducting antenna.

Ans.
$$A = \frac{3\lambda^2}{8\pi}\cdot$$

10.6–9 Solve Problem 10.6–6 when the wire is not perfectly conducting but has an ohmic resistance R_0 per unit length.

Ans.
$$I = \frac{E_0\lambda^2}{160\pi^2 l + \frac{2}{3}R_0\lambda^2}\cdot$$

Note that R_0 may be the controlling factor; see Section 6–2 on the efficiency of antennas.

10.6–10 What is the input resistance of a short antenna when the ohmic resistance is included?

Ans.
$$R_i = \frac{80\pi^2 l^2}{\lambda^2} + \tfrac{2}{3}R_0 l.$$

10.6–11 What is the effective area of a short antenna when the ohmic resistance is included?

Ans.
$$A = \frac{3\lambda^2}{8\pi}\left(1 + \frac{R_0\lambda^2}{120\pi^2 l}\right)^{-1}.$$

11

SELF-RESONANT ANTENNAS

11.1 Half-wave antennas in free space and the corresponding quarter-wave vertical antennas above ground

The elementary theory of self-resonant antennas is much simpler than the general theory, and for this reason we present it in a separate chapter. The terms " half-wave antenna " and " half-wave dipole " are applied loosely to antennas whose length is approximately equal to one-half wavelength (without counting the length of the gap AB, Fig. 11.1a). A

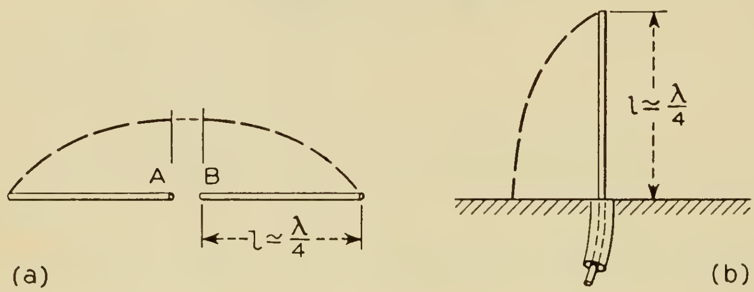


FIG. 11.1 (a) A *half-wave* antenna; (b) a *quarter-wave* vertical antenna.

similar meaning is assigned to the terms " quarter-wave vertical antenna " and " quarter-wave vertical unipole " (Fig. 11.1b). There are two reasons for this slightly indefinite usage. Antennas are usually operated in a certain band of frequencies rather than at a single frequency; hence, the length of an antenna in wavelengths cannot be a specific number. It is customary to place the resonant frequency of the antenna in the center of the operating frequency band; and in Section 8.23 we found that at resonance the antenna is somewhat shorter than $\lambda/2$.

In Section 8.23 we also calculated the approximate effect of the capacitance of the flat outer ends of the antenna on the resonant wavelength; in Section 13.12 we shall obtain the corresponding effect of the excess capacitance *near* the outer ends over the average value. To in-

dicate the magnitude of this shortening we quote the following formulas. Let the length of each antenna arm be

$$l = \tfrac{1}{4}\lambda - \delta; \tag{1}$$

then the first-order approximation given by the mode theory of antennas for circular cylinders is

$$\frac{100\delta}{\lambda/4} = \frac{2700}{K_a + 21} + \frac{20aK_a}{3\pi\lambda}, \tag{2}$$

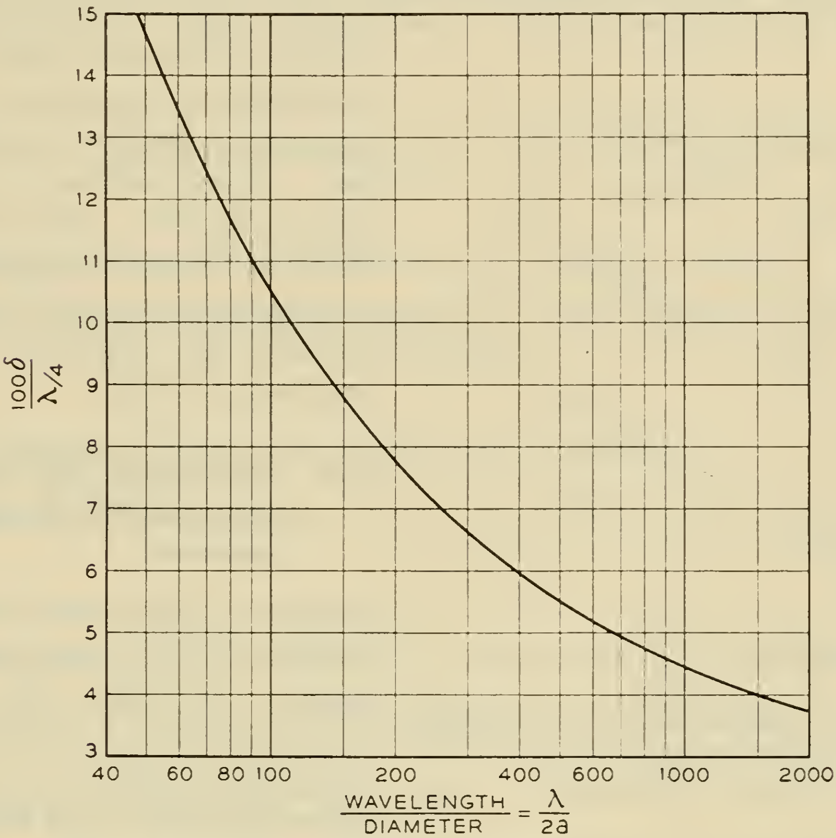


FIG. 11.2 The difference in per cent between $\lambda/4$ and the length of one arm of a self-resonant half-wave antenna.

where K_a is the average characteristic impedance for principal waves, defined by

$$K_a = 120 \log \frac{2l}{a} - 120. \tag{3}$$

Using the method of "balayage" or "sweeping off," (Section 13.12) the corresponding expression is found to be

$$\frac{100\delta}{\lambda/4} = \frac{2700}{2Z_0} + \frac{20a(2Z_0)}{3\pi\lambda}, \tag{4}$$

where Z_0 is given by equation 8–91,

$$2Z_0 = 120\left(\log \frac{\lambda}{2\pi a} + 0.116 + \mathrm{Ci}\,\pi\right) = 120\left(\log \frac{\lambda}{2a} - 0.955\right). \tag{5}$$

Since $l \simeq \lambda/4$, equation 3 gives

$$K_a = 120 \log \frac{\lambda}{2a} - 120; \tag{6}$$

equations 2 and 4 are in agreement except for the small term in the denominator of the first fraction in equation 2. Figure 11.2 shows δ/l in per cent for various ratios of the wavelength to the diameter. The foregoing equations are asymptotic approximations for very large K_a and $2Z_0$. More accurate values of resonant lengths may be obtained from the reactance curves in the vicinity of $l = \lambda/4$ (see Section 13.24).

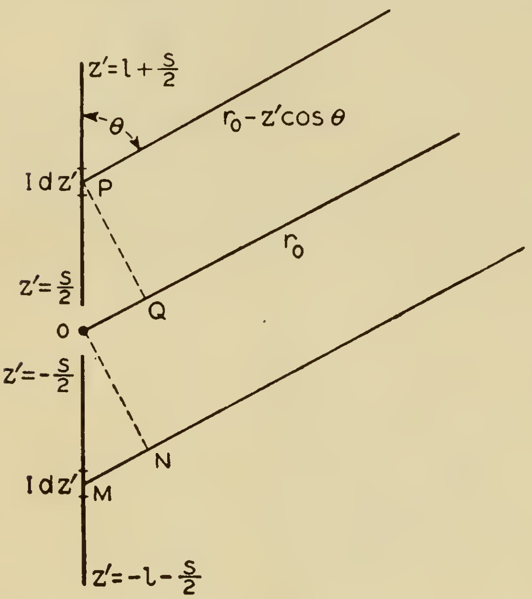


FIG. 11.3 Illustrating the differences in paths from various elements of a linear antenna to a distant point.

11.2 Radiation patterns of sinusoidally distributed currents

Consider a thin current filament and let $I(z')$ be the current at point $z = z'$ (Fig. 11.3). The electric intensity is obtained by integrating equation 5–1 on the assumption that r is so large that it is equal to $r_0 - z' \cos\theta$, and that the second term in this expression may be neglected in the amplitude of the field. Thus,

$$E_\theta = j\,\frac{60\pi}{\lambda r_0}\, e^{-j\beta r_0} \sin\theta \int_{-l-s/2}^{l+s/2} I(z') e^{j\beta z' \cos\theta}\, dz', \tag{7}$$

where s is the length of the gap. If the current distribution is symmetric about the center — as it is in antennas energized at the center — we have $I(-z') = I(z')$. Splitting the range of integration and taking account of the symmetry, we obtain

$$E_\theta = j\,\frac{120\pi}{\lambda r_0}\, e^{-j\beta r_0} \sin\theta \int_{s/2}^{l+s/2} I(z') \cos(\beta z' \cos\theta)\, dz'. \tag{8}$$

We shall now use the approximate current distribution given by equation 8–127:

$$I(z') = I_0 \sin \beta(l + \tfrac{1}{2}s + \delta - z'), \qquad z' > \frac{s}{2}\cdot \tag{9}$$

When the antenna is fed by a two-wire line, the current in the gap is zero. In the case of the vertical antenna (Fig. 11.1*b*), we let $s = 0$. Even in the case of the half-wave antenna we may let $s = 0$ since s is usually small. In fact, in the equatorial plane, where E_θ is maximum under normal operating conditions, s does not affect the field. Evaluating equation 8 for $s = 0$, we obtain

$$E_\theta = \frac{60jI_0[\cos\beta\delta\cos(\beta l\cos\theta) - \cos\beta(l+\delta) - \sin\beta\delta\cos\theta\sin(\beta l\cos\theta)]}{r_0\sin\theta}\, e^{-j\beta r_0}. \tag{10}$$

Substituting in equation 5–10, we find the radiation intensity

$$\Phi = \frac{15I_0^2[\cos\beta\delta\cos(\beta l\cos\theta) - \cos\beta(l+\delta) - \sin\beta\delta\cos\theta\sin(\beta l\cos\theta)]^2}{\pi\sin^2\theta}\cdot \tag{11}$$

11.3 Radiation pattern of a half-wave antenna

For a half-wave antenna, $l + \delta = \lambda/4$, and we find

$$\Phi = \frac{15I_0^2\left\{\cos\beta\delta\cos\left[\left(\frac{\pi}{2}-\beta\delta\right)\cos\theta\right] - \sin\beta\delta\cos\theta\sin\left[\left(\frac{\pi}{2}-\beta\delta\right)\cos\theta\right]\right\}^2}{\pi\sin^2\theta}\cdot \tag{12}$$

In the equatorial plane, $\theta = \pi/2$, and

$$\Phi = \Phi_{\max} = \frac{15}{\pi} I_0^2 \cos^2\beta\delta. \tag{13}$$

Thus, the effect of δ on $\Phi_{\max}$ is a second-order effect. We shall therefore simplify equation 12 by assuming $\delta = 0$,

$$\Phi \simeq \frac{15I_0^2\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\pi\sin^2\theta}\cdot \tag{14}$$

The directive pattern of a half-wave antenna in free space, or of a quarter-wave vertical antenna above a perfect ground, does not differ much from that of a current element (Fig. 11.4).

Over an imperfect ground the directive pattern is changed very radically (Fig. 7.5); and this is true for any other antenna emitting vertically polarized waves.

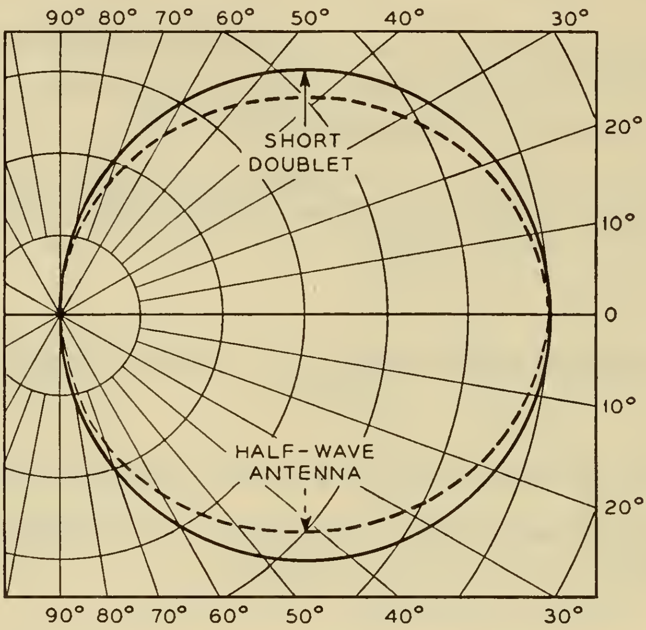


Fig. 11.4 Directive patterns of a short doublet and a half-wave antenna.

11.4 Power radiated by a half-wave antenna

Integrating equation 14, we obtain the power radiated by a half-wave antenna:

$$P = \int_0^{2\pi}\int_0^{\pi} \Phi \sin\theta \, d\theta \, d\varphi = 30I_0^2 \int_0^{\pi} \frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \sin\theta \, d\theta$$

$$= 60I_0^2 \int_0^{\frac{1}{2}\pi} \frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \sin\theta \, d\theta. \tag{15}$$

Introducing a new variable of integration, $t = \cos\theta$, we find

$$P = 60I_0^2 \int_0^1 \frac{\cos^2(\pi t/2)}{1 - t^2} \, dt$$

$$= 30I_0^2 \int_0^1 \frac{\cos^2(\pi t/2)}{1 - t} \, dt + 30I_0^2 \int_0^1 \frac{\cos^2(\pi t/2)}{1 + t} \, dt. \tag{16}$$

Letting $1 - t = u$ in the first integral and $1 + t = u$ in the second, we have

$$P = 30I_0^2 \int_0^2 \frac{\sin^2(\pi u/2)}{u}\, du = 15I_0^2 \int_0^2 \frac{1 - \cos \pi u}{u}\, du$$

$$= 15I_0^2 \int_0^{2\pi} \frac{1 - \cos t}{t}\, dt = 15\, I_0^2 \operatorname{Cin} 2\pi = 36.56 I_0^2. \qquad (17)$$

11.5 Directivity and effective area of a half-wave antenna

From equations 13 and 17 we obtain the directivity,

$$g = \frac{60}{36.56} = 1.64, \qquad G = 2.15 \text{ db}. \qquad (18)$$

This is only 0.39 db greater than the directivity of a current element,

$$g = 1.5, \qquad G = 1.76 \text{ db}. \qquad (19)$$

The effective areas are

$$\begin{aligned} A &= \frac{3}{8\pi}\lambda^2 = 0.12\lambda^2, \text{ for a current element;} \\ A &= 0.13\lambda^2, \text{ for a half-wave antenna.} \end{aligned} \qquad (20)$$

11.6 Input impedance of a half-wave antenna

The asymptotic current in a half-wave antenna (at resonance) is

$$I(z) = I_0 \cos \beta z, \qquad |z| < l, \qquad (21)$$

if z is measured from the corresponding input terminal. Hence, the input current is

$$I_i = I(0) = I_0. \qquad (22)$$

Thus, the input impedance is

$$R_i = \frac{2P}{I_0^2} = 73.12 \text{ ohms}. \qquad (23)$$

This is the resonant impedance of an infinitely thin, perfectly conducting antenna. For theoretical and experimental values for antennas of finite radius see Section 13.23; these values depend on the radius and on the input conditions to the extent of a few ohms. The resonant impedance may be either smaller or larger than 73 ohms.

In accordance with Section 4.18, the input impedance of a quarter-wave vertical antenna above a perfect ground is one half of the impedance of the half-wave antenna in free space: that is, about 36.5 ohms. Over an imperfect ground the impedance is higher owing to the losses in the ground (Section 11.21).

11.7 Impedance matching

In order to match the antenna to the feed line, it may be necessary to raise the input impedance of the antenna. This can be done by feeding the antenna off base, as shown in Fig. 11.5. At resonance the current distribution and the radiated power are not appreciably affected by the position of the voltage source. Hence, from equation 21,

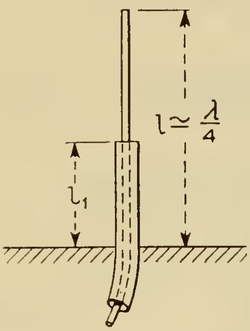


FIG. 11.5 Off-base feeding for impedance matching.

$$P = \tfrac{1}{2}36.5 I_0^2 = \tfrac{1}{2}R_1(l)\, I_0^2 \cos^2 \beta l_1, \tag{24}$$

where $R_1(l)$ is the impedance seen at the point $z = l_1$. Therefore,

$$R_1(l) = \frac{36.5}{\cos^2 \beta l_1}\cdot \tag{25}$$

If $l_1 = \lambda/8$, $\cos^2 \beta l_1 = \tfrac{1}{2}$, and the input impedance is 73 ohms.

11.8 Heat loss in the antenna

If R_0 is the ohmic resistance of the antenna per unit length, the power loss is

$$P_0 = \tfrac{1}{2}R_0 \int_{-l}^{l} |I(z)|^2\, dz. \tag{26}$$

Since R_0 is deliberately kept small, the errors in $I(z)$ are not important in their effect on the total impedance of the antenna. Thus, a satisfactory formula may be obtained by assuming that the current is sinusoidal,

$$\begin{aligned} P_0 &= \tfrac{1}{2}R_0 I_0^2 \int_{-l}^{l} \sin^2 \beta(l + \delta - |z|)\, dz \\ &= \tfrac{1}{2}R_0 l \left[1 - \frac{\sin 2\beta(l+\delta) - \sin 2\beta\delta}{2\beta l}\right] I_0^2. \end{aligned} \tag{27}$$

By the principle of conservation of energy, this power must equal

$$P_0 = \tfrac{1}{2}R_{\text{in}}' I_{\text{in}}^2 = \tfrac{1}{2}R_{\text{in}}' I_0^2 \sin^2 \beta(l+\delta), \tag{28}$$

where R_{in}' is the increase in the input resistance due to heat loss in the antenna. Hence,

$$R_{\text{in}}' = R_0 l \left[1 - \frac{\sin 2\beta(l+\delta) - \sin 2\beta\delta}{2\beta l}\right] \csc^2 \beta(l+\delta). \tag{29}$$

For half-wave antennas this becomes approximately

$$R_{\text{in}}' = \tfrac{1}{4}R_0\lambda. \tag{30}$$

The resistance per unit length is obtained from the skin-effect formulas. Thus, for thick copper conductors (thick compared with the skin depth),

$$R_0 = \frac{\mathcal{R}}{2\pi a},$$

$$\mathcal{R} = 2.61 \times 10^{-7}\sqrt{f} = 8.25 \times 10^{-7}\sqrt{\frac{f}{10}}$$

$$= \frac{4.52 \times 10^{-3}}{\sqrt{\lambda}} = \frac{0.0143}{\sqrt{10\lambda}}. \qquad (31)$$

For example, if $\lambda = 10$ meters and $a = 0.01$ meter, then $R_{in}' = 1.43/8\pi$ ohm — which is a negligible contribution to the antenna impedance.

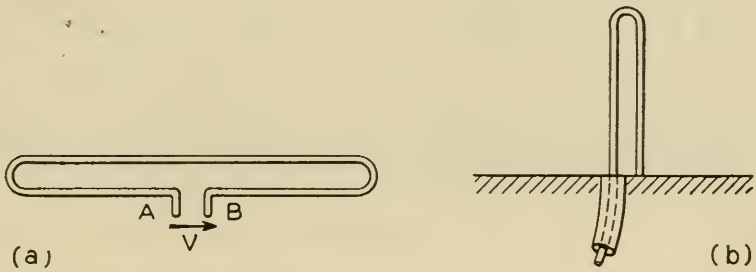


FIG. 11.6 (*a*) A folded dipole; (*b*) a folded unipole.

11.9 Folded dipoles

*Folded dipoles** (Fig. 11.6*a*) and folded unipoles, (Fig. 11.6*b*) are loops in which one dimension is large compared with the other. Because of the proximity of the wires in each half of the loop, the coupling between them is strong. The folded dipole is a two-wire line, short-circuited at both ends. One might think that the currents in these wires would be equal and opposite; and so they are to a large extent, *except when the length of the dipole is near* $\lambda/2$.

To analyze the folded dipole (or unipole) we shall decompose the waves on it into symmetric (Fig. 11.7*a*) and antisymmetric (Fig. 11.7*b*) modes of propagation. *In the symmetric case* we have a dipole antenna consisting of wires 1 and 2 in parallel. The effective radius of this antenna is (see equation 4–44 or Section 8.26)

$$a_{\text{eff}} = \sqrt{as}, \qquad (32)$$

* P. S. Carter, Simple television antennas, *RCA Rev.*, **4**, October 1939, pp. 168–185; R. Guertler, Impedance transformation in folded dipoles, *Proc. Inst. Radio Eng. Australia*, **10**, April 1949, pp. 95–100; also *IRE Proc.*, **38**, September 1950, pp. 1042–1047.

where s is the distance between the axes of the wires. At resonance the input impedance is about 73 ohms, and

$$\frac{V/2}{2I_i^s} = 73, \qquad I_i^s = \frac{V}{292}\cdot \tag{33}$$

In the antisymmetric case (Fig. 11.7*b*), the currents are equal and opposite and the radiation is small; hence,

$$Z_2 = jK_2 \tan \beta l, \qquad K_2 = 120 \log \frac{s}{a}\cdot \tag{34}$$

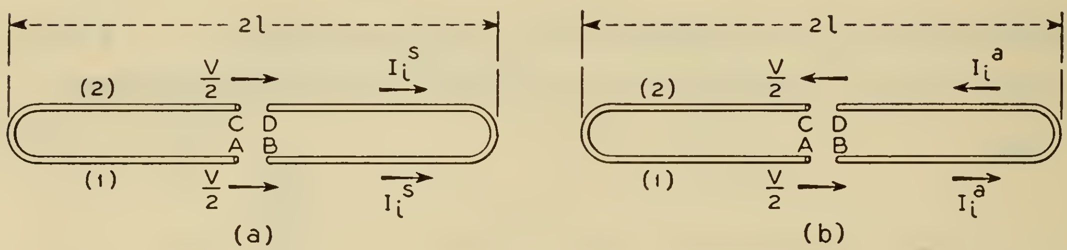


FIG. 11.7 (*a*) Symmetric or push-push and (*b*) antisymmetric or push-pull methods of energizing a two-wire line short-circuited at both ends.

In this case,*

$$I_i^a = \frac{V}{Z_2}\cdot \tag{35}$$

The total input current I_i in the folded dipole antenna is the sum of the symmetric and antisymmetric currents,

$$I_i = I_i^s + I_i^a = \frac{V}{292} + \frac{V}{Z_2}, \tag{36}$$

and the input impedance is

$$Z_i = \frac{V}{I^i} = \frac{292Z_2}{292 + Z_2}\cdot \tag{37}$$

If the resonant length is $l = (\lambda/4) - \delta$, $\tan \beta l \simeq \cot \beta\delta \simeq 1/\beta\delta$; hence,

$$Z_i \simeq \frac{j292K_2}{jK_2 + 292\beta\delta} = \frac{292}{1 - j(292\beta\delta/K_2)}\cdot \tag{38}$$

Since the two modes of propagation are in parallel, simpler general

* Note that the input impedance of a two-wire line is defined as the complex ratio of the total push-pull voltage to the push-pull current in one wire with the sign so chosen that the real part is positive.

formulas are obtained in terms of admittances; thus,

$$Y_i = \frac{1}{292} + Y_2 \simeq \frac{1}{292} - \frac{j\beta\delta}{K_2} \cdot \tag{39}$$

The reactive component is small since $\beta\delta$ is of the order of one tenth and K_2 may be about two or three hundred ohms.

The folded unipole is electrostatically grounded. In some applications (as in railway-car antennas) this is a valuable property. The fact that the effective radius (equation 32) of the folded dipole exceeds the radius of the wire tends to increase the bandwidth in comparison with a simple dipole *made of the same wire.* On the other hand, the energy stored in the push-pull mode tends to decrease the bandwidth. There is a net gain in favor of the folded dipole. However, this type of comparison is not quite fair. We should compare the folded dipole with a simple dipole consisting of *two* wires of the same size and the same distance apart as in the folded dipole but operated in parallel; then we should find that the simple dipole has a *larger* bandwidth than the folded dipole. It is easier to design a 300-ohm, two-wire line than a 73-ohm two-wire line; hence, in a balanced method of feeding, the folded dipole may be preferable to a simple half-wave dipole.

11.10 Half-wave receiving antennas

From equation 9–99 we find that the current through the load in the center of a self-resonant half-wave antenna in response to a uniform electric intensity E_0, parallel to the antenna, is

$$I = \frac{73E_0 \int_{-\lambda/4}^{\lambda/4} Y(0;z)\,dz}{73 + Z}, \tag{40}$$

where Z is the impedance of the load and $Y(0;z)$ is the transfer admittance between $z = 0$ and $z = z$. Since $Y(0;z)$ is the current at $z = z$ per unit voltage at $z = 0$,

$$Y(0;z) = \frac{1}{73} \cos \beta z. \tag{41}$$

Therefore,

$$I = \frac{\lambda E_0}{\pi(73 + Z)} \cdot \tag{42}$$

For maximum reception, $Z = 73$, and

$$I = \frac{\lambda E_0}{146\pi} \cdot \tag{43}$$

The power absorbed by the load is

$$P = \tfrac{1}{2}73I^2 = \frac{E_0{}^2\lambda^2}{584\,\pi^2}\cdot \tag{44}$$

These results may also be obtained from the effective area (equation 20).

11.11 Bent quarter-wave antennas and bent, folded quarter-wave antennas

A bent quarter-wave antenna (Fig. 11.8*a*) may be useful when for some reason the vertical length of the antenna must be limited. For instance, because of bridges and tunnels, the vertical length of an antenna mounted on top of a railway car must not exceed 14 or 15 in. The frequencies allocated to railroads range between 35 and 43 Mc/sec for open-country

FIG. 11.8 (*a*) A bent quarter-wave antenna; (*b*) a bent folded quarter-wave dipole.

operation and between 150 and 160 Mc/sec for urban operation; thus, even at the highest frequency the wavelength is $\lambda = 73.8$ in. Hence, in spite of the relatively high frequencies of operation, the vertical portions of railroad antennas must be comparable to or less than $\lambda/8$. A horizontal section may be provided to increase the effective length of the vertical section and, hence, the power associated with the vertically polarized radiation. This effective length is maximum when the sum of the vertical and horizontal sections is nearly $\lambda/4$. Simultaneously, the reactive component of the input impedance is greatly reduced.

Another very important consideration is that the bandwidth of the antenna is broadened by the top loading, so that the same antenna can be employed for both transmission and reception. A short vertical antenna is substantially a lossy capacitor, with the loss supplied by radiation.* The bandwidth of such an antenna is

$$\Delta = \omega C_a h R, \tag{45}$$

where R is the radiation resistance, C_a is the average capacitance per unit length, and h is the length. The capacitance may be increased by

* Unless the antenna is so short that the radiation resistance becomes comparable to or less than the ohmic resistance.

using rods of larger diameters, and also by providing top loading when $C_a h$ in equation 45 is replaced by $(C_a h + C_{\text{top}})$; simultaneously R is increased.

When the antenna is made self-resonant, the simple formula 45 is no longer correct. In the case of the bent quarter-wave antenna, the condition for natural oscillations is

$$R_1 + K_1 \coth pl\sqrt{LC} = 0, \qquad p = \xi + j\omega, \tag{46}$$

where l is the total length, K_1 is the characteristic impedance (nearly constant), and R_1 is the radiation resistance. Since R_1 is small compared with K_1, $pl\sqrt{LC}$ is nearly equal to $j\pi/2$, and we assume

$$pl\sqrt{LC} = \delta + \tfrac{1}{2}j\pi. \tag{47}$$

Substituting in equation 46, we have

$$\begin{gathered}\coth(\delta + \tfrac{1}{2}j\pi) = \tanh \delta = -\frac{R_1}{K_1}, \\ \delta \simeq -\frac{R_1}{K_1}, \qquad p = \xi + j\omega = \frac{-(R_1/K_1) + (j\pi/2)}{l\sqrt{LC}}.\end{gathered} \tag{48}$$

Hence, for the bandwidth,

$$\Delta_1 = \frac{2|\xi|}{\omega} = \frac{4R_1}{\pi K_1}. \tag{49}$$

For comparison purposes we shall transform equation 45 into a form analogous to equation 49. Thus, if K_a is the average characteristic impedance of the short vertical antenna,

$$\omega C_a = \frac{\beta}{K_a} = \frac{2\pi}{\lambda K_a}. \tag{50}$$

Therefore,

$$\Delta = \frac{2\pi h R}{\lambda K_a}. \tag{51}$$

The radiation resistance R_1 of the bent quarter-wave antenna is larger than R, since the radiation resistance of the vertical portion is $4R$, and there will be a contribution from the horizontal section. The characteristic impedances are of comparable magnitudes. Hence, the increased value of the resistance in equation 49 and the small value of h/λ in equation 51 will make Δ_1 substantially larger than Δ.

Safety considerations require that the antenna be grounded. To meet this requirement W. C. Babcock* designed a bent, folded quarter-

* Mobile radio antennas for railroads, *Bell Labs. Record*, **27**, May 1949, pp. 172–175.

wave antenna (Fig. 11.8*b*). Because of the current reversal, the currents in the vertical parts are in phase, and their fields add. This antenna possesses another desirable feature. For the proportions between the vertical and horizontal parts that seemed to be best for various reasons, the radiation resistance of the bent quarter-wave antenna was only 12 ohms. This impedance is too low for an ordinary coaxial feeder. As we have already seen, the folding magnifies the impedance fourfold and thus raises it to 48 ohms. This impedance provides a better match to the coaxial feed line.

11.12 Full-wave antennas in free space and the corresponding half-wave vertical antennas above ground

The terms " full-wave antenna " and " full-wave dipole " are applied loosely to antennas whose length is approximately equal to one wavelength (Fig. 11.9*a*). A similar meaning is assigned to the terms " half-

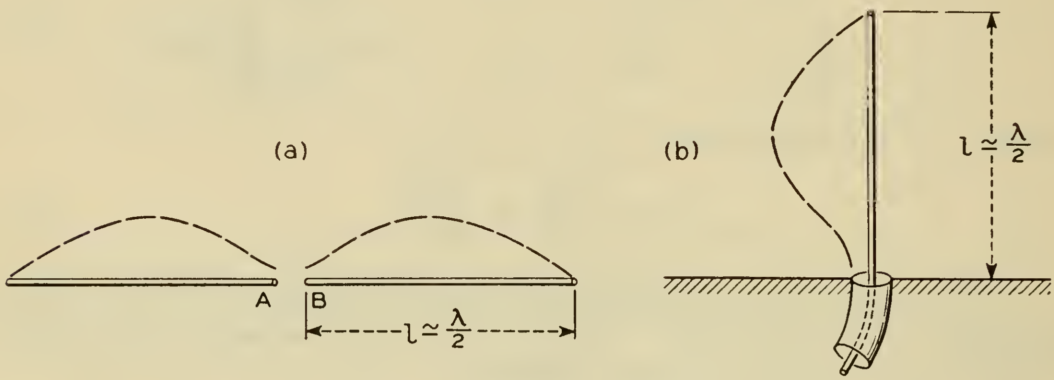


FIG. 11.9 (*a*) A *full-wave* antenna; (*b*) a *half-wave* vertical antenna.

wave vertical antenna " and " half-wave vertical unipole " (Fig. 11.9*b*). Their actual length is usually so adjusted that the antennas are self-resonant at the center of the operating frequency band. The reasons for this usage are the same as those given in Section 11.1.

Let the length of each arm of a full-wave antenna be

$$l = \tfrac{1}{2}\lambda - \delta; \tag{52}$$

then the first-order asymptotic ($K_a \to \infty$) approximation given by the mode theory of antennas (Chapter 13) for circular cylinders is

$$\frac{100\delta}{\lambda/2} = \frac{4000}{K_a + 146} + \frac{10aK_a}{3\pi\lambda}, \tag{53}$$

where K_a is given by equation 3. By the method of " balayage " in

Chapter 13, the corresponding expression is found to be

$$\frac{100\delta}{\lambda/2} = \frac{4000}{2Z_0} + \frac{10a(2Z_0)}{3\pi\lambda}, \tag{54}$$

where Z_0 is given by equation 8–91,

$$2Z_0 = 120\left(\log\frac{\lambda}{2\pi a} + 0.116 + \text{Ci } 2\pi\right) = 120\left(\log\frac{\lambda}{2a} - 1.05\right)\cdot \tag{55}$$

It will be noted that, since $2l \simeq \lambda$,

$$K_a \simeq 120\left(\log\frac{\lambda}{2a} - 0.31\right)\cdot \tag{56}$$

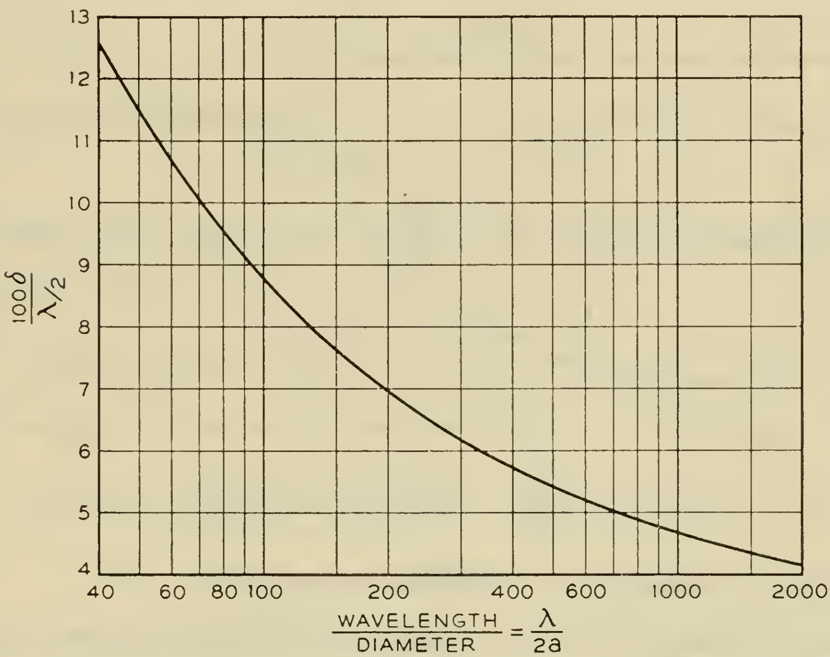


FIG. 11.10 The difference in per cent between $\lambda/2$ and the length of one arm of a self-resonant full-wave antenna.

Thus, the agreement between these two first-order approximations is not as good as in the case of half-wave antennas. Furthermore, if we obtain the antiresonant length graphically from the reactance curves in the vicinity of $l = \lambda/2$, we find that, in the range of practical values of K_a, the shortening is substantially larger than that given by equation 53. This may be seen if we compare Fig. 11.10 obtained from equation 53 with Fig. 13.34 based on the reactance curves given by the mode theory. Thus, there is no simple formula for the antiresonant length.

11.13 Radiation patterns of full-wave antennas

Neglecting δ in equation 11, we obtain the radiation intensity of a full-wave antenna,

$$\Phi = \frac{60I_0^2 \cos^4(\frac{1}{2}\pi \cos \theta)}{\pi \sin^2 \theta}. \tag{57}$$

Comparing this with the radiation intensity (equation 14) of a half-wave antenna, we find

$$\Phi_{\lambda/2} \propto \Phi_{\lambda/4}{}^2 \sin^2 \theta. \tag{58}$$

The squaring and the multiplication by $\sin^2 \theta$ reduce the high angle radiation. Figure 11.11 shows the square root of the relative radiation intensity.

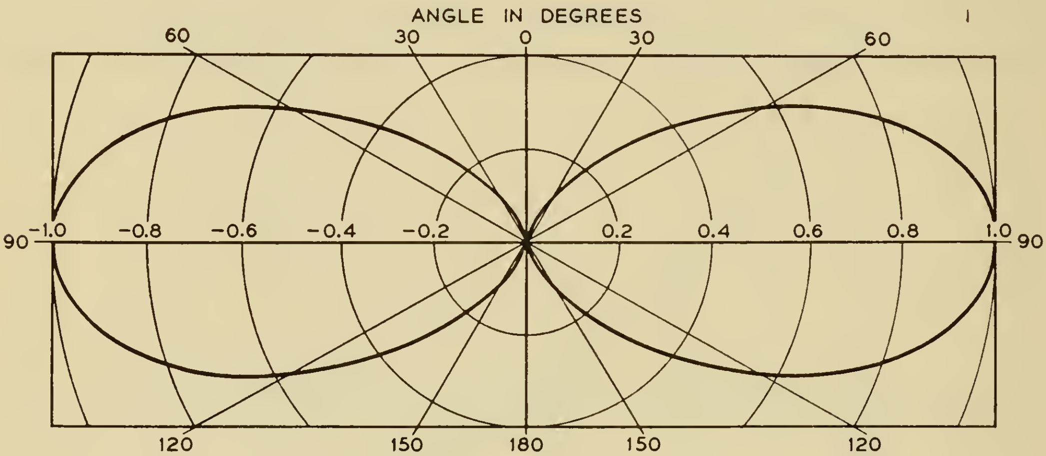


FIG. 11.11 The radiation pattern $\sqrt{\Phi}$ of a full-wave antenna.

11.14 Power radiated by a full-wave antenna

Integrating equation 57 over a unit sphere, we have the power radiated by a full-wave antenna,

$$P = \int_0^{2\pi}\int_0^{\pi} \Phi \sin \theta \, d\theta \, d\varphi = 120I_0^2 \int_0^{\pi} \frac{\cos^4(\frac{1}{2}\pi \cos \theta)}{\sin \theta} \, d\theta$$

$$= 240I_0^2 \int_0^{\frac{1}{2}\pi} \frac{\cos^4(\frac{1}{2}\pi \cos \theta)}{\sin \theta} \, d\theta. \tag{59}$$

Letting $t = \cos \theta$ and expanding $1/(1 - t^2)$ in partial fractions,

$$P = 240I_0^2 \int_0^1 \frac{\cos^4(\frac{1}{2}\pi t)}{1 - t^2} \, dt$$

$$= 120I_0^2 \left[\int_0^1 \frac{\cos^4(\frac{1}{2}\pi t)}{1 + t} \, dt + \int_0^1 \frac{\cos^4(\frac{1}{2}\pi t)}{1 - t} \, dt\right]. \tag{60}$$

Setting $1 + t = u$ in the first integral and $1 - t = u$ in the second, we have

$$P = 120 I_0^2 \int_0^2 \frac{\sin^4(\frac{1}{2}\pi u)}{u}\, du = 120 I_0^2 \int_0^\pi \frac{\sin^4 t}{t}\, dt. \tag{61}$$

Since

$$\begin{aligned}\sin^4 t &= \tfrac{1}{8}(\cos 4t - 4\cos 2t + 3)\\ &= \tfrac{1}{8}[4(1 - \cos 2t) - (1 - \cos 4t)],\end{aligned}$$

we obtain

$$\begin{aligned}P &= 60 I_0^2 \int_0^\pi \frac{1 - \cos 2t}{t}\, dt - 15 I_0^2 \int_0^\pi \frac{1 - \cos 4t}{t}\, dt\\ &= (60 \operatorname{Cin} 2\pi - 15 \operatorname{Cin} 4\pi) I_0^2.\end{aligned} \tag{62}$$

The *radiation resistance R_a with reference to the current antinode* is so defined that

$$P = \tfrac{1}{2} R_a I_0^2. \tag{63}$$

Hence, for a full-wave antenna,

$$R_a = 120 \operatorname{Cin} 2\pi - 30 \operatorname{Cin} 4\pi = 199.1. \tag{64}$$

11.15 Directivity and effective area of a full-wave antenna

From equation 57, we have

$$\Phi_{\max} = \frac{60 I_0^2}{\pi}; \tag{65}$$

hence,

$$\begin{aligned}g &= \frac{4\pi\Phi_{\max}}{P} = \frac{8\pi\Phi_{\max}}{R_a I_0^2} = \frac{480}{199.1} = 2.41,\\ G &= 10 \log_{10} g = 3.82 \text{ db}.\end{aligned} \tag{66}$$

The corresponding effective area is

$$A = \frac{g}{4\pi}\lambda^2 = 0.192\lambda^2. \tag{67}$$

11.16 Input impedance of a full-wave antenna

To obtain the input impedance of a full-wave antenna, we recall that such an antenna is essentially a parallel resonant circuit (Section 9.5). Such a circuit may be represented by either of the two forms shown in Fig. 11.12. These forms are equivalent in the vicinity of resonance. We note that the input current is small and the input voltage is large. The parallel conductance in Fig. 11.12*a* is the input conductance, and it is small. The equivalent resistance in Fig. 11.12*b* in series with the inductance is small. The only features of these circuits that we need for

the present purposes are the following. The radiated power may be expressed in terms of either the input voltage or the maximum current,

$$P = \tfrac{1}{2}G_iV_i^2 = \tfrac{1}{2}R_aI_0^2. \tag{68}$$

At resonance the maximum stored magnetic energy must equal the maximum stored electric energy,

$$\mathcal{E}_m = \mathcal{E}_e. \tag{69}$$

The former may be calculated in terms of I_0 and the latter in terms of V_i; hence, from equations 68 and 69 we can obtain a relation between

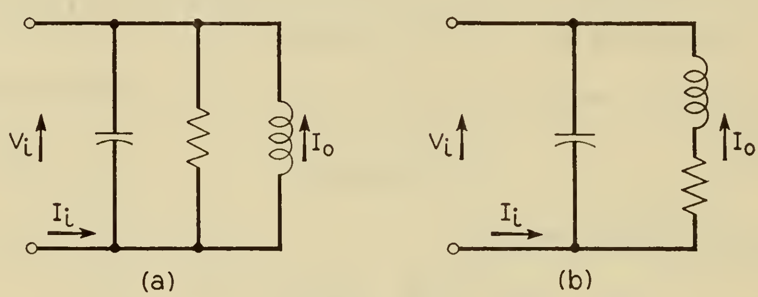


FIG. 11.12 Antiresonant circuits representing a full-wave antenna.

G_i and R_a. Since the latter has been determined, we have also determined the input conductance.

The maximum stored magnetic energy can be obtained from

$$\mathcal{E}_m = \tfrac{1}{2}L\int_{-\lambda/2}^{\lambda/2}[I(z)]^2\,dz, \tag{70}$$

where L is the inductance per unit length of the wire and is given by equation 8–91. Since

$$I(z) = I_0 \sin\beta|z|\,, \tag{71}$$

we find

$$\mathcal{E}_m = \tfrac{1}{4}L\lambda I_0^2. \tag{72}$$

Similarly, the maximum stored electric energy is

$$\mathcal{E}_e = \tfrac{1}{2}C\int_{-\lambda/2}^{\lambda/2}[V(z)]^2\,dz, \tag{73}$$

where $V(z)$ is the potential and C the capacitance per unit length. Since

$$\begin{aligned} V(z) &= \tfrac{1}{2}V_i\cos\beta z, && z > 0; \\ &= -\tfrac{1}{2}V_i\cos\beta z, && z < 0; \end{aligned} \tag{74}$$

we find

$$\mathcal{E}_e = \tfrac{1}{16}C\lambda V_i^2. \tag{75}$$

Thus, the electric and magnetic energies will be equal if

$$\frac{V_i}{I_0} = 2\sqrt{\frac{L}{C}} = 2Z_0. \tag{76}$$

From equation 68, we now have

$$G_i = \frac{R_a}{(2Z_0)^2}, \qquad R_i = \frac{(2Z_0)^2}{R_a} = \frac{(2Z_0)^2}{199.1}. \tag{77}$$

Since we have used the asymptotic forms of $I(z)$, $V(z)$, L, and C, equation 77 is an approximation which improves as $2Z_0$ increases. Likewise, we should not use it if there is a substantial capacitance near the terminals of the antenna (see Section 13.23).

From the mode theory of antennas we have also, for large K_a,

$$R_i = \frac{K_a(K_a - 146)}{199.1}. \tag{78}$$

In the present case,

$$2Z_0 = K_a - 89; \tag{79}$$

hence equation 77 becomes

$$R_i = \frac{(K_a - 89)^2}{199.1} = \frac{K_a^2 - 178K_a + 89^2}{199.1} \simeq \frac{K_a(K_a - 178)}{199.1}. \tag{80}$$

If we use $\hat{K} = 2Z_0(0)$ as given by equation 8–75 instead of the average value $2Z_0$ of $Z(z)$, we obtain $K_a - 155$ as the last factor in this equation. This is in better agreement with equation 78. The curves for R_i are shown in Figs. 13.25 and 13.26.

The impedance of an end-fed, vertical, half-wave antenna over a perfect ground is one half of the impedance of the full-wave antenna in free space (Section 4.18). This impedance is not appreciably affected by ground losses (Section 11.21), since the ground currents near the base of the antenna are small while the power absorbed by the ground at larger distances from the antenna is taken from the radiated power.

11.17 Effect of the gap on the input impedance of a full-wave antenna

In obtaining equation 77, we assumed that, although the capacitive proximity effect between the arms of the antenna is negligible, the distance between the terminals is very small. Both assumptions are satisfied when the radius of the antenna is small and the length s of the gap between the terminals is small but not too small compared with the radius. Suppose now that we keep the radius fixed and begin to separate the terminals. What is the effect on the antiresonant impedance?

To calculate the more general expression for the impedance, we shall first obtain the power radiated by two half-wave antennas whose ends are distance s apart and then use the method of the preceding section to evaluate the input impedance.

The radiation intensity of two half-wave antennas is the product of the radiation intensity of one such antenna and the space factor. Thus,

$$\begin{aligned}\Phi &= \frac{15I_0^2\cos^2(\frac{1}{2}\pi\cos\theta)}{\pi\sin^2\theta}\left\{2\cos\left[\beta\left(\frac{\lambda}{4}+\frac{s}{2}\right)\cos\theta\right]\right\}^2\\ &= \frac{30I_0^2\cos^2(\frac{1}{2}\pi\cos\theta)}{\pi\sin^2\theta}\{1+\cos[(\pi+\beta s)\cos\theta]\}\,. \qquad (81)\end{aligned}$$

The radiation intensity is seen to consist of two terms, one of which is independent of s. This term represents the radiation from two half-wave antennas in free space. The other term is due to the interaction between the antennas. Expressing the radiated power in terms of the maximum amplitude of the current and the corresponding radiation resistances, we have*

$$\begin{aligned}P &= \tfrac{1}{2}R_{11}{}^a\,|I_1|^2 + R_{12}{}^a\,|I_1I_2|\cos\vartheta + \tfrac{1}{2}R_{22}{}^a\,|I_2|^2\\ &= (R_{11}{}^a + R_{12}{}^a)I_0^2, \qquad (82)\end{aligned}$$

since in the present case $R_{22}{}^a = R_{11}{}^a$ and $I_2 = I_1 = I_0$. Consequently, the radiation resistance of the half-wave antennas with reference to the current antinode is

$$R^a = 2(R_{11}{}^a + R_{12}{}^a), \qquad (83)$$

where

$$R_{12}{}^a = \frac{30}{\pi}\int_0^\pi\int_0^{2\pi}\frac{\cos^2(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\cos[(\pi+\beta s)\cos\theta]\sin\theta\,d\theta\,d\varphi, \qquad (84)$$

while $R_{11}{}^a = 73.13$ ohms. Integrating, we find

$$\begin{aligned}R_{12}{}^a = \;&15\left[\log\frac{(\pi+\beta s)^2}{\beta s(2\pi+\beta s)} + \mathrm{Ci}\,2\beta s + \mathrm{Ci}\,(4\pi+2\beta s) - 2\,\mathrm{Ci}\,(2\pi+2\beta s)\right]\cos\beta s +\\ &15[\mathrm{Si}\,2\beta s + \mathrm{Si}\,(4\pi+2\beta s) - 2\,\mathrm{Si}\,(2\pi+2\beta s)]\sin\beta s. \qquad (85)\end{aligned}$$

If $\beta s \ll 1$, then,

$$\begin{aligned}R_{12}{}^a &\simeq 15(\mathrm{Ci}\,4\pi - 2\,\mathrm{Ci}\,2\pi + C + \log\pi) + 15(\mathrm{Si}\,4\pi - 2\,\mathrm{Si}\,2\pi)\beta s\\ &= 26.4 - 127\frac{s}{\lambda}\,; \qquad (86)\end{aligned}$$

* For convenience, we have replaced the subscript a by a corresponding superscript.

therefore,

$$R^a = 199.1 - 127\frac{s}{\lambda}, \tag{87}$$

and the input impedance is

$$R_i = \frac{(2Z_0)^2}{199.1 - 127\dfrac{s}{\lambda}}. \tag{88}$$

The characteristic impedance $2Z_0$ does not change much with s. Hence, if $s = \lambda/20$, the input impedance is higher by about 3 per cent, compared to the impedance when s is negligible.

The mutual radiation resistance between the feeder line and the antenna has been neglected in the above calculations. Further analysis would show that the interaction is primarily reactive and that our assumption is justifiable. When the antenna impedance is measured by the standing-wave method, the radiation from the feeder will be included, together with the radiation from the antenna. This radiation is proportional to $(s/\lambda)^2$ and, hence, is a small quantity of the second order. It tends to compensate, however, for the decrease in R^a and to lower somewhat the input resistance.

11.18 Current distribution in a full-wave antenna

The foregoing conclusions have been based on the asymptotic current form which is given by equation 71 for a full-wave antenna. The input impedance could not be obtained directly, since the input current vanished to this order of approximation and the second term was not available. At resonance we were able to find the input resistance by other means, and from the principle of conservation of energy we obtain

$$R_i|I_i|^2 = R_a|I_0|^2, \tag{89}$$

where I_0 is the current at the antinode. Therefore,

$$\frac{|I_i|}{|I_0|} = \left(\frac{R_a}{R_i}\right)^{1/2} = \frac{R_a}{2Z_0} = \frac{199.1}{2Z_0}. \tag{90}$$

In Section 8.22 we obtained a second approximation to the antenna current which is given by equation 8–120 for the full-wave antenna:

$$I(z) = I_0 \sin\beta|z| + jkI_0(1 + \cos\beta z). \tag{91}$$

Since

$$I_i = I(0) = 2jkI_0,$$

we have, from equation 90,

$$2k = \frac{199.1}{2Z_0}.$$

Hence,

$$I(z) = I_0 \sin \beta|z| + j\,\frac{199.1}{2Z_0}\,\frac{1 + \cos \beta z}{2}\,. \tag{92}$$

Figure 11.13 shows the first term (dashed line), the quadrature term (dotted line), and the magnitude (the solid line) for the case $2Z_0 = 800$.

The "blunted" form of the current at the ends is due to the capacitive effect associated with the ends. To include this effect, we should

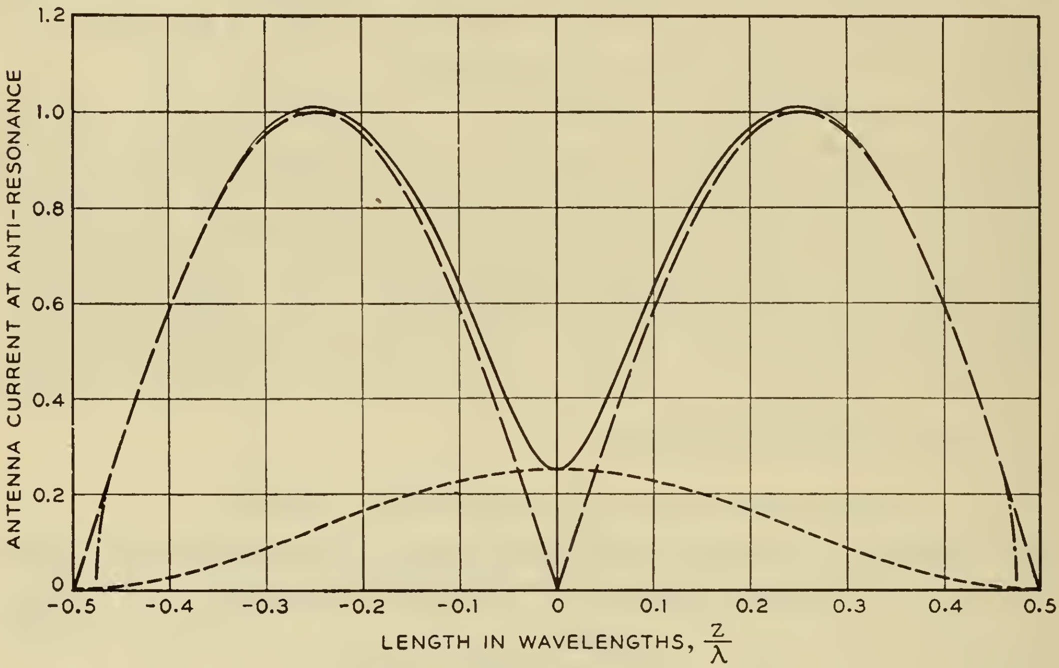


FIG. 11.13 Current distribution in full-wave self-resonant antennas: the dotted line represents the current in phase with the impressed voltage, the dashed line the quadrature current, and the solid line the magnitude.

replace the actual l of each antenna arm by its effective length $l + \delta$. At resonance $l + \delta = \lambda/2$, and we obtain equation 92, except that z does not extend to $\lambda/2$ but only to $(\lambda/2) - \delta$. The effect of δ on R_a and, hence, on R_i is negligible as long as δ is small.

The first term in equation 92 represents the main antenna current in the course of resonant oscillations. The power lost by radiation has to be supplied to the antenna, or else the oscillations will die down. The second term in equation 92 is the current required to supply this power. The old timers in radio call it the "feed current." This current is in quadrature with the main current, and it represents the principal difference between the actual antenna current and the "sinusoidal approximation" to it. The radiation from the feed current is proportional to $|I_i|^2$. As we have seen in Section 5.20, there is no mutual radiation between current elements operating 90° out of phase; hence, there is no

radiation arising out of interaction between the fields associated with the main current and the feed current. Therefore, the radiation from the feed current bears the same ratio to the radiation from the main current as the square of the input current to the square of the maximum current. This must be remembered when the radiated power is calculated from *measured* current distributions; both the amplitude and the *phase* of the antenna current should be measured. Otherwise, the error will be substantial. In fact, even though the measured *amplitude distribution* may show a serious departure from the simple sinusoidal form, the effect of this departure on the radiated power is relatively small.

A full-wave antenna fed at the center should not be confused with a full-wave antenna fed a quarter wavelength from one of its ends. In the first case the currents in the two halves of the antenna flow in the same direction (Fig. 8.8*b*); in the second case they flow in opposite directions (Fig. 8.8*a*). The radiation resistances of these antennas are quite different because the signs of the mutual radiation resistance between the halves of the antenna are opposite. Since the radiation resistance of the center-fed antenna is 199 ohms and that of each half 73 ohms, the mutual radiation resistance is 199 − 146 = 53 ohms. Hence, the radiation resistance of the full-wave antenna fed a quarter wavelength from one end is 146 − 53 = 93 ohms.

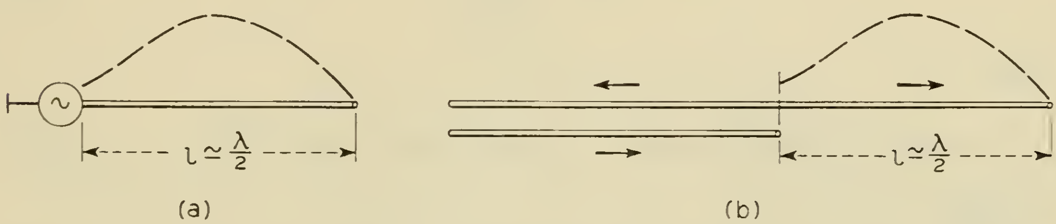


FIG. 11.14 End-fed antennas: (*a*) an antenna fed against the capacitance of the generator; (*b*) an antenna fed by a two-wire line.

11.19 End-fed antennas

Figure 11.14 shows end-fed, half-wave, self-resonant antennas, one fed against the capacitance of the generator and the other by a two-wire line. In this case $R_a = 73$ ohms, and the resistance of the antenna is

$$R_i = \frac{Z_0{}^2}{73}\cdot \tag{93}$$

Since Z_0 is very nearly the same for the half-wave and full-wave antennas, we find

$$\frac{2R_i\text{ (half-wave)}}{R_i\text{ (full-wave)}} = \frac{398}{292} = 1.36. \tag{94}$$

The 36 per cent increase in the input impedance of the "full-wave antenna" when s is large is due to decoupling between the arms of the antenna. The antiresonant impedances of end-fed antennas given by the mode theory are larger. Thus, for one antenna the above ratio is 1.51.

11.20 Quality factors (Q) of antennas

The quality factor Q of a resonant physical circuit is defined by equation 9–57,

$$Q = \frac{\omega\mathcal{E}}{P}, \tag{95}$$

where $\mathcal{E}$ is the total stored energy and P is the average dissipated power. The bandwidth is the reciprocal of Q.

To obtain the Q of the half-wave antenna, we calculate the maximum stored magnetic energy (see equation 70),

$$\mathcal{E}_m = \tfrac{1}{2}L\int_{-\lambda/4}^{\lambda/4} I_0^2 \cos^2 \beta z \, dz = \tfrac{1}{8}\lambda L I_0^2. \tag{96}$$

When the stored magnetic energy is maximum, there is no electric energy, and $\mathcal{E}_m$ represents the total stored energy. Since $P = 36.6 I_0^2$,

$$Q = \frac{\omega\lambda L}{292.5}. \tag{97}$$

Since

$$\left(\frac{L}{C}\right)^{1/2} = Z_0, \qquad \omega(LC)^{1/2} = \beta = \frac{2\pi}{\lambda}, \tag{98}$$

we have

$$\omega L = \frac{2\pi Z_0}{\lambda}; \tag{99}$$

hence,

$$Q = \frac{2\pi Z_0}{292.5} = \frac{2Z_0}{93} = \frac{Z_0}{46.5}. \tag{100}$$

For the full-wave antenna we use equation 72 and $P = \tfrac{1}{2}\cdot 199.1 I_0^2$; thus,

$$Q = \frac{\pi Z_0}{199.1} = \frac{Z_0}{63.5} = \frac{2Z_0}{127}. \tag{101}$$

To the same order of approximation $2Z_0$ in the foregoing formulas may be replaced by $\hat{K}$.

According to the mode theory (Section 13.18), at resonance,

$$Q = \frac{K_a - 6}{93} - \frac{aK_a^2}{4380\lambda}; \tag{102}$$

and, at antiresonance,

$$Q = \frac{K_a + 106}{127} - \frac{aK_a^2}{12{,}000\lambda}. \tag{103}$$

11.21 Influence of ground on antenna impedance

In Section 4.18 we have found that the input impedance of a grounded vertical antenna (Fig. 4.29) over a perfect ground is exactly half the input impedance in free space of the antenna system formed by the given

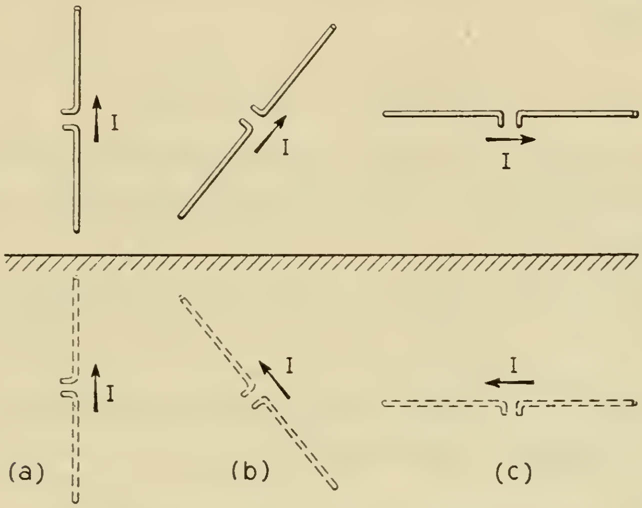


FIG. 11.15 Antennas and their images.

antenna and its image. In the case of an ungrounded antenna (Fig. 11.15) the *antenna impedance equals the sum of its free-space impedance* Z_{11} *and the mutual impedance* Z_{12} *between the antenna and its image,*

$$Z = Z_{11} + Z_{12}. \tag{104}$$

This follows immediately from equations 9–74, since in the image antenna $I_2 = I_1 = I$ (Fig. 11.15).

When this formula is used, it is essential to be careful about the sign of the mutual impedance Z_{12}, for it depends on the relative directions of the currents in the free-space antennas assumed in the calculation of Z_{12}. Equation 104 may be written as

$$Z = Z_{11} - Z_{12}', \tag{105}$$

if, in calculating the mutual impedance Z_{12}, the currents in the lower set of antennas are reversed. *In obtaining the mutual impedance of parallel antennas, it is usual to assume that the currents flow in the same direction in*

both antennas; then equation 104 *must be used for vertical antennas and equation* 105 *for horizontal antennas.*

The mutual impedance may be obtained by successive approximations, of which the first is usually sufficient. To obtain this approximation, one antenna is considered as a receiving antenna in the field of the other, and the current distribution in the first antenna is assumed to be unaffected by the presence of the second antenna.

The admittance of the antenna is

$$Y = Y_{11} + Y_{12}, \tag{106}$$

where Y_{12} is the mutual admittance between the given antenna and its image. Since the antenna and its image form a symmetric structure, $Y_{11} = Y_{22}$, and

$$Y_{11} + Y_{12} = \frac{1}{Z_{11} + Z_{12}}\cdot \tag{107}$$

This is not unexpected since the input admittance is always the reciprocal of the input impedance. In terms of fields, however, this equation is far from obvious.

The quantities Z_{12} and Y_{12} are, of course, also the mutual impedance and admittance between the antenna and the ground. If these quantities are properly determined, equations 104 and 106 will apply to imperfect ground. The analytical problem involved in the calculations of Z_{12} and Y_{12} is greatly complicated, however, by the finite conductivity of the ground.

A perfect ground affects only the radiation properties of the antenna; but an imperfect ground absorbs power as well. Power absorbed at fairly large distances from the antenna has already been radiated from the antenna, and has no effect on the antenna impedance. On the other hand, the power absorbed in the region of the reactive field of the antenna must appear as an increase in the input resistance. There is no clear line of demarcation between the two regions.

If the antenna is grounded and if the base current is large, the ground loss will also be large. For this reason half-wave broadcasting towers are more efficient than quarter-wave ones, unless a good ground system is provided.

The effect of a plane homogeneous earth on the antenna impedance may be expressed easily enough in terms of Sommerfeld's integrals;* but the evaluation of these integrals is not simple.† A much simpler method depends on the equivalence principle.‡ The field of the ground currents

* *Electromagnetic Waves*, pp. 431–434.

† W. L. Barrow, On the impedance of a vertical half-wave antenna above an earth of finite conductivity, *IRE Proc.*, 23, February 1935, pp. 150–167.

‡ *Electromagnetic Waves*, p. 158; see also Section 16.3.

may be evaluated from the tangential components of E and H over the surface of the ground; from this field we may obtain the induced voltage across the terminals of the antenna, and, therefore, the mutual impedance. Since we are interested in the change in the impedance caused by finite conductivity, we may subtract from the actual magnetic intensity that part of it which corresponds to a perfectly conducting earth. The field over the surface of the earth may be obtained approximately from optical reflection formulas. Even this method involves laborious calculations. And the problem becomes still more complex when the earth is not uniform — either for natural reasons or because of artificial ground systems.

Experimental data* indicate that the effect of finite conductivity of the earth on the impedance of a half-wave vertical antenna, the center of which is one-quarter wavelength or more above the surface, is negligible. The effect on the impedance of a horizontal half-wave antenna is negligible only as long as the height of the antenna is greater than one fifth of the wavelength. This effect begins to increase very rapidly as the height falls below 0.2λ; when the height is 0.1λ, the actual impedance is more than twice as large as it would have been over a perfectly conducting earth. For smaller heights the impedance ratio increases still more rapidly.

In 1933, W. M. Sharpless measured the power gain of a half-wave vertical antenna (just above the ground) with reference to a quarter-wave vertical antenna at 18.30 Mc/sec. He found this gain to be 4 db. There would be some gain even over a perfectly conducting ground, because of the greater directivity of the half-wave antenna; but this gain would be only 1.67 db. The difference of 2.33 db is due to smaller ground currents in the case of the half-wave antenna. If we assume that the ground loss is negligible for the half-wave antenna, as indicated by these experiments, we can evaluate the ground resistance of the quarter-wave antenna from equation 6–8. Thus, the ratio of the directivity gain g_d to the power gain g_p is

$$\frac{g_d}{g_p} = \frac{R_{GR} + R_{\text{rad}}}{R_{\text{rad}}} = \frac{R_{GR}}{R_{\text{rad}}} + 1 = 10^{0.233},$$

or

$$R_{GR} = 0.71 R_{\text{rad}},$$

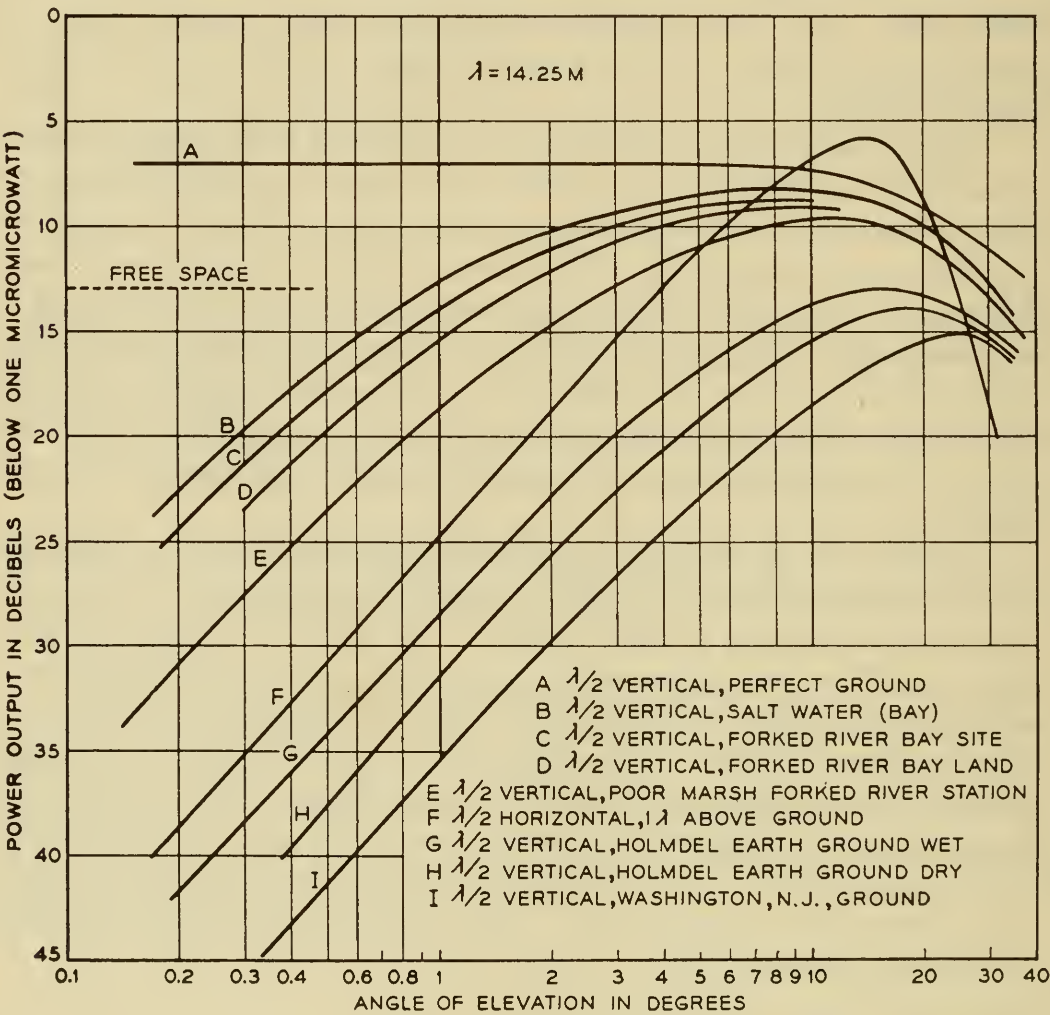
where R_{rad} is the impedance of the quarter-wave antenna over a perfect ground.

* H. T. Friis, C. B. Feldman, and W. M. Sharpless, The determination of the direction of arrival of short radio waves, *IRE Proc.*, **22**, January 1934, pp. 47–78, Fig. 3.

When two copper screens $10' \times 10'$ were placed on the ground under the quarter-wave antenna, its efficiency was increased by 0.7 db. This gives

$$R_{GR} = 0.455 R_{\text{rad}}.$$

Figure 11.16 illustrates the effect of different ground conditions on the received power for different angles of arrival, when the electric inten-



Feldman, Courtesy IRE

FIG. 11.16 Effect of different ground conditions on the received power.

sity of the incident wave is one microvolt per meter (effective) and $\lambda = 14.25$ meters. Ground constants were measured and the curves calculated by Feldman.* A few scattered experimental observations were consistent with these curves.

* See his paper cited in Section 7.3.

REFERENCE

R. A. Smith, *Aerials for Metre and Decimetre Wave-Lengths*, University Press, Cambridge, 1949.

PROBLEMS

11.2–1 What is the electric intensity of a half-wave antenna at distant points in its equatorial plane?

Ans.
$$E_z = -E_\theta = -\frac{60jI_0}{d}\cos\beta\delta e^{-j\beta d} \simeq -\frac{60jI_0}{d}e^{-j\beta d},$$

where d is the distance from the antenna.

11.4–1 Express the electric intensity of a half-wave antenna at distant points in its equatorial plane in terms of the radiated power.

Ans.
$$E_z = -\frac{60}{d}\sqrt{\frac{P}{36.6}}\,je^{-j\beta d} = -\frac{\sqrt{98.4P}}{d}\,je^{-j\beta d} \simeq -\frac{10\sqrt{P}}{d}\,je^{-j\beta d}.$$

11.4–2 Express the electric intensity of a current element at distant points in its equatorial plane in terms of the radiated power.

Ans.
$$E_z = -\frac{\sqrt{90P}}{d}\,je^{-j\beta d}.$$

11.10–1 Calculate the maximum power received by a half-wave self-resonant antenna from its effective area.

11.10–2 What is the current distribution in a self-resonant wire (about one-half wavelength) in a uniform field E_0 parallel to the wire?

Ans.
$$I_z = \frac{\lambda E_0}{73\pi}\cos\beta(z+\delta),$$

where δ is a small effective increase in the length of the wire due to the capacitive end effect.

11.10–3 Assume that the wire in the preceding problem is broken in the center and that the length of the gap is small but not too small compared with the diameter. What is the voltage induced across the gap?

Ans.
$$V = \frac{\lambda E_0}{\pi}\cdot$$

11.13–1 Solve Problem 11.2–1 for a full-wave antenna.

Ans.
$$E_z = -\frac{120jI_0}{d}e^{-j\beta d}.$$

11.14–1 Solve Problem 11.4–1 for a full-wave antenna.

Ans.
$$E_z = -\frac{12j\sqrt{P}}{d}e^{-j\beta d}.$$

11.19–1 Explain why the radiation resistance of an end-fed quarter-wave antenna in free space should be somewhat greater than one quarter of the radiation resistance of a half-wave antenna.

11.21–1 Consider a quarter-wave antenna in the center of a perfectly conducting circular disk serving as the ground. Show that at large distances in the ground plane, well beyond the conducting disk, the field intensity is half that which would have existed at the same distance if the ground were infinite in extent.

11.21–2 Consider a thin vertical half-wave antenna just above a "ground" consisting of a horizontal wire, one wavelength long. Assume that the antenna is over the center of the ground. Find the electric intensity on the axis of the antenna below the artificial ground and show that this ground provides good shielding.

Ans.
$$E_z = -\frac{15j\lambda I_0}{r(r+\frac{1}{2}\lambda)}e^{-j\beta r} - \frac{15j\lambda I_0}{rR}e^{-j\beta R},$$

where r is the distance from the center of the ground, $R = (r^2 + \frac{1}{4}\lambda^2)^{1/2}$ is the distance from one of its ends, and I_0 is the maximum amplitude of the antenna current.

12

GENERAL THEORY OF LINEAR ANTENNAS

12.1 General formula for the radiation intensity of a system of current elements

We have seen that, when the elements of a given current distribution are parallel to the z axis, the distant electric field has only one component E_θ, which can easily be obtained by adding directly the corresponding fields of the individual elements. And, of course, if the elements are parallel, we can always choose the z axis in their common direction. When the current elements are not parallel, as in rhombic antennas, the direct addition of fields is complicated, and it is best to take advantage of the expressions for the field derived in Sections 8.5 and 8.6. The dynamic component F of the electric intensity is always parallel to the current element and is given by the simple expression 8–17,

$$F = -j\omega\mu \frac{pe^{-j\beta r}}{4\pi r}, \tag{1}$$

where p is the moment of the element. If the element is at the point (r', θ', φ') in Fig. 12.1, then the distance r from it to a distant point in the direction (θ, φ) is less than the distance r_0 from the origin by the length OQ of the projection of OP; that is,

$$r = r_0 - r' \cos \psi, \tag{2}$$

where

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'). \tag{3}$$

Substituting in equation 1, we have

$$F = -j \frac{\omega\mu}{4\pi} pe^{j\beta r' \cos \psi} \frac{e^{-j\beta r_0}}{r_0}, \tag{4}$$

since $r' \cos \psi$ has a negligible effect on the amplitude.

Let us now define the *radiation vector* N of a current element of moment p at point (r', θ', φ') with respect to $(0, 0, 0)$,

$$N = pe^{j\beta r' \cos \psi}. \tag{5}$$

This equation represents a *rule of translation* of the effective point of radiation. The radiation vector of the element with respect to its actual position is simply its moment p. As far as any distant point is con-

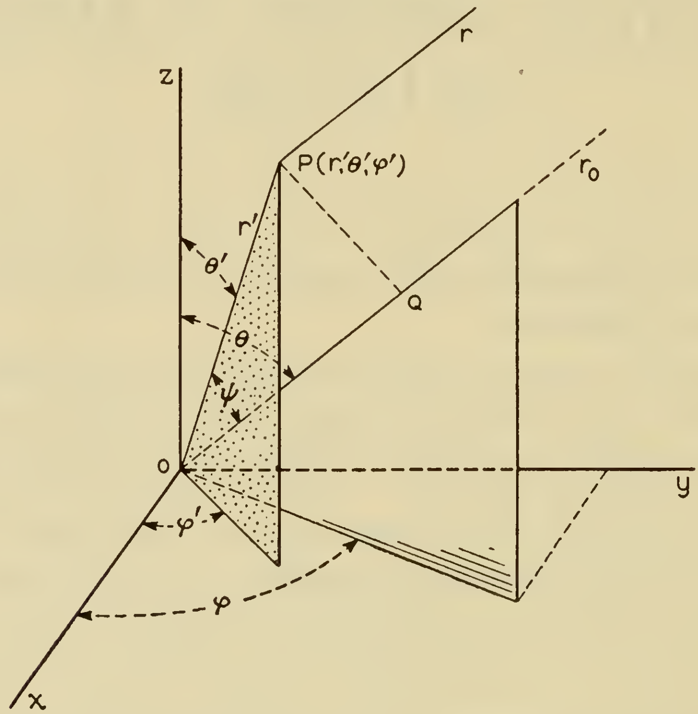


FIG. 12.1 Illustrating a general method for obtaining the distant field of any current distribution.

cerned, we may equally well think of the radiation as coming from any other point, provided we make a proper allowance for the phase difference due to the difference in paths; the exponential factor in equation 5 has this effect. In terms of the radiation vector, we have

$$F = -j \frac{\omega\mu}{4\pi} N \frac{e^{-j\beta r_0}}{r_0}. \tag{6}$$

From this formula for a typical element, we obtain the general equation by summation (or integration in the case of a continuous distribution). The general procedure of calculating the radiation intensity by this method consists of the following steps:

1. Evaluation of the Cartesian components of the radiation vector of a given system of current elements with respect to some fixed point

(which may be taken as the origin of the coordinate system). The equations are:

$$\begin{aligned} N_x &= \sum e^{j\beta r_n' \cos\psi_n} p_{n,x}, \\ N_y &= \sum e^{j\beta r_n' \cos\psi_n} p_{n,y}, \\ N_z &= \sum e^{j\beta r_n' \cos\psi_n} p_{n,z}, \\ \cos\psi_n &= \cos\theta\cos\theta_n' + \sin\theta\sin\theta_n'\cos(\varphi - \varphi_n'). \end{aligned} \tag{7}$$

For a continuous current filament the moment is tangential to the filament and is equal to $I(x, y, z)\,ds$, where ds is the element of length. Hence, the components of the moment are:

$$dp_x = I(r', \theta', \varphi')\,dx, \quad dp_y = I(r', \theta', \varphi')\,dy, \quad dp_z = I(r', \theta', \varphi')\,dz. \tag{8}$$

2. Evaluation of the θ and φ components of the radiation vector,

$$\begin{aligned} N_\theta &= N_x\cos\theta\cos\varphi + N_y\cos\theta\sin\varphi - N_z\sin\theta, \\ N_\varphi &= -N_x\sin\varphi + N_y\cos\varphi. \end{aligned} \tag{9}$$

3. Evaluation of the radiation intensity,

$$\Phi = \frac{15\pi}{\lambda^2}(N_\theta N_\theta^* + N_\varphi N_\varphi^*). \tag{10}$$

This final equation is obtained if we note that, at great distances from the source, the quasistatic θ and φ components of the electric intensity vary inversely as the square of the distance and thus vanish in comparison with the dynamic components; hence, at great distances,

$$E_\theta = F_\theta, \qquad E_\varphi = F_\varphi. \tag{11}$$

Substituting from equation 6, we have

$$\begin{aligned} E_\theta &= -j\frac{\omega\mu}{4\pi}N_\theta\frac{e^{-j\beta r_0}}{r_0} = -j\frac{60\pi}{\lambda}N_\theta\frac{e^{-j\beta r_0}}{r_0}, \\ E_\varphi &= -j\frac{\omega\mu}{4\pi}N_\varphi\frac{e^{-j\beta r_0}}{r_0} = -j\frac{60\pi}{\lambda}N_\varphi\frac{e^{-j\beta r_0}}{r_0}. \end{aligned} \tag{12}$$

Substituting in equation 5–10, we obtain equation 10.

A more general formula may be derived by including radiation from magnetic currents, as we shall see in Chapters 16 and 17. Magnetic currents do not exist in nature; but the calculation of radiation from horns and slots is greatly simplified if we introduce certain distributions of hypothetical electric and magnetic currents over the apertures.

12.2 Formulas for the radiated power

From equations 5–5 and 10 we have a general formula for the radiated power,

$$P = \frac{15\pi}{\lambda^2}\int_0^{2\pi}\int_0^{\pi} (N_\theta N_\theta^* + N_\varphi N_\varphi^*) \sin\theta \, d\theta \, d\varphi. \tag{13}$$

It is sometimes convenient to consider one system of sources as consisting of several parts. Suppose, for instance, that the system is divided into two parts; then,

$$N_\theta = N_{\theta,1} + N_{\theta,2}, \qquad N_\varphi = N_{\varphi,1} + N_{\varphi,2}. \tag{14}$$

Substituting in equation 13, we have

$$P = P_{11} + 2P_{12} + P_{22}, \tag{15}$$

where

$$\begin{aligned} P_{11} &= \frac{15\pi}{\lambda^2}\int_0^{2\pi}\int_0^{\pi} (N_{\theta,1}N_{\theta,1}^* + N_{\varphi,1}N_{\varphi,1}^*) \sin\theta \, d\theta \, d\varphi, \\ 2P_{12} &= \frac{15\pi}{\lambda^2}\int_0^{2\pi}\int_0^{\pi} (N_{\theta,1}N_{\theta,2}^* + N_{\theta,1}^*N_{\theta,2} + N_{\varphi,1}N_{\varphi,2}^* + \\ &\qquad N_{\varphi,1}^*N_{\varphi,2}) \sin\theta \, d\theta \, d\varphi \\ &= \frac{30\pi}{\lambda^2} \operatorname{re}\int_0^{2\pi}\int_0^{\pi} (N_{\theta,1}N_{\theta,2}^* + N_{\varphi,1}N_{\varphi,2}^*) \sin\theta \, d\theta \, d\varphi, \\ P_{22} &= \frac{15\pi}{\lambda^2}\int_0^{2\pi}\int_0^{\pi} (N_{\theta,2}N_{\theta,2}^* + N_{\varphi,2}N_{\varphi,2}^*) \sin\theta \, d\theta \, d\varphi. \end{aligned} \tag{16}$$

The term $2P_{12}$ is the mutual radiated power.

Another general formula expresses the radiated power in terms of the local field. One of its advantages is that it gives the reactive power as well as the power lost by radiation (Section 5.18). Thus, for thin filaments, we have the complex power

$$\Psi = -\tfrac{1}{2}\int E_s(s)I^*(s)\, ds, \tag{17}$$

where $I(s)\, ds$ is the moment of a typical current element and $E_s(s)$ the electric intensity tangential to it.

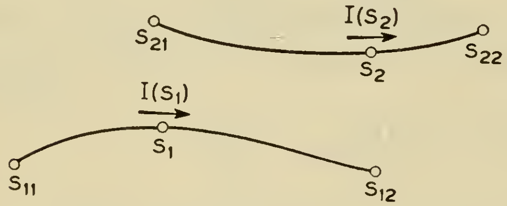


FIG. 12.2 Two thin current filaments.

In the case of two thin filaments (Fig. 12.2), we subdivide E_s into components $E_{s,1}$, $E_{s,2}$ associated with each filament and divide the range of integration accordingly. Thus,

$$\Psi = \Psi_{11} + 2\Psi_{12} + \Psi_{22}, \tag{18}$$

where

$$\Psi_{11} = -\tfrac{1}{2}\int_{s_{11}}^{s_{12}} E_{s,1}(s_1)\, I_1^*(s_1)\, ds_1,$$

$$\Psi_{22} = -\tfrac{1}{2}\int_{s_{21}}^{s_{22}} E_{s,2}(s_2)\, I_2^*(s_2)\, ds_2, \tag{19}$$

$$2\Psi_{12} = -\tfrac{1}{2}\int_{s_{11}}^{s_{12}} E_{s,2}(s_1)\, I_1^*(s_1)\, ds_1 - \tfrac{1}{2}\int_{s_{21}}^{s_{22}} E_{s,1}(s_2)\, I_2^*(s_2)\, ds_2.$$

In these equations s_1 is the distance along the first filament and s_2 that along the second filament; points s_{11} and s_{12} mark the beginning and the end of the first filament and similarly s_{21} and s_{22} mark the ends of the second filament.

There exists a simple relationship between Ψ for a single filament and Ψ_{12} for two typical generators* of the filament,

$$\Psi = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \Psi_{12}\, d\varphi_1\, d\varphi_2. \tag{20}$$

To obtain this formula we subdivide the current $I(s)$ into infinitely thin filaments $I(s)\, d\varphi/2\pi$. The mutual power between two such filaments is $2\Psi_{12}\, d\varphi_1\, d\varphi_2/4\pi^2$. When this is integrated around the original finite filament, this mutual power is counted twice; hence, we must divide the result in half. By symmetry the integral of $\Psi_{12}\, d\varphi_1$ is independent of φ_2; hence, the second integration reduces to multiplication by 2π, and

$$\Psi = \frac{1}{2\pi}\int_0^{2\pi} \Psi_{12}\, d\varphi. \tag{21}$$

Therefore, *the average value of* Ψ_{12} around the filament *gives the required complex power.*

12.3 Input impedance and mutual impedance

Two antennas form a four-terminal transducer. Let $I_{i,1}$, $I_{i,2}$ be the input currents in response to the impressed voltages V_1, V_2. In accordance with equations 9–74, we have

$$V_1 = Z_{11}I_{i,1} + Z_{12}I_{i,2}, \qquad V_2 = Z_{12}I_{i,1} + Z_{22}I_{i,2}. \tag{22}$$

Multiplying the first equation by $\tfrac{1}{2}I_{i,1}^*$, the second by $\tfrac{1}{2}I_{i,2}^*$, and adding, we obtain the complex power

$$\begin{aligned}\Psi &= \tfrac{1}{2}Z_{11}I_{i,1}I_{i,1}^* + \tfrac{1}{2}Z_{12}(I_{i,1}I_{i,2}^* + I_{i,1}^*I_{i,2}) + \tfrac{1}{2}Z_{22}I_{i,2}I_{i,2}^* \\ &\equiv \Psi_{11} + 2\Psi_{12} + \Psi_{22}.\end{aligned} \tag{23}$$

* A *generator of a filament* is a curve which generates the surface of the filament when translated parallel to itself.

Therefore,

$$Z_{11} = \frac{2\Psi_{11}}{I_{i,1}I_{i,1}^*}, \qquad Z_{22} = \frac{2\Psi_{22}}{I_{i,2}I_{i,2}^*},$$
$$Z_{12} = \frac{4\Psi_{12}}{I_{i,1}I_{i,2}^* + I_{i,1}^*I_{i,2}}. \tag{24}$$

Since Z_{12} is independent of either the amplitudes or the phases of the input currents, we may assume that the phases are the same and write

$$Z_{12} = \frac{2\Psi_{12}}{I_{i,1}I_{i,2}}. \tag{25}$$

12.4 Second set of formulas for the mutual impedance and the input impedance

In accordance with equations 22, the mutual impedance Z_{12} is the voltage which must be impressed across the terminals of the second antenna to counteract the voltage induced there by the first antenna when the input current in the first antenna is unity. Let $I_1(0; s_1)$ be the current in the first antenna due to an applied voltage at $s_1 = 0$ and let $E_{s,1}(s_2)$ be the electric intensity produced by this current along the second antenna. By equation 9–100 the voltage induced across the terminals of the second antenna is*

$$\frac{1}{I_{i,2}} \int_{s_{21}}^{s_{22}} E_{s,1}(s_2)\, I_2(0; s_2)\, ds_2, \tag{26}$$

where $I_2(0; s_2)$ is the current in the second antenna due to an applied voltage across its terminals at $s_2 = 0$. Taking the negative and dividing by $I_{i,1}$, we find the mutual impedance,

$$Z_{12} = -\frac{\int_{s_{21}}^{s_{22}} E_{s,1}(s_2)\, I_2(0; s_2)\, ds_2}{I_{i,1}I_{i,2}}. \tag{27}$$

Since $I_{i,1}$ and $I_{i,2}$ are a pair of fixed values, the current distributions $I_1(0; s_1)$ and $I_2(0; s_2)$ are to be calculated on the assumption that the impedances of the generators driving the antennas are infinite; hence, $E_{s,1}(s_2)$ is to be calculated on the same assumption.

Let $T(s_1, s_2)$ be the *free-space transmission factor* between two antenna elements, ds_1 and ds_2, defined as the ratio of the electric intensity at s_2 along the tangent to the element ds_2 to the moment of the

* This important formula was first obtained by P. S. Carter, Circuit relations in radiating systems and applications to antenna problems, *IRE Proc.*, **20**, June 1932, pp. 1004–1041.

current at s_1 flowing through the element ds_1; that is,

$$E_{s,1}(s_2) = T(s_1, s_2)\, I_1(0; s_1)\, ds_1. \tag{28}$$

The T function is obtained from equations 4–82 and is found to be a symmetric function,

$$T(s_2, s_1) = T(s_1, s_2), \tag{29}$$

as it should, in view of the reciprocity theorem. Substituting in equation 27, we have

$$Z_{12} = -\frac{\int_{s_{11}}^{s_{12}} \int_{s_{21}}^{s_{22}} T(s_1, s_2)\, I_1(0; s_1)\, I_2(0; s_2)\, ds_1\, ds_2}{I_{i,1} I_{i,2}}. \tag{30}$$

To obtain the input impedance we subdivide the antenna into infinitely thin filaments of angular density $I_{i,1}\, d\varphi/2\pi$ across the input terminals. The voltage across the terminals of a typical elementary filament is

$$\frac{1}{2\pi} \int_0^{2\pi} Z_{12} I_{i,1}\, d\varphi. \tag{31}$$

This is also the voltage across the terminals of the entire antenna. Dividing by the input current, we find the input impedance,

$$Z_{11} = \frac{1}{2\pi} \int_0^{2\pi} Z_{12}\, d\varphi. \tag{32}$$

12.5 Mutual admittance

In accordance with equations 9–75, the mutual admittance $Y_{12} = Y_{21}$ is the current through the short-circuited terminals of one antenna divided by the voltage impressed on the other by a generator of zero impedance. Suppose that we have a fixed voltage V_1 across the terminals of the first antenna and a zero voltage across the terminals of the second. From the definition and from equation 9–99, we have

$$Y_{12} = \frac{1}{V_1} \int_{s_{21}}^{s_{22}} E_{s,1}(s_2)\, Y_2(0; s_2)\, ds_2, \tag{33}$$

where the transfer admittance $Y_2(0; s_2)$ is the current in response to a unit voltage across the terminals of the second antenna. If $V_1 = 1$, the current producing $E_{s,1}(s_2)$ is the transfer admittance $Y_1(0; s_1)$ along the first antenna; hence,

$$E_{s,1}(s_2) = T(s_1, s_2)\, Y_1(0; s_1)\, ds_1, \tag{34}$$

and equation 33 becomes

$$Y_{12} = \int_{s_{11}}^{s_{12}} \int_{s_{21}}^{s_{22}} T(s_1, s_2)\, Y_1(0; s_1)\, Y_2(0; s_2)\, ds_1\, ds_2. \tag{35}$$

We may also express Y_{12} in terms of current distributions corresponding to arbitrary but fixed impressed voltages V_1 and V_2; thus,

$$Y_{12} = \frac{1}{V_1 V_2} \int_{s_{11}}^{s_{12}} \int_{s_{21}}^{s_{22}} T(s_1, s_2)\, I_1(0; s_1)\, I_2(0; s_2)\, ds_1\, ds_2. \tag{36}$$

Since the terminal conditions involved in equation 30 for Z_{12} and in equation 36 for Y_{12} are different, the functions $I_1(0; s_1)$ and $I_2(0; s_2)$ are different in the two cases. They are approximately the same when the antennas are so far apart that the interaction between them is small. In this case,

$$V_1 \simeq Z_{i,1} I_{i,1}, \qquad V_2 \simeq Z_{i,2} I_{i,2}, \tag{37}$$

where $Z_{i,1}$ and $Z_{i,2}$ are the input impedances of the respective antennas in free space. Then equation 36 becomes

$$Y_{12} \simeq -\frac{Z_{12}}{Z_{i,1} Z_{i,2}}. \tag{38}$$

In view of equations 9–76, this equation neglects Z_{12}^2 in comparison with $Z_{i,1}$, $Z_{i,2}$.

As we shall see in the next chapter the above expressions for Z_{12} and Y_{12} are particularly useful in calculations by successive approximations.

12.6 Local field of a straight current filament

Relatively simple formulas for the local field of a straight current filament were obtained by Pistolkors and Bechman.* Here we shall confine our attention to the electric intensity parallel to the filament; for a more complete discussion the reader is referred elsewhere.† One general method of calculation is summarized in Section 8.5. It consists of integrating the scalar potential of an element of charge over the region occupied by the charge, then integrating the dynamic component of the electric intensity of a current element over the filament, and, finally, differentiating to obtain E and H. In the present case, however, it is more convenient to integrate directly the field produced by the current element.

Thus from equation 8–28, we have the electric potential of the

* See references in Section 5.18.

† *Electromagnetic Waves*, Chapter 9.

element (Fig. 12.3),

$$V = -\frac{I(z')\,dz'}{j\omega\varepsilon}\frac{\partial\psi}{\partial z}, \qquad \psi = \frac{e^{-j\beta r}}{4\pi r}, \qquad r = \sqrt{\rho^2 + (z - z')^2}. \quad (39)$$

Substituting in equation 8–45 and using equation 8–17, we obtain the electric intensity of the element in the direction parallel to the filament,

$$E_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial^2\psi}{\partial z^2} + \beta^2\psi\right) I(z')\,dz'. \quad (40)$$

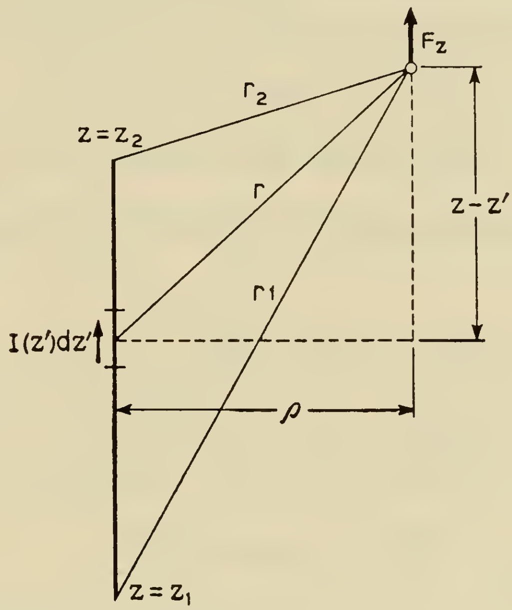


Fig. 12.3 Illustrating the calculation of the field of a straight current filament.

Finally, we obtain the electric intensity of the entire filament by integrating between $z' = z_1$ and $z' = z_2$,

$$E_z = \frac{1}{j\omega\varepsilon}\int_{z_1}^{z_2}\left(\frac{\partial^2\psi}{\partial z^2} + \beta^2\psi\right) I(z')\,dz'. \quad (41)$$

Noting that

$$\frac{\partial\psi}{\partial z} = -\frac{\partial\psi}{\partial z'}, \qquad \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\psi}{\partial z'^2}, \quad (42)$$

we integrate the first term in equation 41 by parts twice. Thus,

$$\int_{z_1}^{z_2}\frac{\partial^2\psi}{\partial z^2} I(z')\,dz' = \int_{z_1}^{z_2} I(z')\frac{\partial^2\psi}{\partial z'^2}\,dz' = \int_{z_1}^{z_2} I(z')\,d\frac{(\partial\psi)}{(\partial z')}$$

$$= I(z')\frac{\partial\psi}{\partial z'}\Big|_{z_1}^{z_2} - \int_{z_1}^{z_2}\frac{d\,I(z')}{dz'}\frac{\partial\psi}{\partial z'}\,dz'$$

$$= \left[I(z')\frac{\partial\psi}{\partial z'} - I'(z')\psi\right]_{z'=z_1}^{z'=z_2} + \int_{z_1}^{z_2}\frac{d^2\,I(z')}{dz'^2}\psi\,dz'. \quad (43)$$

Substituting in equation 41 and using the first equation in the set 42, we have

$$E_z = \frac{j}{\omega\varepsilon}\left[I'(z')\,\psi(z,z') + I(z')\frac{\partial\psi}{\partial z}\right]_{z'=z_1}^{z'=z_2} + \frac{1}{j\omega\varepsilon}\int_{z_1}^{z_2}\left[\frac{d^2 I(z')}{dz'^2} + \beta^2 I(z')\right]\psi\,dz'. \qquad (44)$$

12.7 Local field of a straight sinusoidal current filament

If the current between $z' = z_1$ and $z' = z_2$ is of the form

$$I(z') = A\cos\beta z' + B\sin\beta z', \qquad (45)$$

or

$$I(z') = Ce^{-j\beta z'} + De^{j\beta z'}, \qquad (46)$$

the bracketed expression in the integrand of equation 44 vanishes. Hence,

$$\begin{aligned} E_z &= \frac{j}{\omega\varepsilon}\left[I'(z')\,\psi(z,z') + I(z')\frac{\partial\psi}{\partial z}\right]_{z'=z_1}^{z'=z_2} \\ &= \frac{j}{4\pi\omega\varepsilon}\left[I'(z')\frac{e^{-j\beta r}}{r} + I(z')\frac{\partial}{\partial z}\frac{e^{-j\beta r}}{r}\right]_{z'=z_1}^{z'=z_2} \\ &= \frac{j}{4\pi\omega\varepsilon}\left[I'(z_2)\frac{e^{-j\beta r_2}}{r_2} - I'(z_1)\frac{e^{-j\beta r_1}}{r_1} + I(z_2)\frac{\partial}{\partial z}\frac{e^{-j\beta r_2}}{r_2} - I(z_1)\frac{\partial}{\partial z}\frac{e^{-j\beta r_1}}{r_1}\right], \end{aligned} \qquad (47)$$

where r_1 and r_2 are the distances from the beginning and the end of the filament (Fig. 12.3),

$$r_1 = [\rho^2 + (z - z_1)^2]^{1/2}, \qquad r_2 = [\rho^2 + (z - z_2)^2]^{1/2}. \qquad (48)$$

Two of the terms in equation 47 depend on the currents at the ends of the filament and two on their derivatives. If the filament consists of several sinusoidal segments with either the current or its derivative discontinuous at the junctions, each segment should be treated separately.

12.8 Methods of antenna analysis

In Section 8.2 we concluded that practical approximations for the radiation pattern of a given antenna, and for the radiated power in terms of the maximum amplitude of the antenna current, may be obtained even from a relatively poor approximation to the detailed current distribution. Hence, if the antenna is driven at a point where the amplitude of

the current is maximum or nearly maximum, a rather good approximation can be obtained for the input resistance (Section 8.3). Later in Chapter 8 we concluded that the current distribution in a thin wire excited at a single point is approximately sinusoidal. This fact has been known for more than half a century,* and much of antenna analysis has been based on the sinusoidal approximation to the antenna current. However, in order to obtain the antenna impedance over a wide range of frequencies, a better approximation to the current is needed. There are three methods for solving this problem.

If the current is distributed on the surface of a hollow cylinder, we can subdivide it into filaments of angular density $I(z')/2\pi$ and angular width $d\varphi'$. The electric intensity is then the average value of the longitudinal intensity (equation 41) around the cylinder,

$$E_z = \frac{1}{j\omega\varepsilon}\int_{z_1}^{z_2}\left(\frac{\partial^2\Gamma}{\partial z^2} + \beta^2\Gamma\right) I(z')\,dz', \tag{49}$$

where

$$\Gamma(z, z') = \frac{1}{2\pi}\int_0^{2\pi} \psi(z, z'; \varphi, \varphi')\,d\varphi'. \tag{50}$$

If the cylinder is a perfect conductor and if $E_z^i(z)$ is the impressed field, then,

$$E_z + E_z^i(z) = 0. \tag{51}$$

Substituting in equation 49, we have

$$\frac{1}{j\omega\varepsilon}\int_{z_1}^{z_2}\left(\frac{\partial^2\Gamma}{\partial z^2} + \beta^2\Gamma\right) I(z')\,dz' = -E_z^i(z). \tag{52}$$

The unknown current is under the integral sign. Such equations are called *integral equations*. Equation 52 is a " circuit equation " for the antenna. To bring out the analogy between this equation and Kirchhoff's equations for lumped networks, we note that the integral is the limit of a sum,

$$\sum_{n=1}^{n=N} \frac{1}{j\omega\varepsilon}\left[\frac{\partial^2\,\Gamma(z_n, z)}{\partial z^2} + \beta^2\,\Gamma(z_n, z)\right]\Delta z_n\,I(z_n) = -E_z^i(z), \tag{53}$$

where N is the number of elements in the sum. If $z = z_n$ is the midpoint of a typical element, we have N equations:

$$\sum_{n=1}^{n=N} Z_{mn}\,I(z_n) = -E_z^i(z_m), \qquad m = 1, 2, 3, \cdots N, \tag{54}$$

* H. C. Pocklington, Electrical oscillations in wires, *Cambridge Phil. Soc. Proc.*, **9**, October 25, 1897, pp. 324–332.

where

$$Z_{mn} = \frac{1}{j\omega\varepsilon}\left[\frac{\partial^2 \Gamma(z_n, z_m)}{\partial {z_m}^2} + \beta^2 \Gamma(z_n, z_m)\right]\Delta z_n. \tag{55}$$

Thus equation 52 is "Kirchhoff's equation" for the antenna. There exist other forms of integral equations for antennas, one of which, due to Hallén, is particularly well suited to treatment by successive approximations. Antenna theory based on integral equations may be conveniently referred to as the circuit theory of antennas.

We can combine the method of successive approximations with the general formulas derived in the preceding sections. On the surface of a perfectly conducting transmitting antenna, for example, E_z vanishes everywhere except in a small region occupied by the source of power. If we integrate equation 49 by parts as we did equation 41, we obtain an equation analogous to equation 44 with $\Gamma(z, z')$ in place of $\psi(z, z')$. Hence, if the antenna arms extend from $z = z_1$ to $z = z_2$ and from $z = z_3$ to $z = z_4$, then for each antenna arm we have

$$\int_{z_1}^{z_4}\left[\frac{d^2 I}{dz'^2} + \beta^2 I\right]\Gamma(z, z')\,dz' = \left[I'(z')\,\Gamma(z, z') + I(z')\frac{\partial \Gamma}{\partial z}\right]_{z'=z_1}^{z'=z_2} + \left[I'(z')\,\Gamma(z, z') + I(z')\frac{\partial \Gamma}{\partial z}\right]_{z'=z_3}^{z'=z_4}. \tag{56}$$

As the radius of the antenna approaches zero, the right-hand side approaches a constant limit for every z not equal to z_1, z_2, z_3, z_4 — that is, everywhere except at the ends of the antenna arms. Sufficiently far from the ends this limit is small. The factor $\Gamma(z, z')$ in the integrand is infinite at $z' = z$ and large in its vicinity; hence, the other factor must be small. Thus, as the radius of the antenna approaches zero,

$$\frac{d^2 I}{dz^2} + \beta^2 I \to 0; \tag{57}$$

that is, $I(z)$ approaches sinusoidal form. This is another way of proving the result already derived in Chapter 8. Substituting the sinusoidal current forms in equations 27 or 30, we can obtain asymptotic expressions for the mutual impedance. Equation 27 is particularly convenient, since the electric intensity of a sinusoidal current filament has been obtained in closed form, and we need to perform only one additional integration.

There is one theoretical question that we should discuss. Equation 47 gives the *exact* expression for the electric intensity parallel to a sinu-

soidal current filament. This expression does not vanish along the filament. It would seem, therefore, that the current in a transmitting antenna cannot be distributed sinusoidally, *even in the limiting case of zero radius,* for the boundary condition $E_z = 0$ on the surface of the antenna is apparently violated. This apparent contradiction arises from a false assumption that the convergence of E_z as the radius approaches zero is uniform.* Actually the convergence is nonuniform at $\rho = 0$, where ρ is the distance from the filament. What happens is that if the antenna radius is equal to a, there is a remainder term $I_1(z)$ in addition to the sinusoidal current; and this term is just sufficient to make $E_z(a)$ equal to zero. As a approaches zero, the remainder term also approaches zero. Hence, its contribution to E_z for any *fixed value of* ρ, *greater than* a, will approach zero with a. Thus, *in the limit* E_z *is given exactly by equation* 47 *for any* ρ *greater than zero; but at* $\rho = 0$ *equation* 47 *does not hold since* $E_z(a)$ *equals zero for any* a *and thus remains zero when* $a = 0$.

This seemingly academic point has an important bearing on the physical interpretation of what happens to the lines of power flow (Figs. 4.18 and 4.19) as the antenna becomes thinner. Since the power emerges from the generator, which we have assumed to occupy a small region, all power-flow lines should start from it. Since the antenna has been assumed to be a perfect conductor, the tangential component of E and, hence, the normal component of the Poynting vector vanish; thus, no lines of flow originate on the antenna. In the limiting case of an infinitely thin antenna, E_z is given by equation 47 when $\rho > 0$, *no matter how small* ρ *is*; this gives a nonvanishing contribution to the normal component of the Poynting vector, and the lines of flow *appear* to originate on the antenna. This appearance is due to the fact that we cannot look at the lines of flow through a microscope with infinite magnification and see how these lines turn sharply parallel to the antenna and go to the source. We can, however, calculate the lines of power flow for various finite radii and observe their tendency to hug the antenna as the radius diminishes (Figs. 4.18 and 4.19).

In the next Chapter we shall use the sinusoidal approximation to the current to obtain general formulas for the mutual admittance and the mutual radiated power in some important special cases. The same basic method can be used to obtain higher-order approximations. The outline of this second method of antenna analysis is: (1) Starting with the sinusoidal approximation to the current in a transmitting antenna of a

* For a discussion of uniform convergence of series and sequences of successive approximations with graphical illustrations, see I. S. Sokolnikoff, *Advanced Calculus,* McGraw-Hill, New York, 1939, pp. 253–255.

given *finite* radius, we evaluate the tangential field E_z by equation 47; (2) we subject the antenna to the compensating impressed field $-E_z$ and evaluate the correction terms by using the results of the first approximation. Essentially this is an application to the antenna problem of Poincaré's method *du balayage*, the " sweeping-off " method, in potential theory. The aim is to " sweep off " the surface of the antenna the residual tangential electric intensity by applying an equal and opposite intensity. This is probably the most elementary method of antenna analysis.

The third method of analysis is based on the solution of Maxwell's field equations subject to the prescribed boundary conditions at the surface of the antenna and the surface of the source of power. This method consists in first calculating special types of waves or *modes of propagation* consistent with the boundary conditions at the lateral surface of the antenna, then the modes consistent with free-space propagation, and, finally, in combining them so as to satisfy all the boundary conditions. In the end, the current in the antenna is obtained as the sum of two components, the TEM or principal wave, and the complementary wave consisting of all higher-order waves. We shall refer to this theory as the *mode theory of antennas.* The propagation of the principal mode is governed by ordinary transmission-line equations (see Chapter 4) in terms of distributed series inductance and shunt capacitance. Near the source of power only the principal mode is important. The phenomena occurring at the junction between feed lines and antennas are particularly easy to understand from the point of view of the mode theory. Generally, however, it is best to look at any phenomenon from several points of view.

12.9 Antennas, local circuits, and feed lines

Between the source of power and the transmitting antenna (or between the receiving antenna and the load) there may be local circuits and connecting transmission lines or waveguides. Unless all the sections of this local network are matched, the impedance looking toward the antenna will vary from point to point in the network. Circuit theory and transmission-line theory provide rules for impedance transformations between different points. Under certain restrictions these rules are applicable to waveguides. These local circuits are designed so that the radiation from them is small. The fields associated with the local circuits on the one hand and with the antenna on the other are kept as separate as possible. The entire transmission system may thus be treated as composed of two independent parts, the " feed system " and the " antenna proper." The small coupling that may exist between these

two parts may be evaluated later from the current and charge distribution obtained by neglecting the coupling.

Although it is practically impossible to eliminate the coupling between the antenna and the feed line, there are methods for reducing it; and, of course, theoretically this coupling can always be made to vanish. If the antenna is fed at the center with a two-wire line (Fig. 12.4), the interaction between the antenna and the feed line is reduced by decreasing the distance between the wires of the line and by twisting them around each other. When the distance between the wires is decreased, the impedance relations will be upset unless the radii of the wires are also

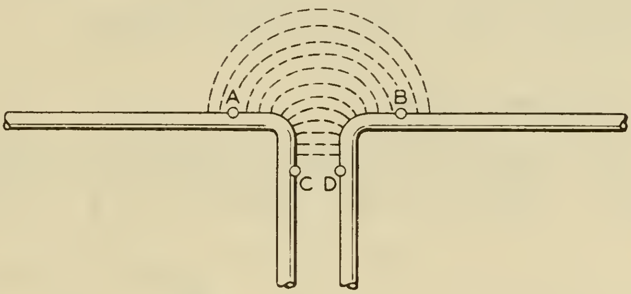


FIG. 12.4 A two-wire line feeding an antenna and the electric lines of force associated with the principal or TEM mode of propagation.

diminished, and properly tapered in the elbow junction. Theoretically, however, the impedance relations may be maintained indefinitely, so that the antenna may be electrically separated from the feed line. When the interaction between the antenna and the feeder is not negligible, it depends on the ratio of the antenna impedance to the characteristic impedance of the feeder; for this ratio determines the distribution of the current and charge on the feeder.

In free space, the two-wire antenna-feed system is perfectly balanced, the currents in the wires are equal and opposite,* and there is little radiation from the feed line; but, in the presence of the earth, the system is balanced only if it is horizontal. If the antenna is vertical, the field reflected from the earth is impressed on the feed line, and a parasitic antenna circuit is created. Since the field is impressed equally on both feed wires, the new circuit is composed of these wires in parallel with the earth, which acts as a return conductor. In this circuit the antenna becomes a shunt-excited antenna, in addition to being series-excited by the feed line operating in the push-pull mode. The amount of shunt excitation depends on the characteristic impedance of the ground return circuit. Theoretically this impedance can be made infinite; but practi-

* We assume, of course, a balanced antenna.

cally the ideal is unattainable because of limitations on the practicable size of the wires.

If the antenna is fed by a coaxial transmission line, the inner conductor is connected to one arm of the antenna and the outer conductor to the other. In this case there exists a parasitic radiating circuit even if the antenna system is in free space. The voltage transmitted along the coaxial line is impressed, not only between the arms of the antenna, but also between the outer surface of the outer conductor and the antenna arm connected to the inner conductor. Such parasitic radiating circuits affect the impedance seen at the end of the feed line and the radiation characteristics of the antenna system. In Chapter 15 we shall discuss some methods for breaking parasitic circuits.

12.10 Antenna input regions*

The *antenna input region* is a region of transition between the feed line and the antenna. The elbow region in Fig. 12.4 is an example. It is the

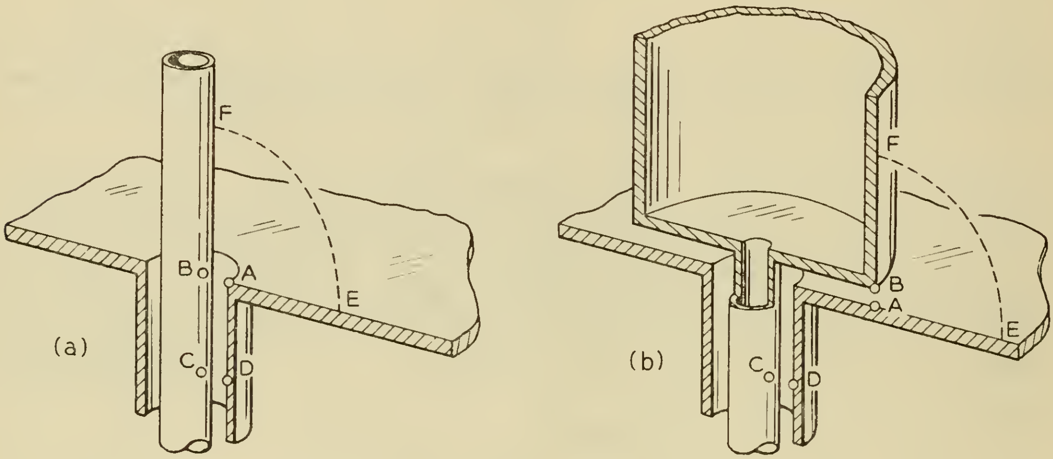


FIG. 12.5 Illustrating various input regions.

region beginning where the feed line ceases to be uniform and extending to where the antenna becomes more or less uniform and the waves guided by it become substantially spherical. This region is more conspicuous in a system such as that shown in Fig. 12.5*b*. Plane waves guided between the coaxial cylinders are forced to change into cylindrical waves between the base of the antenna cylinder and the ground plane. If the distance AB is small compared with the radius of the antenna, these waves emerge from the gap as waves similar to those between plane conductors constituting a wedge or a dihedral horn. Gradually, at dis-

* The reader should also consult J. R. Whinnery, The effect of input configuration on antenna impedance, *Jour. Appl. Phys.*, **21**, October **1950**, pp. 945–956.

tances from A, B large compared with the diameter, these waves become substantially spherical. In the broadcast antenna tower the input region includes the base insulator on which the tower is resting.

The limits of the antenna region are chosen arbitrarily for the purpose of analysis. The feed lines and antennas in many systems may be essentially the same, but the transition parts may be different; it is convenient, therefore, to consider different parts of the system separately. In such simple cases as that shown in Fig. 12.4, the input region may be considered merely as an irregularity in the feed line or in the antenna; even then, however, it is more convenient to deal with it separately. The dimensions involved in the input region are frequently so small compared with the wavelength that the region may be represented by a simple T network (Fig. 12.6) with series inductive branches and a shunt capacitive branch — or by a II network. For instance, in the impedance measurements reported by Brown and Woodward* the wavelength was 5 meters, and the distance AB between the base of a coaxially fed an-

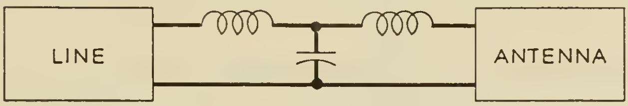


Fig. 12.6 A T-network may be used to represent a small input region between the line and the antenna.

tenna and the ground plane was $\lambda/360 = 1.39$ cm; the largest diameter was 27.8 cm; hence, in all cases the dimensions of the input region were small compared with λ. In the largest cylinder the length of the gap AB was quite small compared with the circumference of the antenna (87.3 cm.).

If the dimensions of the input region are large, it may still be to our advantage to consider it separately from the antenna and the feed line; but the equivalent T network becomes more complicated.

Whenever the series impedance of the network representing the input region is very small compared with the impedance of the antenna, and the shunt impedance is very large, it is permissible to speak of the "output terminals" of the feed line and the "input terminals" of the antenna. Otherwise, there are no antenna input terminals in the proper sense. The impedances are then obtained by measuring the standing waves in the feed line. From these measurements we can deduce the effective impedance that is presented to the waves in the plane passing through the points C, D normal to the feed line (Fig. 12.5), or in the

* Experimentally determined impedance characteristics of cylindrical antennas, *IRE Proc.*, 33, April 1945, pp. 257–262.

plane through the end of the coaxial line, or in the cylindrical surface passing through points A, B in the gap, or in the surface coinciding with the wavefront at E, F. In this particular case the end of the coaxial line is the most convenient reference plane for the measured impedance; but in theory it is just as easy to use any other position of reference.

Experimentally it is difficult to separate the input region from the rest of the antenna. The antenna may be cut away so as to leave only the input stub; then, however, there will be an end effect associated with the stub which will alter the capacitance of the stub. Corrections for this effect will have to be made theoretically. In the case of microwave antennas it is possible to measure the parameters of the input region by adding a conducting spherical shell concentric with the region. This shell is provided to eliminate the radiation without disturbing the shape of the lines of force in the input region. The shell may be chosen to provide either an effective short circuit at the place selected as the end of the input region or an effective open circuit. From proper measurements it will then be possible to deduce the image parameters of the input region.

Exact calculations of the characteristics of the input region are involved, although the fact that in this case the principal waves alone are important makes the problem more manageable. Approximate calculations are not so difficult and are usually sufficient for ordinary purposes. In the overhanging cylinder, for instance (Fig. 12.5b), there is a direct capacitance between the base of the cylinder and the ground plane. Neglecting the fringing, we have

$$C = \frac{\varepsilon\pi(a^2 - b^2)}{h}, \tag{58}$$

where

a = radius of antenna,
b = radius of the outer conductor of the coaxial line,
h = length of gap.

The inductance of this region is

$$L = \frac{\mu h}{2\pi} \log \frac{b}{a} \cdot \tag{59}$$

It is assumed, of course, that a is small compared with $\lambda/8$; otherwise, this region should be treated as a "disk transmission line."*

In the region extending beyond A, B toward E, F, it is convenient

* *Electromagnetic Waves*, p. 261.

to use the coordinate system* shown in Fig. 12.7. For the principal waves we may assume that the electric lines are circles of radius r. Then the magnetic intensity is

$$H_\varphi = \frac{I(r)}{2\pi\rho} = \frac{I(r)}{2\pi(a + r\sin\theta)}, \tag{60}$$

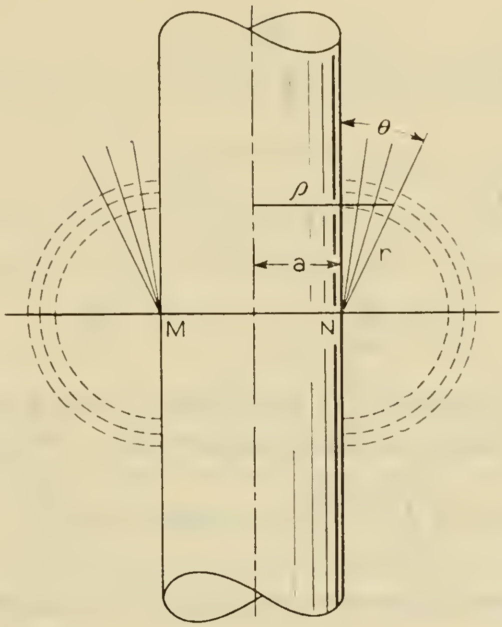


FIG. 12.7 A coordinate system suitable for studying waves on a cylinder.

where $I(r)$ is the current in the cylinder at distance r from the "origin circle." Hence, the magnetic displacement per unit length in the radial direction is

$$L\,I(r) = \int_0^{\pi/2} \mu H_\varphi r\,d\theta = \frac{\mu\,I(r)}{2\pi}\int_0^{\pi/2}\frac{r\,d\theta}{a + r\sin\theta}, \tag{61}$$

where L is the inductance per unit length. Thus,

$$L = \frac{\mu}{2\pi}\int_0^{\pi/2}\frac{r\,d\theta}{a + r\sin\theta}\cdot \tag{62}$$

Similarly, we obtain the capacitance per unit length and find that

$$LC = \mu\varepsilon, \qquad C = \frac{\mu\varepsilon}{L}\cdot \tag{63}$$

A more complete discussion on the basis of Maxwell's equations may be found elsewhere.†

* S. A. Schelkunoff, Principal and complementary waves in antennas, *IRE Proc.*, 34, January 1946, pp. 23P–32P.

† *Ibid.*

Evaluating the integral in equation 62, we find

$$
\begin{aligned}
L &= \frac{\mu(r/a)}{\pi\sqrt{1-(r^2/a^2)}} \tan^{-1}\sqrt{\frac{a-r}{a+r}}, && r < a, \\
&= \frac{\mu}{2\pi}, && r = a, \\
&= \frac{\mu(r/a)}{\pi\sqrt{(r^2/a^2)-1}} \tanh^{-1}\sqrt{\frac{r-a}{r+a}}, && r > a, \\
&= \frac{\mu(r/a)}{\pi\sqrt{(r^2/a^2)-1}} \log\frac{\sqrt{r+a}+\sqrt{r-a}}{\sqrt{2a}}, && r > a. \qquad (64)
\end{aligned}
$$

When $r \ll a$,

$$L \simeq \frac{\mu r}{4a}, \qquad C \simeq \frac{4\varepsilon a}{r}. \tag{65}$$

In this region the inductance is negligible, but the capacitance may be large. The total charge on the cylinder from $r = r_1$ to $r = r_2$ is

$$q = V\int_{r_1}^{r_2} \frac{4\varepsilon a}{r}\,dr = 4\varepsilon a V \log\frac{r_2}{r_1}. \tag{66}$$

Hence, the difference in the current along the cylinder is

$$
\begin{aligned}
I(r_2) - I(r_1) &= -j\omega q = -j4\omega\varepsilon a V \log\frac{r_2}{r_1} \\
&= -j\frac{aV}{15\lambda}\log\frac{r_2}{r_1}. \qquad (67)
\end{aligned}
$$

Thus, when $r \ll a$, the current in the cylinder varies as $\log r$. The input current per unit voltage is infinite if the gap is vanishingly small. Thus, in theoretical calculations of the input impedance it is *absolutely essential* to give proper consideration to the length of the gap and to questions of convergence.* The input impedance of an antenna consisting of two *hollow*† cylindrical shells is a function of the length l of each antenna arm, the wavelength λ, the radius a, and the length of the gap s; thus,

$$Z_i = f(l, \lambda, a, s). \tag{68}$$

This function possesses no unique limit as $a \to 0$ *and* $s \to 0$. If Z_i is regarded as a function of two variables, a and s, the limit depends on the

* L. Infeld, The influence of the width of the gap upon the theory of antennas, *Quart. Appl. Math.*, 5, July 1947, pp. 113–132.

† In order to eliminate the obvious effect of the direct capacitance between the flat ends of the solid cylinders.

manner in which a and s approach zero. Thus, *if s is kept constant* and l differs from an odd multiple of $\lambda/4$,

$$Z_i \to \infty \quad \text{as} \quad a \to 0; \tag{69}$$

but, *if a is kept constant,*

$$Z_i \to 0 \quad \text{as} \quad s \to 0. \tag{70}$$

If a and s both approach zero while the ratio s/a is constant, then,

$$Z_i \to \infty. \tag{71}$$

The general conditions may be described more readily for the input admittance which may be expressed as a sum of three functions,

$$Y_i = F_1(l, \lambda, a, s) + k\frac{a}{\lambda}\left[\log\frac{a}{s} + F_3(l, \lambda, a, s)\right], \tag{72}$$

where k is a constant. Normally only the first term is important. When s approaches zero, this term approaches a constant limit. But when we pass to a limit we must keep in mind that: (1) The first term slowly approaches zero as a approaches zero; (2) the second term approaches zero much more rapidly as a approaches zero, but it slowly approaches infinity when s approaches zero.

In asymptotic formulas for the input admittance, it is assumed that the radius approaches zero. This assumption automatically excludes the second term in the above equation and makes the admittance relatively independent of the length of the gap. These formulas may be applied to antennas of finite radius, but only to the extent to which the second term has been ascertained to be negligible: that is, only if the length of the gap is not too small compared with the radius. If the gap happens to be considerably smaller than the radius, a shunt capacitive admittance should be added to the asymptotic input admittance.

Thus, we have an added reason for explicit consideration of the input region and its representation by an equivalent network (Fig. 12.6). In theory, substantial simplifications can be made if the dimensions of the source of power are assumed to be small; but, in applying the results, we must either prove that our assumption does not affect the conclusions or else show how to amend these conclusions. In the mode theory of antennas, which is summarized in the following chapter, the simplification is made by altering the shape of the input region to conform to a pair of conical tips. This is the only case in which the dimensions of the source of power can be made infinitely small without radically affecting the input impedance. The effect of this alteration in shape can subsequently

be included — if it happens to be significant* — either by adding or by subtracting the difference in the capacitances of the actual and assumed input regions.

If we do not alter the shape of the input region and assume an infinitely narrow ring source at $r = 0$ (Fig. 12.7), the current in response to a unit voltage discontinuity at the source has a logarithmic singularity. This response is the " Green's function " of our antenna problem, and from it we can obtain by integration the response to any given impressed voltage. In this case the input admittance is given substantially by $I(s/2)$, the current at the ends of the gap. For a zero gap, the exact admittance is infinite. If, however, our method of solution depends on the hypothesis that the radius of the antenna approaches zero, so that the solution is asymptotic in character, the input region is automatically eliminated from the solution, and the solution is valid only to the extent to which the capacitance of the input region is negligible.

It should be understood that, when we speak of the " antenna gap," we mean the region between the assumed terminals of the antenna. The gap is empty when the antenna is fed by transmission lines. In long-wave antennas, the gap may include the building with the generator, the tuning coils, etc. If for theoretical purposes we exclude the feeder, the gap is the region of applied electric forces. In this case we assume that these forces are capable of transferring the electric charge back and forth between the antenna arms. At our convenience we may assume that the internal impedance of the gap is either zero or infinity. In the first case the applied voltage is equal and opposite to the voltage developed across the ends of the gap by the charges and currents in the antenna. In the second case the total applied voltage is infinite, but we consider only that part which appears across the terminals of the gap. This is the usual method for separating the internal impedance of a generator from that of a passive structure connected to it.

12.11 Modes of propagation in cage antennas

On several occasions we have noted that in a long pair of parallel wires of equal diameters there are two principal modes of propagation: (1) the push-pull mode in which the currents in the wires are equal but oppositely directed, (2) the push-push mode in which the currents are equal and similarly directed. In the first case the waves are plane. In the second case they are plane only if the wires are parallel to the surface of

* In broadcast antennas, the capacitance of the base insulator is important, but the near-base capacitance is likely to be unimportant. In thin microwave antennas, the near-base capacitance is also unimportant, but it is important in fat antennas.

the ground; otherwise, the waves are spherical. With each principal mode there are associated complementary modes which enable us to satisfy the boundary conditions at the ends and at other discontinuities. If the diameters of the wires are unequal, there are still two principal modes of propagation; but the currents in the push-push mode are unequal.

In antenna analysis this mode of thinking is very useful when the distance between the antennas is small compared with the length. The simplification is due to the fact that the above modes of propagation are independent of each other, and each is symmetrical in its own way. Furthermore, the push-pull mode has very simple properties. But, as the distance between the antennas becomes larger, the coupling between them becomes smaller; then it is more convenient to base our thinking on two coupled modes, each of which is simply the principal mode of propagation in an isolated antenna.

In the case of n parallel closely spaced wires there are n independent principal modes of propagation, of which one is spherical and $n - 1$ are plane. As the distances between the wires become larger, we may still think in terms of these modes, except that the modes will be neither spherical nor plane; in such a case it is better to think in terms of n coupled modes each of which is merely the principal mode in an isolated antenna. When the $(n - 1)$ modes are substantially plane, their theory is simple; it involves Maxwellian potential coefficients and inductance coefficients. These modes are the ones that play such an important part in the theory of "transmission lines" at low frequencies.

If wires of equal diameter are equispaced on a cylindrical surface to form a cage, it is particularly easy to determine from symmetry considerations a set of n independent modes. In the case of three wires, for example, we have the three modes shown in Fig. 12.8. If we satisfy the boundary conditions at the surface of one wire when the currents are equal, the boundary conditions at the other wires are satisfied automatically. In cases b and c, one wire is in the neutral plane of the other two. Similarly we obtain four modes (Fig. 12.9) for a cage with four wires.

In the general case of n wires we may start with the relative currents shown in Fig. 12.10. For the phase angle ψ we take

$$\psi = \frac{2m\pi}{n}, \qquad m = 0, 1, 2, \cdots, n - 1. \tag{73}$$

If we replace I by $I \exp(j\psi)$, we merely rotate the current distribution; hence, if we satisfy the boundary conditions at the surface of one wire, we shall satisfy them at the surface of every other wire. Now the currents

in the wires are

$$I,\ Ie^{j\psi},\ \cdots,\ Ie^{(n-1)j\psi}. \tag{74}$$

If we rotate the phase in the opposite direction, the same set of modes is expressed by

$$I,\ Ie^{-j\psi},\ Ie^{-2j\psi},\ \cdots,\ Ie^{-(n-1)j\psi}. \tag{75}$$

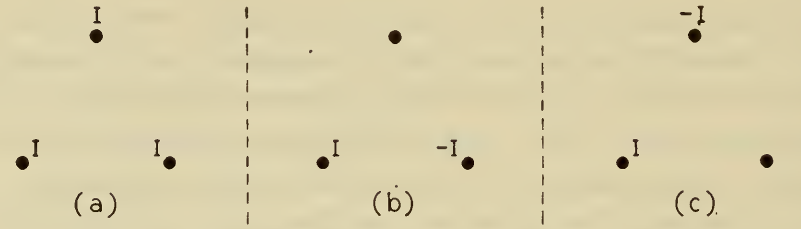


FIG. 12.8 Relative currents in three independent modes of propagation in a cage consisting of three equispaced parallel wires of equal diameters.

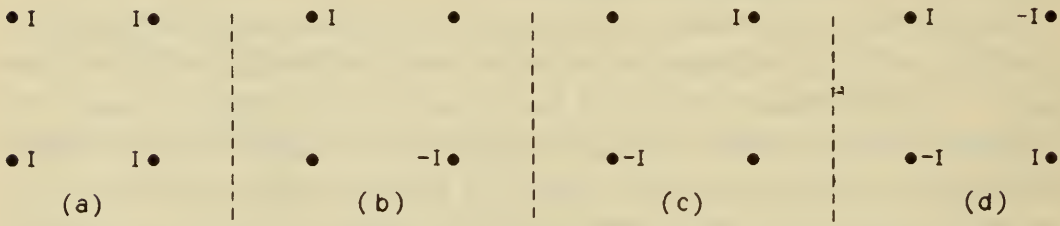


FIG. 12.9 Relative currents in four independent modes of propagation on four parallel wires.

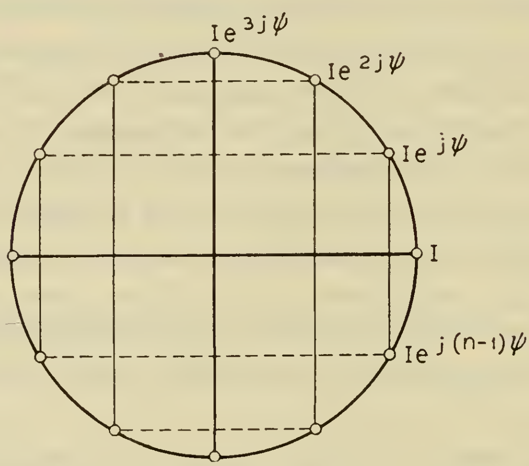


FIG. 12.10 Relative currents in n parallel wires equispaced on the surface of a cylinder.

From these we obtain two possible sets of real currents which may exist on the wires,

$$\begin{aligned} &I,\ I\cos\psi,\ I\cos 2\psi,\ \cdots,\ I\cos(n-1)\psi; \\ &0,\ I\sin\psi,\ I\sin 2\psi,\ \cdots,\ I\sin(n-1)\psi. \end{aligned} \tag{76}$$

We may now choose our independent set of modes from these current distributions.

If $n = 3$, for instance,

$$\psi = \frac{2\pi}{3} m, \qquad m = 0, 1, 2. \tag{77}$$

A set of independent modes is then given by

$$\begin{array}{lll} I, & I, & I; \\ I, & -\frac{1}{2}I, & -\frac{1}{2}I; \\ 0, & \frac{\sqrt{3}}{2}I, & -\frac{\sqrt{3}}{2}I. \end{array} \tag{78}$$

The amplitude factor in each mode is arbitrary, of course. The first and third of these modes are among the set shown in Fig. 12.8; the second is one half the sum of b and c. Vice versa, the modes b and c may be obtained from the second and third modes of the present set. The relative amplitudes of various modes may be obtained graphically from the projections of a radius on two mutually perpendicular diameters (Fig. 12.10).

It is important to observe that the *sum of the currents in all wires of the cage is zero for any of the above modes except the one in which all currents are equal.* In these modes one set of wires provides a complete return circuit for the current in the remaining wires. For small distances between the wires the end effects are small, and such modes have the properties of those considered in multiple-wire transmission lines. The end effects do not really complicate the picture; but we must not forget them when they make significant contributions.

All possible conditions of excitation of a cage may be expressed in terms of independent modes of propagation. For example, let a voltage V be impressed at the center of a single wire of the cage. This condition may be expressed as the sum of n modes:

$$\begin{array}{lllll} \frac{1}{n}V, & \frac{1}{n}V, & \frac{1}{n}V, & \cdots, & \frac{1}{n}V; \\ \frac{1}{n}V, & \frac{1}{n}Ve^{j\vartheta}, & \frac{1}{n}Ve^{2j\vartheta}, & \cdots, & \frac{1}{n}Ve^{(n-1)j\vartheta}; \\ \frac{1}{n}V, & \frac{1}{n}Ve^{2j\vartheta}, & \frac{1}{n}Ve^{4j\vartheta}, & \cdots, & \frac{1}{n}Ve^{2(n-1)j\vartheta}; \\ \frac{1}{n}V, & \frac{1}{n}Ve^{(n-1)j\vartheta}, & \frac{1}{n}Ve^{2(n-1)j\vartheta}, & \cdots, & \frac{1}{n}Ve^{(n-1)^2j\vartheta}. \end{array} \tag{79}$$

The sum gives

$$V, \quad 0, \quad 0, \quad \cdots, \quad 0. \tag{80}$$

Suppose now that the length of the cage is $\lambda/2$ and that the ends are connected together; then the waves in all modes of propagation except the first encounter effective short circuits at the ends.* The generators in the center see infinite impedances and the corresponding currents vanish. In effect, the voltage V impressed on one wire is equivalent to a voltage V/n impressed uniformly around the cage. If the diameter of the cage

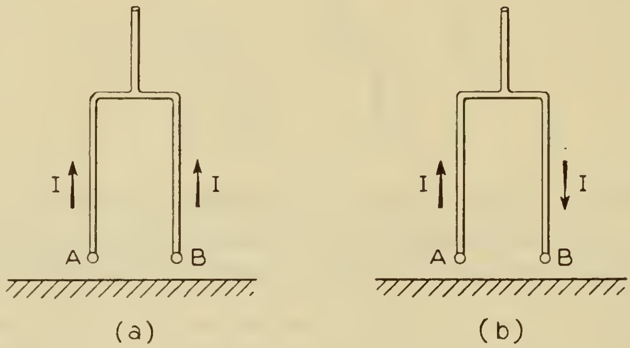


FIG. 12.11 Two independent modes of excitation of a wire network.

is small, the radiation resistance is about† 73 ohms; therefore, the total current in the cage is $V/73n$. The current in the wire where V is actually impressed is $V/73n^2$; hence, the impedance seen by the generator is $73n^2$. We have thus multiplied the impedance in the ratio n^2 to 1.

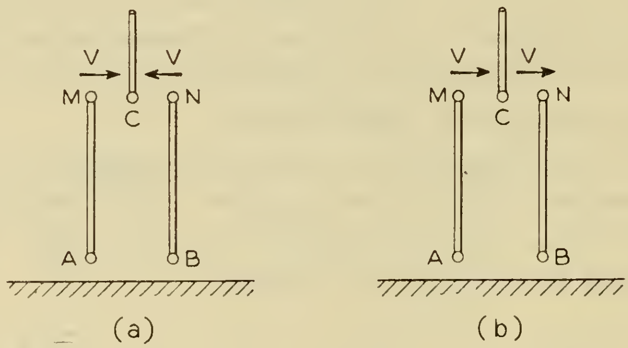


FIG. 12.12 Another pair of modes of excitation of a wire network.

This is a generalization of the result obtained in Chapter 11 for folded dipoles. When the length of the antenna differs from $\lambda/2$, it is more convenient to consider the admittance seen by the generator, since the various modes of propagation are electrically in parallel.

* Except for a small radiation resistance.

† Actually it is higher, depending on the radius. The resistance is about 73 ohms for a *self-resonant* cage; but then we have some reactance from the other modes.

Considerations of this kind may be extended to such structures as those shown in Figs. 12.11 and 12.12. The second case is more general since it provides for different sets of connections at the points M, C, N. Thus, we may short-circuit these terminals as in Fig. 12.11; then we have to consider only two modes corresponding to the generators at the base. Also we may connect only two points, M and C, or only C and N; then we must consider the two modes of excitation shown in Fig. 12.12 in addition to the modes of excitation at the base. In the mode shown in Fig. 12.11*b*, no current is flowing in the single wire at the top. If the distance between the parallel wires is small, the input admittance for this mode is substantially equal to that of a short-circuited transmission line. When the length of this line is $\lambda/4$, the input admittance is zero. If we now connect B to ground, the impedance from A to ground is four times the impedance A, B to ground. Incidentally, when B is connected to ground and the input is at A, we have essentially a shunt-excited antenna.

12.12 Loop antennas and shunt-excited antennas

Equations 24 are applicable to loop antennas (Fig. 12.13) and to shunt-excited antennas (Fig. 12.14). Thus, the input impedance is

$$Z_i = -\frac{\oint E_s(s)\, I^*(s)\, ds}{I_i I_i^*}. \tag{81}$$

Another formula is found from the Faraday-Maxwell equation for the voltage around a closed loop. Assuming that the wires are perfectly conducting, we find that the only contribution to the line integral comes from the gap AB. Thus, the impressed voltage from A to B is

$$V^i = -\frac{\partial \Phi}{\partial t} = -j\omega\Phi, \tag{82}$$

where Φ is the magnetic displacement toward the reader through the area of the loop (Fig. 12.13). An equivalent expression is in terms of the dynamic component of E (or the vector potential),

$$V^i = \int_{(ACDEFBA)} F_s\, ds. \tag{83}$$

Hence,

$$Z_i = -\frac{j\omega\Phi}{I_i} = \frac{\int F_s\, ds}{I_i}. \tag{84}$$

All these formulas are exact, although various approximations are usually made in their applications. In fact, when we know the exact

current distribution, we also know the potential distribution and have no need for formulas such as those above. It is only when we do not know the exact current distribution that these formulas become useful, for, in

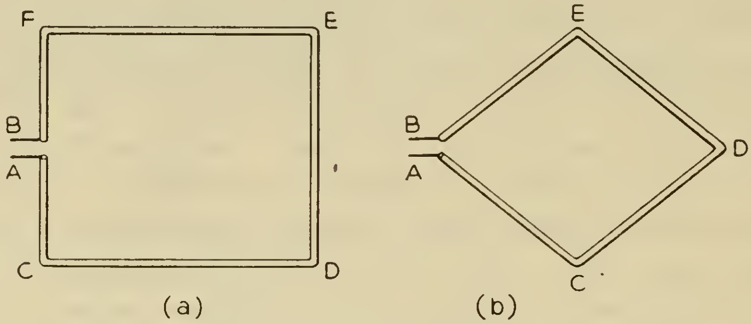


FIG. 12.13 Loop antennas.

effect, they yield a second approximation to the input impedance. We know, for instance, that in the first approximation the current and potential are distributed sinusoidally; the impedance obtained directly from the potential difference between the input terminals and the input

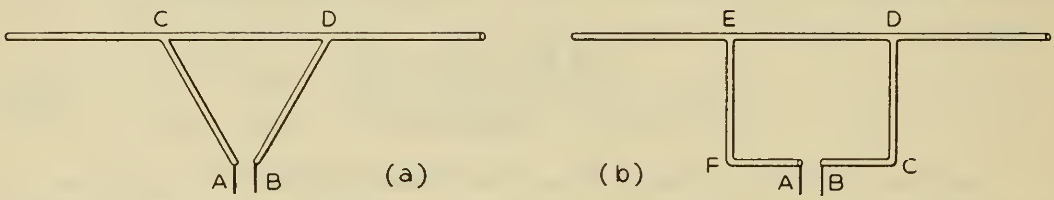


FIG. 12.14 Shunt-excited antennas.

current is not adequate for practical purposes; but, if we substitute the first approximation to the current in equation 81 or 84, we obtain a better result. The principal error occurs in the denominator when its value is small — that is, near antiresonance. This region can be studied by the method given in Section 11.16.

12.13 Elliptically polarized waves

Electromagnetic waves are said to be *linearly polarized* at a given point if the electric vector lies on a fixed straight line at all times. In general, however, the end point of the electric vector describes an ellipse, and the waves are said to be *elliptically polarized*. Such waves may be generated by two crossed current elements (either in free space or inside a horn) if we operate them in quadrature (see Section 6.5). We have seen that, at great distances from the source, the electric vector is perpendicular to the direction of propagation; hence, the plane of the ellipse is also perpendicular to the direction of propagation. The general expression for the

electric vector is then

$$E = E_\theta i_\theta + E_\varphi i_\varphi \,, \tag{85}$$

where i_θ and i_φ are unit vectors in the θ and φ directions. If E is linearly polarized, the ratio $\tilde{E}_\theta/\tilde{E}_\varphi$ of the instantaneous values of its components must be independent of time; hence, E_θ and E_φ must be in phase or 180° out of phase. For any other phase relationship the wave is elliptically polarized.

We can choose the origin of time so that the initial phase of E_θ is zero; then we may set

$$E_\theta = A, \qquad E_\varphi = Be^{j\vartheta}, \tag{86}$$

where A and B are the amplitudes of E_θ and E_φ and ϑ is the phase lead of E_φ with respect to E_θ. The instantaneous values are

$$\tilde{E}_\theta = A \cos \omega t, \qquad \tilde{E}_\varphi = B \cos(\omega t + \vartheta). \tag{87}$$

If $B = A$ and $\vartheta = \pm \pi/2$, we have

$$\tilde{E}_\theta = A \cos \omega t, \qquad \tilde{E}_\varphi = \mp A \sin \omega t \tag{88}$$

and

$$\tilde{E}_\theta^2 + \tilde{E}_\varphi^2 = A^2. \tag{89}$$

Hence, the locus of the end points is a circle, and the wave is *circularly polarized*. If $\vartheta = \pi/2$, then, at $t = 0$,

$$\tilde{E}_\theta(0) = A, \qquad \tilde{E}_\varphi = 0, \tag{90}$$

and one quarter period later,

$$\tilde{E}_\theta(\tfrac{1}{4}T) = 0, \qquad \tilde{E}_\varphi = -A. \tag{91}$$

Therefore, as we look in the direction of propagation (positive r direction) the vector appears to rotate counterclockwise. If $\vartheta = -\pi/2$, the vector rotates clockwise.

If $\vartheta = \pm \pi/2$ but $B \neq A$, the locus of the end points is the ellipse

$$\frac{\tilde{E}_\theta^2}{A^2} + \frac{\tilde{E}_\varphi^2}{B^2} = 1. \tag{92}$$

In general, eliminating t from equations 87, we find

$$\frac{\tilde{E}_\theta^2}{A^2} - \frac{2\tilde{E}_\theta\tilde{E}_\varphi \cos \vartheta}{AB} + \frac{\tilde{E}_\varphi^2}{B^2} = \sin^2 \vartheta. \tag{93}$$

This is an ellipse with its major axis inclined to E_θ.

12.14 Radiation and reception of elliptically polarized waves

In Chapter 6 we derived several simple formulas for the amount of power which may be transmitted between two antennas on the assumption that, as transmitting antennas, they radiate linearly polarized waves. We shall now obtain more general formulas. Equations 12 express the electric intensity in terms of the radiation vector. In a nondissipative medium of arbitrary intrinsic impedance η, these equations are

$$E_\theta = -\frac{j\eta}{2\lambda r} N_\theta e^{-j\beta r}, \qquad E_\varphi = -\frac{j\eta}{2\lambda r} N_\varphi e^{-j\beta r}. \tag{94}$$

Consider a current element of length s tangential to a typical meridian. The voltage induced in the element is

$$V_1 = -\frac{js\eta}{2\lambda r} N_\theta e^{-j\beta r}. \tag{95}$$

By the reciprocity theorem, this is also the voltage induced across the terminals of the given antenna if the current in the element equals the input current I_i in the given antenna required to create the field (equation 94). The field of this element at the given antenna is

$$E_\theta{}^i = \frac{j\eta I_i s}{2\lambda r} e^{-j\beta r}. \tag{96}$$

Hence, the induced voltage is

$$V_1 = -\frac{E_\theta{}^i N_\theta}{I_i}. \tag{97}$$

Similarly, we may consider a current element tangential to a typical parallel and obtain the voltage induced in the given antenna by a φ-polarized wave

$$V_2 = -\frac{B_\varphi{}^i N_\varphi}{I_i}. \tag{98}$$

The total induced voltage is

$$V = V_1 + V_2 = -\frac{E_\theta{}^i N_\theta + E_\varphi{}^i N_\varphi}{I_i}. \tag{99}$$

Thus, we have expressed the receiving properties of the antenna in terms of its radiating properties as defined by the radiation vector N.

Consider now two antennas with radiation vectors N_1 and N_2, one used as a transmitting antenna and the other as a receiving antenna.

The field created by one and the voltage induced by it in the other are:

$$E_{\theta,1} = -\frac{j\eta}{2\lambda r} N_{\theta,1} e^{-j\beta r}, \qquad E_{\varphi,1} = -\frac{j\eta}{2\lambda r} N_{\varphi,1} e^{-j\beta r},$$

$$\begin{aligned} V &= -\frac{E_{\theta,1}{}^i N_{\theta,2} + E_{\varphi,1}{}^i N_{\varphi,2}}{I_{i,2}} \\ &= \frac{j\eta}{2\lambda r} \frac{N_{\theta,1} N_{\theta,2} + N_{\varphi,1} N_{\varphi,2}}{I_{i,2}}, \end{aligned} \tag{100}$$

where $I_{i,2}$ is the input current which would have to be supplied to the receiving antenna, if it were used as a transmitting antenna, to create the field given by N_2.

If the receiving antenna is terminated into its conjugate impedance, the received power is

$$P_{\text{rec}} = \frac{VV^*}{8R_{i,2}}. \tag{101}$$

The power delivered to the transmitting antenna is

$$P_{\text{tr}} = \tfrac{1}{2} R_{i,1} I_{i,1} I_{i,1}^*. \tag{102}$$

Thus, the power ratio is

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{VV^*}{4R_{i,1}R_{i,2}I_{i,1}I_{i,1}^*}. \tag{103}$$

Substituting from equations 100, we have

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{\eta^2 (N_{\theta,1}N_{\theta,2} + N_{\varphi,1}N_{\varphi,2})\,(N_{\theta,1}^*N_{\theta,2}^* + N_{\varphi,1}^*N_{\varphi,2}^*)}{16\lambda^2 r^2 R_{i,1}R_{i,2}I_{i,1}I_{i,1}^*I_{i,2}I_{i,2}^*}. \tag{104}$$

We note that

$$P_1 = \tfrac{1}{2} R_{i,1} I_{i,1} I_{i,1}^*, \qquad P_2 = \tfrac{1}{2} R_{i,2} I_{i,2} I_{i,2}^*, \tag{105}$$

represent the power inputs to the antennas required to produce fields given by the radiation vectors N_1 and N_2. Hence,

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{\eta^2 |N_{\theta,1}N_{\theta,2} + N_{\varphi,1}N_{\varphi,2}|^2}{64\lambda^2 r^2 P_1 P_2}. \tag{106}$$

If the heat losses in the antennas are negligible,

$$\begin{aligned} P_1 &= \frac{\eta}{8\lambda^2}\int_0^{2\pi}\int_0^{\pi} (N_{\theta,1}N_{\theta,1}^* + N_{\varphi,1}N_{\varphi,1}^*)\, d\Omega, \\ P_2 &= \frac{\eta}{8\lambda^2}\int_0^{2\pi}\int_0^{\pi} (N_{\theta,2}N_{\theta,2}^* + N_{\varphi,2}N_{\varphi,2}^*)\, d\Omega, \end{aligned} \tag{107}$$

and equation 106 may be written as

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{\lambda^2(N_{\theta,1}N_{\theta,2}+N_{\varphi,1}N_{\varphi,2})(N_{\theta,1}^*N_{\theta,2}^*+N_{\varphi,1}^*N_{\varphi,2}^*)}{r^2\int_0^{2\pi}\int_0^{\pi}(N_{\theta,1}N_{\theta,1}^*+N_{\varphi,1}N_{\varphi,1}^*)\,d\Omega\int_0^{2\pi}\int_0^{\pi}(N_{\theta,2}N_{\theta,2}^*+N_{\varphi,2}N_{\varphi,2}^*)\,d\Omega}. \tag{108}$$

This formula depends only on the current distributions of the antennas when each is used as a transmitting antenna. In equation 104 we have to know the values of the input resistances of the antennas and of the input currents. When the input currents are small, as in antiresonant antennas, we have to know these values very accurately, or else the error will be large. On the other hand, equation 108 is not sensitive to the errors in the current where the current is small.

In the numerator of equation 108, the same coordinate system must be used for both antennas. We can make it independent of the coordinate systems by noting that the quantities in parentheses are scalar products of those components $\overline{N}_1$, $\overline{N}_2$ of the radiation vectors N_1 and N_2 which are normal to the radii drawn from the corresponding sources. Thus,

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{\lambda^2(\overline{N}_1\cdot\overline{N}_2)(\overline{N}_1^*\cdot\overline{N}_2^*)}{r^2\int_0^{2\pi}\int_0^{\pi}\overline{N}_1\cdot\overline{N}_1^*\,d\Omega\int_0^{2\pi}\int_0^{\pi}\overline{N}_2\cdot\overline{N}_2^*\,d\Omega}. \tag{109}$$

Let us define the *polarization vectors* as the following unit complex vectors

$$n_1 = \overline{N}_1(\overline{N}_1\cdot\overline{N}_1^*)^{-1/2}, \qquad n_2 = \overline{N}_2(\overline{N}_2\cdot\overline{N}_2^*)^{-1/2}. \tag{110}$$

These are " unit " vectors in the sense that

$$n_1\cdot n_1^* = 1, \qquad n_2\cdot n_2^* = 1. \tag{111}$$

Equation 109 becomes

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = \frac{\lambda^2(\overline{N}_1\cdot\overline{N}_1^*)(\overline{N}_2\cdot\overline{N}_2^*)(n_1\cdot n_2)(n_1^*\cdot n_2^*)}{r^2\int_0^{2\pi}\int_0^{\pi}(\overline{N}_1\cdot\overline{N}_1^*)\,d\Omega\int_0^{2\pi}\int_0^{\pi}(\overline{N}_2\cdot\overline{N}_2^*)\,d\Omega}. \tag{112}$$

Since $N\cdot N^*$ is proportional to the radiation intensity, we have

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = k\frac{\lambda^2\Phi_1\Phi_2}{r^2\int_0^{2\pi}\int_0^{\pi}\Phi_1\,d\Omega\int_0^{2\pi}\int_0^{\pi}\Phi_2\,d\Omega}, \tag{113}$$

where

$$k = (n_1\cdot n_2)(n_1^*\cdot n_2^*) \tag{114}$$

is the *polarization loss factor.*

Polarization does not enter into the remainder of equation 113. The definition of directivity (equation 6–2) does not involve polarization considerations. Hence, we may express equation 113 as follows:

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = kg_1g_2\left(\frac{\lambda}{4\pi r}\right)^2 \frac{\Phi_1}{\Phi_{1,\max}} \frac{\Phi_2}{\Phi_{2,\max}}$$

$$= k\frac{A_1A_2}{\lambda^2r^2} \frac{\Phi_1}{\Phi_{1,\max}} \frac{\Phi_2}{\Phi_{2,\max}}, \tag{115}$$

where the effective areas are defined by $A = g\lambda^2/4\pi$, irrespective of polarization. If $k = 1$, $\Phi_1 = \Phi_{1,\max}$, $\Phi_2 = \Phi_{2,\max}$, we have the maximum possible transmission of power between antennas with directivities equal to g_1 and g_2. The factor $\Phi_1/\Phi_{1,\max}$ represents the loss if the transmitting antenna is not beamed on the receiving antenna, $\Phi_2/\Phi_{2,\max}$ is the loss if the receiving antenna is not beamed on the transmitting antenna, and k is the loss caused by the polarization differences. If $n_2 = \pm n_1^*$, then $k = 1$; otherwise, $k < 1$.

Any unit complex vector may be expressed in the form

$$n = i_\theta \cos\alpha + i_\varphi e^{j\vartheta}\sin\alpha, \tag{116}$$

where i_θ and i_φ are orthogonal unit vectors and α is not greater than 90°. The factors $\cos\alpha$ and $\sin\alpha$ represent the relative amplitudes of the components of E in the two polarizations and ϑ is the phase difference. Substituting in equation 114, we find

$$\begin{aligned} k &= |\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2\, e^{j(\vartheta_1+\vartheta_2)}|^2 \\ &= \cos^2\alpha_1\cos^2\alpha_2 + 2\cos\alpha_1\cos\alpha_2\sin\alpha_1\sin\alpha_2\cos(\vartheta_1+\vartheta_2) + \\ &\quad \sin^2\alpha_1\sin^2\alpha_2 \\ &= \cos^2(\alpha_1-\alpha_2) - \sin 2\alpha_1\sin 2\alpha_2\sin^2\tfrac{1}{2}(\vartheta_1+\vartheta_2). \end{aligned} \tag{117}$$

Since α_1 and α_2 are not greater than 90°, the second term is never negative; hence, k attains its maximum value, unity, when $\alpha_1 = \alpha_2$ and the second term vanishes; the second requirement is satisfied either when $\alpha_1 = \alpha_2 = 0$, 90° or when $\vartheta_2 = -\vartheta_1$.

12.15 Directivity vectors and effective lengths of antennas

We can also express equation 109 in the following form, which resembles equation 6–28,

$$\frac{P_{\text{rec}}}{P_{\text{tr}}} = |\hat{N}_1\cdot\hat{N}_2|^2\left(\frac{\lambda}{4\pi r}\right)^2, \tag{118}$$

where the *directivity vector* $\hat{N}$ is defined by

$$\hat{N} = \left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \bar{N}\cdot\bar{N}^*\,d\Omega\right]^{-1/2}\bar{N}. \tag{119}$$

All the above formulas for the power ratio are based on the assumption that the receiving antenna is matched, that is, terminated into its conjugate impedance. In the formulas that follow equation 106, it is further assumed that the heat losses in the antennas are negligible. If the heat losses are not negligible, we should introduce the efficiency factors; and, if the receiving antenna is mismatched, we should include the mismatch loss. If we do not wish to separate the various effects, we should go back to equations 100 which give the field created by the transmitting antenna and the voltage induced across the open terminals of the receiving antenna. In such calculations it may be convenient to introduce the *generalized effective length* of the antenna defined as a complex vector*

$$h = \frac{\overline{N}}{I_i}, \tag{120}$$

where I_i is the input current required to create the field represented by $\overline{N}$. Note that $\overline{N}$ is the component of the radiation vector normal to the direction of propagation. The product $\overline{N} = I_i h$ is the *effective moment* of the transmitting current distribution in the direction (θ, φ). The electric intensity of the wave generated by the transmitting antenna may then be expressed as†

$$E = \left(\frac{j\eta}{2\lambda r}\right) I_{i,1} h_1 e^{-j\beta r}. \tag{121}$$

The voltage induced across the terminals of the receiving antenna is

$$V = E \cdot h_2 = \left(\frac{j\eta}{2\lambda r}\right) I_{i,1} h_1 \cdot h_2 e^{-j\beta r}. \tag{122}$$

This is a direct generalization of the transmission formula for two parallel current elements perpendicular to the line joining them, in which case h_1 and h_2 are the actual lengths of the elements.

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† The negative signs in equations 100 are due to the fact that the positive directions of the spherical components N_θ and N_φ are opposite to the positive directions of the generating currents I_z and I_φ.

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PROBLEMS

12.1–1 Consider two current elements of moment $I\,dz$, one at the origin and the other at one of the points whose Cartesian coordinates are: $(x_0, 0, 0)$, $(0, y_0, 0)$ $(0, 0, z_0)$, (x_0, y_0, z_0). Find the radiation vectors.

Solution. With respect to its own position the radiation vector of either element is $I\,dz$. With respect to the origin the radiation vectors of the elements at $(0, 0, 0)$ and $(x_0, 0, 0)$ are, respectively,

$$N_{1,z} = I\,dz, \qquad N_{2,z} = I\,dz e^{j\beta x_0 \sin\theta\cos\varphi}.$$

To obtain the second vector we use the translation rule 5 and note that, if $x_0 > 0$, then $r' = x_0$, and the direction from the origin to the second element is given by $\theta' = 90^\circ$, $\varphi' = 0$. Hence, from equation 3 we have $\cos\psi = \sin\theta\cos\varphi$. If x_0 is negative, then $r' = -x_0$ (since the radial coordinate is essentially positive) and $\theta' = 90^\circ$, $\varphi' = 180^\circ$.

The translation factor remains unchanged since $\cos\psi = \sin\theta\cos(\varphi - \pi) = -\sin\theta\cos\varphi$. The total radiation vector is

$$N_z = I\,dz(1 + e^{j\beta x_0 \sin\theta\cos\varphi}).$$

We can equally well translate the element at the origin to $(x_0, 0, 0)$. Then,

$$N_{1,z}' = I\,dz e^{-j\beta x_0 \sin\theta\cos\varphi}, \qquad N_{2,z}' = I\,dz,$$

$$\begin{aligned} N_z' &= I\,dz(e^{-j\beta x_0 \sin\theta\cos\varphi} + 1) \\ &= I\,dz(1 + e^{j\beta x_0 \sin\theta\cos\varphi})e^{-j\beta x_0 \sin\theta\cos\varphi}. \end{aligned}$$

We note that the two vectors differ only by a phase factor,

$$N_z' = N_z e^{-j\beta x_0 \sin\theta\cos\varphi},$$

so that the squares of the absolute values are equal,

$$N_z' N_z'^* = N_z N_z^*.$$

The radiation intensity is not affected.

We can refer the radiation vector of both elements to the mid-point; then,

$$\begin{aligned} N_z &= I\,dz(e^{\frac{1}{2}j\beta x_0 \sin\theta\cos\varphi} + e^{-\frac{1}{2}j\beta x_0 \sin\theta\cos\varphi}) \\ &= 2(I\,dz)\cos(\tfrac{1}{2}\beta x_0 \sin\theta\cos\varphi). \end{aligned}$$

In the case of the elements at $(0, 0, 0)$ and $(0, y_0, 0)$ we have

$$N_z = I\,dz(1 + e^{j\beta y_0 \sin\theta\sin\varphi})$$

since $r_0 = y_0$, $\theta' = 90^\circ$, $\varphi' = 90^\circ$. And, for the elements at $(0, 0, 0)$ and $(0, 0, z_0)$, we have $r_0 = z_0$, $\theta' = 0$, and

$$N_z = I\,dz(1 + e^{j\beta z_0 \cos\theta}).$$

If the elements are at (0, 0, 0) and (x_0, y_0, z_0), we can translate the second point of radiation to the origin directly by means of equation 5. In this case,

$$r' = (x_0^2 + y_0^2 + z_0^2)^{1/2}, \qquad \cos\theta' = \frac{z_0}{r'}, \qquad \tan\varphi' = \frac{y_0}{x_0}.$$

It is often convenient, however, from the point of view of subsequent integrations, to use the *rule of successive translations* of the effective point of radiation. Thus, we can translate from (x_0, y_0, z_0) to $(0, y_0, z_0)$ then to $(0, 0, z_0)$ and finally to (0, 0, 0). In this way we obtain

$$N_{2,z} = I\,dz e^{j\beta x_0 \sin\theta\cos\varphi} e^{j\beta y_0 \sin\theta\sin\varphi} e^{j\beta z_0\cos\theta}.$$

This rule is equivalent to the equation,

$$r'\cos\psi = x_0\sin\theta\cos\varphi + y_0\sin\theta\sin\varphi + z_0\cos\theta,$$

which states that the projection of the vector from (0, 0, 0) to (x_0, y_0, z_0) on a particular direction equals the sum of the projections of its three mutually perpendicular components.

12.1–2 Find the radiation vector of an array of four elements of moment $I\,dz$ at the following points: (0, 0, 0), $(a, 0, 0)$, $(0, b, 0)$, $(a, b, 0)$.

Solution. First we obtain the radiation vector of the first two elements with respect to the mid-point $(\frac{1}{2}a, 0, 0)$,

$$N_{1,z} = 2I\,dz\cos(\tfrac{1}{2}\beta a\sin\theta\cos\varphi).$$

Next we obtain the radiation vector of the last two elements with respect to the mid-point $(\frac{1}{2}a, b, 0)$ between them,

$$N_{2,z} = 2I\,dz\cos(\tfrac{1}{2}\beta a\sin\theta\cos\varphi) = N_{1,z}.$$

These two effective points of radiation form an array of identical elements. The radiation vector of this array with respect to the mid-point $(\frac{1}{2}a, \frac{1}{2}b, 0)$ is

$$\begin{aligned} N_z &= 2N_{1,z}\cos(\tfrac{1}{2}\beta b\sin\theta\sin\varphi) \\ &= 4I\,dz\cos(\tfrac{1}{2}\beta a\sin\theta\cos\varphi)\cos(\tfrac{1}{2}\beta b\sin\theta\sin\varphi). \end{aligned}$$

12.1–3 Obtain the radiation vector of three current elements of moments $I\,dz$, $2I\,dz$, $I\,dz$ at (0, 0, 0), $(0, 0, l)$, $(0, 0, 2l)$.

Ans.
$$\begin{aligned} N_z &= I\,dz(1 + 2e^{ju} + e^{2ju}) = I\,dz(1 + e^{ju})^2 \\ &= 4I\,dz\cos^2(\tfrac{1}{2}u)\exp(ju), \quad u = \beta l\cos\theta. \end{aligned}$$

12.1–4 Obtain the radiation vector of $n + 1$ current elements whose moments are proportional to the coefficients of the binomial expansion $(a + b)^n$. Assume that they are along the z axis and that the distance between the successive elements is l.

Ans.
$$\begin{aligned} N_z &= I\,dz(1 + e^{ju})^n = 2^n I\,dz\cos^n(\tfrac{1}{2}u)\exp(\tfrac{1}{2}jnu), \\ u &= \beta l\cos\theta. \end{aligned}$$

12.1–5 Solve the preceding problem for the case in which the moments are proportional to the coefficients of the various powers of a in the expansion $(1 - ja)^n$.

Ans. $N_z = 2^n I\,dz\cos^n(\tfrac{1}{2}u - \tfrac{1}{4}\pi)\exp(\tfrac{1}{2}jnu), \qquad u = \beta l\cos\theta.$

Note that, if $l = \lambda/4$, $\beta l = \pi/2$, and, except for the phase factor,

$$N_z = 2^n I\, dz \cos^n \tfrac{1}{4}\pi(1 - \cos\theta).$$

Hence, the radiation vectors of all the elements add in phase in the direction $\theta = 0$; in the direction $\theta = \pi$, $N_z = 0$.

12.1–6 Obtain the radiation vector of four equal current elements parallel to the z axis and equispaced in the xy plane on a circle of radius a. Assume that one element is at $(a, 0, 0)$.

Ans.
$$\begin{aligned} N_z &= \left(e^{j\beta a \sin\theta \cos\varphi} + e^{j\beta a \sin\theta \cos(\varphi - \frac{1}{2}\pi)} + e^{j\beta a \sin\theta \cos(\varphi - \pi)} + e^{j\beta a \sin\theta \cos(\varphi - \frac{3}{2}\pi)}\right) I\, dz \\ &= 4(I\, dz) J_0(\beta a \sin\theta) + 8I\, dz \sum_{n=1}^{\infty} J_{4n}(\beta a \sin\theta) \cos 4n\varphi. \end{aligned}$$

Note. A clover-leaf pattern in the equatorial plane is obtained when $\beta a = 2\pi a/\lambda = 2.40$, the first zero of $J_0(x)$.

12.1–7 Obtain the radiation vector of a uniform current filament, I amperes, extending along the z axis from $(0, 0, 0)$ to $(0, 0, l)$.

Ans. $$N_z = I \int_0^l e^{j\beta z \cos\theta}\, dz = \frac{2I \sin(\frac{1}{2}\beta l \cos\theta)}{\beta \cos\theta} \exp(\tfrac{1}{2} j\beta l \cos\theta).$$

12.1–8 Obtain the radiation vector of a blade of current elements of moment $p_x = C\, dx\, dz$ extending from $(0, 0, 0)$ to $(0, 0, l)$.

Ans. $$N_x = C\, dx \int_0^l e^{j\beta z \cos\theta}\, dz.$$

(Compare with the answer in the preceding problem).

12.1–9 Obtain the radiation vector of a circular ribbon of radius a and height dz carrying current I at right angles to the plane of the circle.

Ans.
$$\begin{aligned} N_z &= \frac{I\, dz}{2\pi} \int_0^{2\pi} e^{j\beta a \sin\theta \cos(\varphi - \varphi')}\, d\varphi' \\ &= \frac{I\, dz}{2\pi} \int_0^{2\pi} e^{j\beta a \sin\theta \cos\varphi'}\, d\varphi' \\ &= (I\, dz)\, J_0(\beta a \sin\theta). \end{aligned}$$

12.1–10 Obtain the radiation vector of a circular current loop of radius a in the xy plane. *Hint:* Express N_x and N_y in the form of integrals; *before* actual integration obtain N_ρ and N_φ.

Ans. $$N_\varphi = j2\pi a I\, J_1(\beta a \sin\theta), \qquad N_\rho = N_z = 0.$$

12.1–11 Obtain the radiation vector of a cylindrical uniform current sheet of radius a and height l. Assume that the current is axial.

Ans. $$N_z = \frac{2I\, J_0(\beta a \sin\theta) \sin(\frac{1}{2}\beta l \cos\theta)}{\beta \cos\theta}.$$

12.1–12 Solve the preceding problem on the assumption that the current is circulating.

Ans.
$$N_\varphi = j\,\frac{4\pi a I\, J_1(\beta a \sin\theta)\,\sin(\tfrac{1}{2}\beta l \cos\theta)}{\beta l \cos\theta}\,.$$

12.1–13 Solve Problem 5.15–3 by the method given in this section.

12.1–14 Consider a helical progressive current filament wound on a cylinder of radius a. Let h be the pitch of the helix. Let the filament start at $(a, 0, 0)$ and end at $(a, 0, nh)$. Let the phase constant along the filament be $\bar{\beta}$. Find the radiation vector.

Ans.
$$N_\varphi = I_0 a \int_0^{2n\pi} e^{j(\vartheta_1+\vartheta_2)} \cos(\varphi - \varphi')\, d\varphi',$$

$$N_\theta = I_0 a \cos\theta \int_0^{2n\pi} e^{j(\vartheta_1+\vartheta_2)} \sin(\varphi - \varphi')\, d\varphi' - \frac{I_0 h}{2\pi} \sin\theta \int_0^{2n\pi} e^{j(\vartheta_1+\vartheta_2)}\, d\varphi',$$

$$\vartheta_1 = \beta a \sin\theta \cos(\varphi - \varphi'), \qquad \vartheta_2 = \left[\frac{h}{\lambda} - \bar{\beta} L\right] \varphi',$$

$$L = \left[a^2 + \left(\frac{h}{2\pi}\right)^2\right]^{1/2}.$$

Note: To evaluate the integrals, express $\exp(j\vartheta_1)$ as a Fourier-Bessel series. If $\vartheta_2 = 0$, the final answer is simple:

$$N_\varphi = 2jn\pi a I_0\, J_1(\beta a \sin\theta), \qquad N_\theta = -nhI_0\, J_0(\beta a \sin\theta).$$

In general, the final answer is given by a Fourier-Bessel series.

12.1–15 Obtain the radiation vector of a circular blade of electric current I flowing uniformly in the radial direction from $\rho = a$ to $\rho = a + s$, where s is small.

Ans.
$$N_\rho = jIs\, J_1(\beta a \sin\theta), \qquad N_\varphi = N_z = 0;$$

hence,

$$N_\theta = jIs\, J_1(\beta a \sin\theta) \cos\theta, \qquad N_r = N_\varphi = 0.$$

12.1–16 Consider a conical sheet coaxial with the z axis. Let its length along the generators be l, and let ψ be the angle between the generators and the axis. Assume that the current flows along the generators and that at distance r from the apex it is $I(r) = I_0 \sin\beta(l - r)$. Find the radiation vector.

Ans.
$$N_z = I_0 \cos\psi \int_0^l \sin\beta(l - r)\, e^{j\beta r \cos\theta \cos\psi}\, J_0(\beta r \sin\theta \sin\psi)\, dr,$$

$$N_\rho = jI_0 \sin\psi \int_0^l \sin\beta(l - r)\, e^{j\beta r \cos\theta \cos\psi}\, J_1(\beta r \sin\theta \sin\psi)\, dr.$$

Note. To integrate, expand J_0 and J_1 in power series. The series converge rapidly if ψ is small.

12.1–17 Solve the preceding problem for a double cone formed by a single cone and its image in the equatorial plane. Assume that the current distribution is symmetric.

Ans. The same as in the preceding problem if the exponential factor is replaced by $2\cos(\beta r \cos\theta \cos\psi)$.

12.1–18 Obtain the radiation vector of a current filament beginning at the origin and lying in the xy plane. The angle between the filament and the x axis is φ', the length l, and the current $I(s) = I_0 \sin\beta(l - s)$, where s is measured from the origin and the positive direction of current is from the origin.

Ans.
$$N_x = N_0 \cos\varphi', \qquad N_y = N_0 \sin\varphi',$$
$$N_0 = \frac{I_0[\exp(j\beta l \cos\psi) - \cos\beta l - j\sin\beta l \cos\psi]}{\beta \sin^2\psi},$$
$$\cos\psi = \sin\theta\cos(\varphi - \varphi').$$

12.1–19 Calculate the radiation vector of a V-filament formed by adding to the current filament in the preceding problem a similar filament making an angle $-\varphi'$ with the x axis and carrying the current toward the origin.

Ans.
$$N_x = (N_1 - N_2)\cos\varphi', \qquad N_y = (N_1 + N_2)\sin\varphi',$$

where N_1 equals N_0 in the preceding problem and N_2 is obtained from N_0 by reversing the sign of φ'.

12.1–20 Obtain the radiation intensity of two current elements, $I_1\,ds_1$ and $I_2\,ds_2$. Let the first be along the z axis at (0, 0, 0) and the second parallel to the x axis, at (0, 0, l). Assume that the phase lead of I_2 with respect to I_1 is ϑ.

Ans.
$$\Phi = \frac{15\pi}{\lambda^2}\big[|I_1\,ds_1|^2\sin^2\theta + |I_2\,ds_2|^2(1 - \sin^2\theta\cos^2\varphi) -$$
$$(I_2I_1^*e^{j\beta l\cos\theta} + I_1I_2^*e^{-j\beta l\cos\theta})\,ds_1\,ds_2\sin\theta\cos\theta\cos\varphi\big]$$
$$= \frac{15\pi}{\lambda^2}\big[|I_1\,ds_1|^2\sin^2\theta + |I_2\,ds_2|^2(1 - \sin^2\theta\cos^2\varphi) -$$
$$2|I_1\,ds_1\,I_2\,ds_2|\cos(\beta l\cos\theta + \vartheta)\sin\theta\cos\theta\cos\varphi\big].$$

12.1–21 Consider two elements $I_1\,dx_1$ and $I_2\,dy_2$ at $(l_1, 0, 0)$ and $(0, l_2, 0)$, respectively, and assume that $\mathrm{ph}(I_2/I_1) = \vartheta$. Calculate the radiation intensity.

Ans.
$$\Phi = \frac{15\pi}{\lambda^2}\big[|I_1\,dx_1|^2(\cos^2\theta\cos^2\varphi + \sin^2\varphi) +$$
$$|I_2\,dy_2|^2(\cos^2\theta\sin^2\varphi + \cos^2\varphi) -$$
$$2|I_1\,dx_1\,I_2\,dy_2|\cos(\beta l_1\sin\theta\cos\varphi - \beta l_2\sin\theta\sin\varphi - \vartheta)\big].$$

12.4–1 Show that the input impedance of a linear antenna may be expressed as follows:

$$Z_i = -\frac{1}{I_iI_i^*}\iint T(s_1, s_2)\,I(s_1)\,I^*(s_2)\,ds_1\,ds_2,$$

where $T(s_1, s_2)$ is an appropriate space transmission factor between two typical current elements.

12.4–2 Along the surface of a perfectly conducting antenna the tangential component of E vanishes except in the region of the impressed voltage. Hence, from equation 17 we find

$$\Psi = -\tfrac{1}{2}\int_{s_0-\frac{1}{2}\delta}^{s_0+\frac{1}{2}\delta} E_s(s)\, I^*(s)\, ds,$$

where δ is the length over which the impressed voltage is distributed. If $I(s)$ does not vary rapidly in this region, then,

$$\Psi = -\tfrac{1}{2}I^*(s_0)\int_{s_0-\frac{1}{2}\delta}^{s_0+\frac{1}{2}\delta} E_s(s)\, ds = \tfrac{1}{2}V^i I_i^*.$$

This is what we should have at the input terminals of a concealed circuit.

If the exact antenna current is known, the equations of Section 12.4 are trivial since over practically the entire range of integration the integrals vanish. But in this case we have no need of these equations because we obtain the impedance merely by dividing the impressed voltage by the input current. The equations are useful for approximate calculations.

The equations of this section can be misinterpreted, and it is essential that they be thoroughly understood. All radiated energy comes from the generator connected to the antenna, and it may be expressed in terms of the voltage and current at the terminals of the antenna. On the other hand, each current element radiates energy; hence, the radiated energy may be expressed in terms of the currents at various points along the antenna. This seems to suggest that the antenna is a system of distributed sources of radiation. So it is; but almost all of the elements of the antenna radiate borrowed power. They receive power from the generator and reradiate it. If the antenna is not perfectly conducting, each element retains a small fraction of the received power and transforms it in heat. The mechanism is the same as in the case of two coupled circuits with the source of power in one. If there is no loss in the coupling mechanism, the power dissipated in both circuits is

$$P = \tfrac{1}{2}R_1 I_1 I_1^* + \tfrac{1}{2}R_2 I_2 I_2^*,$$

where R_1 and R_2 are the resistances of the circuits. The same power can also be expressed as

$$P = \tfrac{1}{2}(R_1 + R_c) I_1 I_1^*,$$

where R_c is the resistance of the passive circuit coupled into the active circuit.

Consider now two perfectly conducting capacitor antennas, that is, current elements connecting pairs of parallel plates. Let one antenna be active and the other passive. Let R_{11} and R_{22} be the radiation resistances of the antennas when isolated from each other. Let R_{12} be the mutual radiation resistance. The radiated power may then be expressed in the following form:

$$P = \tfrac{1}{2}[R_{11}I_1I_1^* + R_{12}(I_1I_2^* + I_1^*I_2) + R_{22}I_2I_2^*].$$

Note that this expression does not distinguish between active and passive antennas. Show that this expression reduces to

$$P = \tfrac{1}{2}\text{re}\, V_1 I_1^*$$

when the second antenna is passive.

12.7–1 Obtain the magnetic intensity of a sinusoidal current filament extending from $z = z_1$ to $z = z_2$.

Ans.
$$4\pi j\beta\rho H_\varphi = [I'(z_2) - j\beta I(z_2)\cos\theta_2]\,e^{-j\beta r_2} - [I'(z_1) - j\beta I(z_1)\cos\theta_1]\,e^{-j\beta r_1},$$

where θ_1 and θ_2 are the angles made by r_1 and r_2 with the segment (z_1, z_2).

12.7–2 Obtain the radial component of the electric intensity for a sinusoidal current filament.

Ans.
$$4\pi j\omega\varepsilon\rho E_\rho = I'(z_2)\,e^{-j\beta r_2}\cos\theta_2 - I'(z_1)\,e^{-j\beta r_1}\cos\theta_1 + I(z_1)\left(j\beta\cos^2\theta_1 - \frac{\sin^2\theta_1}{r_1}\right)e^{-j\beta r_1} - I(z_2)\left(j\beta\cos^2\theta_2 - \frac{\sin^2\theta_2}{r_2}\right)e^{-j\beta r_2}.$$

12.7–3 Show that the formulas for the electric intensity of a sinusoidal current filament may be expressed as follows:

$$4\pi j\omega\varepsilon E_z = I'(z_1)\,\frac{e^{-j\beta r_1}}{r_1} - I'(z_2)\,\frac{e^{-j\beta r_2}}{r_2} - \frac{\partial V}{\partial z},$$

$$4\pi j\omega\varepsilon\rho E_\rho = [I'(z_2)\cos\theta_2 - j\beta\, I(z_2)]e^{-j\beta r_2} - [I'(z_1)\cos\theta_1 - j\beta\, I(z_1)]e^{-j\beta r_1} - \frac{\partial V}{\partial \rho},$$

where V is the electric potential of the end charges. Hence, in any connected network V contributes nothing to E.

12.12–1 Consider a folded dipole consisting of two parallel wires whose radii are a_1 and a_2. Assume that each cylinder is of length $2l$. Show that the admittance seen at the center of the first wire is approximately

$$Y_i = -\tfrac{1}{2}jA_1\cot\beta l + Y_p k_2^2(k_1 + k_2)^{-2},$$

where A_1 is the characteristic impedance of the wires energized in push-pull, Y_p is the input admittance of the two wires connected in parallel, and $k_1 = \log(s/a_1)$, $k_2 = \log(s/a_2)$, where the interaxial distance s is assumed to be larger than $2(a_1 + a_2)$.

Note. To obtain this result, consider first two infinitely long wires surrounded by a cylindrical shield whose radius R is very large and which is substantially coaxial with either wire. Show then that, if q_1 and q_2 are the linear charge densities, the potentials are

$$V_1 = \frac{q_1}{2\pi\epsilon}\log\frac{R}{a_1} + \frac{q_2}{2\pi\epsilon}\log\frac{R}{s},$$

$$V_2 = \frac{q_1}{2\pi\epsilon}\log\frac{R}{s} + \frac{q_2}{2\pi\epsilon}\log\frac{R}{a_2}.$$

From these expressions show that, in the push-pull mode of propagation, in which the charge densities are equal and of opposite signs, the potentials are in the ratio $-k_1/k_2$, while in the push-push mode, in which the potentials are equal, the charges are in the ratio k_2/k_1.

12.13–1 Evaluate the square of the instantaneous length of the E vector (equation 85).

Ans. $\frac{1}{2}(A^2 + B^2) + \frac{1}{2}(A^2 + B^2 \cos 2\vartheta) \cos 2\omega t - \frac{1}{2}B^2 \sin 2\vartheta \sin 2\omega t.$

12.13–2 Using the result of the preceding problem, obtain the instants at which the instantaneous length of the E vector is either maximum or minimum. Calculate the maximum and minimum lengths.

Ans. $$t = \frac{1}{2\omega} \tan^{-1} \frac{-B^2 \sin 2\vartheta}{A^2 + B^2 \cos 2\vartheta};$$

$$[\tfrac{1}{2}(A^2 + B^2) \pm \tfrac{1}{2}(A^4 + 2A^2B^2 \cos 2\vartheta + B^4)^{1/2}]^{1/2}.$$

13

IMPEDANCE OF DIPOLE ANTENNAS

13.1 Interaction between antennas

In Section 12.3 we derived general expressions for the transducer parameters (see equations 9–74 and 9–75) of two linear antennas in terms of the antenna current distributions. If one of these antennas is used for transmission and the other for reception, the distance between them is so large that the impedance of either antenna is not affected by the conditions at the terminals of the other. Hence, $Z_{11} = Z_1$ and $Z_{22} = Z_2$, where Z_1 and Z_2 are the input impedances of the antennas when each is alone. The mutual impedance Z_{12} is by definition the voltage induced across the open terminals of one antenna when the current across the input terminals of the other is unity. To obtain Z_{12} we should calculate the field of the second antenna at the first and from it obtain the required induced voltage. Similarly, $Y_{11} = Y_1 = 1/Z_1$ and $Y_{22} = Y_2 = 1/Z_2$. The mutual admittance Y_{12} is, by definition, the current across the short-circuited terminals of one antenna when the voltage across the terminals of the other is unity. In any particular case we have to obtain only the Z's or the Y's since one set of these parameters may be expressed in terms of the other (Section 9.7).

When the antennas are fairly close as in an array, the interaction between them may not be negligible. The transducer parameters may then be obtained by successive approximations. To begin with we neglect the interaction and calculate, let us say, the Z's. In addition, we calculate the currents induced at various points of one antenna by the first-order currents in the other. The induced currents should be obtained on the assumption that the impedances across the terminals of both antennas are infinite, since, in calculating the Z's, we must keep the input currents unchanged. From the induced currents we obtain the corrections to the voltages induced across the terminals, that is, the corrections to the Z's. Theoretically we can continue this process

indefinitely: from the first-order induced currents we can obtain the second-order induced currents and second corrections to the Z's, etc.

If we wish to calculate the Y's, we may use the same method. This time, however, we must keep the voltages across the terminals unchanged. Therefore, the induced currents should be obtained on the assumption that the impedances across the input terminals are equal to zero.

In practice, these calculations are very complicated, and at best we have to be satisfied with one or at most two successive approximations. Only in some idealized cases the results are fairly simple. Thus, if the antennas are infinitely thin, the current induced in one antenna

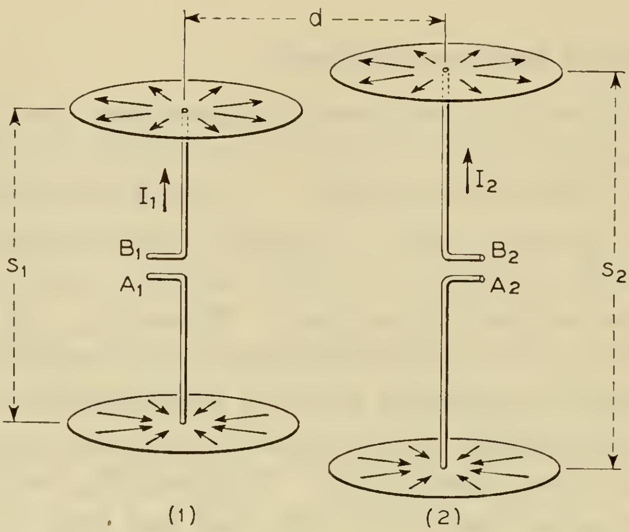


FIG. 13.1 Capacitor antennas.

by a finite current in the other is zero. Hence, the current distributions in both antennas are sinusoidal. In this case we can obtain the exact values of Z_{12}, R_{11}, R_{22}, and the asymptotic values of X_{11} and X_{22}.

Another case is that of two "capacitor antennas." We shall consider it in order to illustrate the method of successive approximations outlined above. A *capacitor antenna* is an antenna so heavily loaded with capacitance at the ends that the current in it is substantially constant. Its field, therefore, is nearly identical with the field of an electric current element of the same length. The currents in the loading plates flow in all directions from the main radiating element, and their fields tend to cancel. In what follows we shall neglect these currents. Figure 13.1 shows two such antennas, and we shall assume that the distance d between them is large compared with the length of the longer capacitor antenna. The main characteristics of a capacitor antenna are: (1) The current at various points equals the input current;

(2) the current vanishes when the terminals are disconnected. These properties simplify the theory of interaction between the capacitor antennas, particularly when the impedances across their terminals are infinite.

Let the *free-space transmission* factor T be

$$T = -\frac{60\pi j}{\lambda d}\left(1 + \frac{1}{j\beta d} - \frac{1}{\beta^2 d^2}\right) e^{-j\beta d}. \tag{1}$$

With this definition, the electric intensity of the first antenna along the second antenna (in the direction of I_2) is

$$E_{s,1} = TI_1 s_1. \tag{2}$$

Hence, the voltage impressed on the second antenna by the first, or the induced voltage, is

$$V_2{}^i = TI_1 s_1 s_2. \tag{3}$$

The voltage which we have to impress on the second antenna in order to counteract this voltage is $-V_2{}^i$; hence, the mutual impedance is

$$Z_{12} = -Ts_1 s_2. \tag{4}$$

The self-impedance Z_{11} is the input impedance of the first antenna when the terminals of the second are floating; then there is no current in the second antenna and no reaction from it on the first antenna. Consequently, Z_{11} equals the input impedance Z_1 of the antenna in free space. Thus,

$$Z_{11} = Z_1 = R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right),\ Z_{22} = R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right). \tag{5}$$

The resistive components include the ohmic resistances of the wires and the radiation resistances. Thus,

$$R_1 = \bar{R}s_1 + 80\pi^2\left(\frac{s_1}{\lambda}\right)^2, \tag{6}$$

where $\bar{R}$ is the ohmic resistance per unit length. For the inductance we find

$$L_1 = \frac{\mu s_1}{2\pi}\left(\log\frac{s_1}{a_1} - 1\right). \tag{7}$$

The capacitance will depend on the loading.

From Z_{11}, Z_{12}, Z_{22} we can find Y_{11}, Y_{12}, Y_{22}. The problem of finding the admittances directly from the field equations is somewhat more complicated, because in this case the impedances across the antenna terminals must be assumed equal to zero and the induced

currents do not vanish. We begin by neglecting the interaction altogether. The self-admittances of the antennas are then equal to their respective input admittances in free space,

$$Y_{11}^{(1)} = Y_1 = \frac{1}{Z_1}, \qquad Y_{22}^{(1)} = Y_2 = \frac{1}{Z_2}, \tag{8}$$

where Z_1 and Z_2 are given by equation 5. By our assumption the first approximation to the mutual admittance is zero,

$$Y_{12}^{(1)} = 0. \tag{9}$$

To obtain the second approximation to Y_{12}, we impress a unit voltage on the first antenna and calculate the current in the second when the terminals of the latter are short-circuited. The current in the first antenna is Y_1, its moment $Y_1 s_1$, its electric intensity along the second antenna $Y_1 s_1 T$, its voltage impressed on the second antenna $Y_1 s_1 T s_2$, and, finally, the current produced in the second antenna is $Y_1 s_1 T s_2 Y_2$. Hence,

$$Y_{12}^{(2)} = Y_1 Y_2 s_1 T s_2. \tag{10}$$

To obtain the next approximation, we note that the current given by equation 10 induces a current in the first antenna which in turn alters the current in the second antenna. Thus, the current of moment $Y_{12}^{(2)} s_2$ produces an electric intensity $Y_{12}^{(2)} s_2 T$ along the first antenna, a voltage $Y_{12}^{(2)} s_2 T s_1$, and a current $Y_{12}^{(2)} s_2 T s_1 Y_1$. The moment of this current is $Y_{12}^{(2)} s_2 T s_1 Y_1 s_1$, its electric intensity along the second antenna $Y_{12}^{(2)} s_2 T s_1 Y_1 s_1 T$, its voltage $Y_{12}^{(2)} s_2 T s_1 Y_1 s_1 T s_2$, and the induced current $Y_{12}^{(2)} s_2 T s_1 Y_1 s_1 T s_2 Y_2$. This current is induced in addition to that given by equation 10; hence,

$$Y_{12}^{(3)} = Y_{12}^{(2)} + Y_{12}^{(2)} (s_2 T s_1)^2 Y_1 Y_2. \tag{11}$$

The current represented by the second term produces a field at the first antenna, and induces a current in it, which in turn induces a current in the second antenna. The multiplication factor is evidently the same as in the preceding case, $(s_2 T s_1)^2 Y_1 Y_2$; hence,

$$Y_{12}^{(4)} = Y_{12}^{(2)} + Y_{12}^{(2)} (s_2 T s_1)^2 Y_1 Y_2 + Y_{12}^{(2)} (s_2 T s_1)^4 (Y_1 Y_2)^2. \tag{12}$$

Continuing the sequence and using equations 8 and 10, we find the exact value of Y_{12},

$$\begin{aligned} Y_{12} &= \frac{s_1 T s_2}{Z_1 Z_2} \left[1 + \frac{(s_1 T s_2)^2}{Z_1 Z_2} + \frac{(s_1 T s_2)^4}{(Z_1 Z_2)^2} + \frac{(s_1 T s_2)^6}{(Z_1 Z_2)^3} + \cdots \right] \\ &= \frac{s_1 T s_2}{Z_1 Z_2} \left[1 - \frac{(s_1 T s_2)^2}{Z_1 Z_2} \right]^{-1}. \end{aligned} \tag{13}$$

In view of equation 4, we have,

$$Y_{12} = -\frac{Z_{12}}{Z_1Z_2 - Z_{12}^2}, \tag{14}$$

an equation obtainable directly from the transducer equations.

In the present case of two capacitor antennas, the calculation of the impedance coefficients is quick and simple because no current is induced in such antennas when their terminals are floating. In general, this is not the case, and the method of calculating the impedances is similar to the above method of obtaining the admittances. We start with the conditions existing in the antennas when they are in free space and calculate successive interactions. The procedure is the same as in the case of reflection from two impedance discontinuities.*

13.2 Asymptotic fields of straight antennas

Consider a sinusoidal current filament whose slope is discontinuous in the center; if the filament extends from $z = -l$ to $z = l$, the current is

$$I(z) = I_0 \sin \beta(l - |z|). \tag{15}$$

The derivative is

$$I'(z) = \mp\beta I_0 \cos \beta(l - |z|), \qquad z \gtrless 0. \tag{16}$$

Therefore,

$$\begin{aligned} I(-l) &= I(l) = 0, & I(-0) &= I(0) = I_0 \sin \beta l, \\ I'(-l) &= \beta I_0, & I'(-0) &= \beta I_0 \cos \beta l, \\ I'(+0) &= -\beta I_0 \cos \beta l, & I'(l) &= -\beta I_0. \end{aligned} \tag{17}$$

Applying equation 12–47 to the two segments, we have

$$E_z = 30jI_0 \left(2\frac{e^{-j\beta r_0}}{r_0} \cos \beta l - \frac{e^{-j\beta r_1}}{r_1} - \frac{e^{-j\beta r_2}}{r_2} \right), \tag{18}$$

where r_0, r_1, r_2 are the distances from the center and the ends of the current filament (Fig. 13.2).

We shall now expand the proof given in Section 12.8, showing that, as the radius of a perfectly conducting transmitting antenna fed at the center approaches zero, the longitudinal electric intensity is, in the limit, given by equation 18 at any distance from the antenna greater than zero, and that, at $\rho = 0$, $E_z = 0$ (except at $z = 0$). In Chapter 8 we have shown that the current in such an antenna approaches the form given by equation 15. The residual current is just sufficient to make E_z equal to zero on both arms of the antenna. Suppose now that we fix our attention on a point P *outside* the antenna and at distance ρ from its

* *Electromagnetic Waves*, pp. 224–225.

axis. The field at this point equals the average value of equation 18 around the surface of the wire plus the field of the residual current. As the radius approaches zero, the average value of equation 18 approaches its value for the entire current concentrated along the axis of the antenna. The residual current and its field at P vanish in the limit. This is true for any ρ different from zero. Our argument applies only to points outside the antenna; hence, the points on the axis of the antenna remain always excluded. Since, however, $E_z(a) = 0$ at all points on each antenna arm for any a, we have in the limit $E_z(0) = 0$. Thus, the limit function is discontinuous at $\rho = 0$.

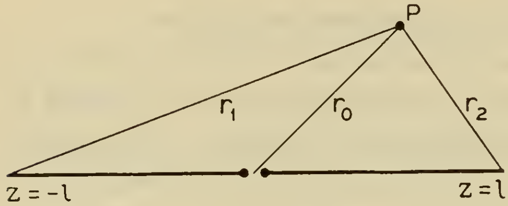


FIG. 13.2 A center-fed antenna.

It is left to the student to derive the following expressions for the electric intensity parallel to the symmetric and antisymmetric current distributions (equations 8–109 and 8–110):

Symmetric mode (Fig. 8.9*a*):

$$E_z = 30jA\left\{\left[\frac{\exp(-j\beta r_\xi)}{r_\xi} + \frac{\exp(-j\beta r_{-\xi})}{r_{-\xi}}\right]\cos\beta l - \left[\frac{\exp(-j\beta r_l)}{r_l} + \frac{\exp(-j\beta r_{-l})}{r_{-l}}\right]\cos\beta\xi\right\}, \qquad \rho > 0. \tag{19}$$

Antisymmetric mode (Fig. 8.9*b*):

$$E_z = 30jA\left\{\left[\frac{\exp(-j\beta r_\xi)}{r_\xi} - \frac{\exp(-j\beta r_{-\xi})}{r_{-\xi}}\right]\sin\beta l + \left[\frac{\exp(-j\beta r_{-l})}{r_{-l}} - \frac{\exp(-j\beta r_l)}{r_l}\right]\sin\beta\xi\right\}, \qquad \rho > 0. \tag{20}$$

The subscripts indicate the points from which the distances are measured.

13.3 Asymptotic expression for the mutual impedance of two parallel center-fed antennas

Consider two parallel center-fed antennas (Fig. 13.3). Substituting from equations 15 and 18 in equation 12–27, we obtain an asymptotic expression for the mutual impedance of these antennas,

$$Z_{12} = \frac{60j}{\sin^2\beta l}\int_0^l \left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} - 2\frac{e^{-j\beta r_0}}{r_0}\cos\beta l\right) \times \sin\beta(l - z)\,dz. \tag{21}$$

This integral may be evaluated in terms of sine and cosine integrals. Thus, noting that

$$\sin \beta(l - z) = \sin \beta l \cos \beta z - \cos \beta l \sin \beta z$$
$$= \tfrac{1}{2}(e^{j\beta z} + e^{-j\beta z}) \sin \beta l + \tfrac{1}{2}j(e^{j\beta z} - e^{-j\beta z}) \cos \beta l, \qquad (22)$$

we obtain

$$Z_{12} = \frac{Z_{12}{}^a}{\sin^2 \beta l} = \frac{R_{12}{}^a + jX_{12}{}^a}{\sin^2 \beta l}, \qquad (23)$$

where

$$\begin{aligned} R_{12}{}^a = {} & 60[2 \operatorname{Ci} \beta\rho - \operatorname{Ci} \beta(r_{04} + l) - \operatorname{Ci} \beta(r_{04} - l)] + \\ & 30[2 \operatorname{Ci} \beta\rho - 2 \operatorname{Ci} \beta(r_{04} + l) - 2 \operatorname{Ci} \beta(r_{04} - l) + \\ & \operatorname{Ci} \beta(r_{14} + 2l) + \operatorname{Ci} \beta(r_{14} - 2l)] \cos 2\beta l + \\ & 30[2 \operatorname{Si} \beta(r_{04} - l) - 2 \operatorname{Si} \beta(r_{04} + l) + \\ & \operatorname{Si} \beta(r_{14} + 2l) - \operatorname{Si} \beta(r_{14} - 2l)] \sin 2\beta l, \end{aligned} \qquad (24)$$

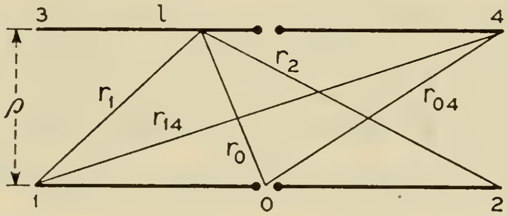


FIG. 13.3 Two parallel antennas.

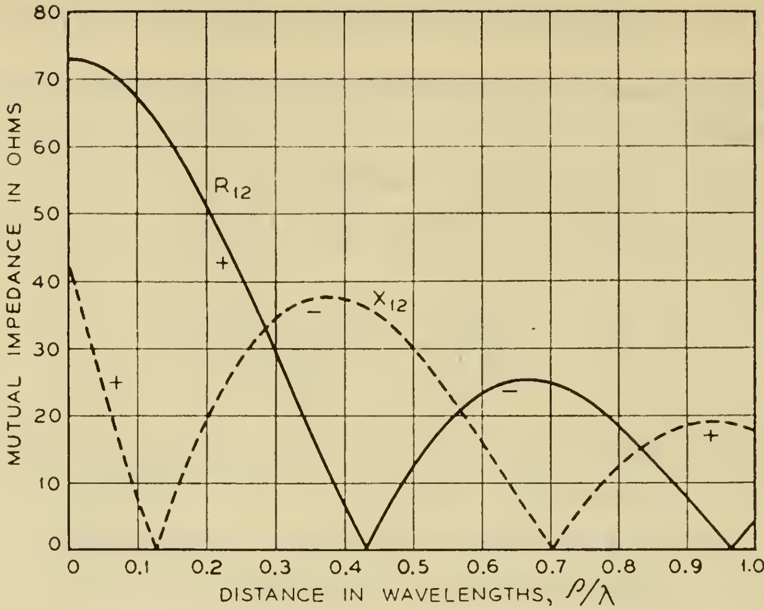
$$\begin{aligned} X_{12}{}^a = {} & 60 [\operatorname{Si} \beta(r_{04} + l) + \operatorname{Si} \beta(r_{04} - l) - 2 \operatorname{Si} \beta\rho] + \\ & 30[2 \operatorname{Si} \beta(r_{04} + l) + 2 \operatorname{Si} \beta(r_{04} - l) - 2 \operatorname{Si} \beta\rho - \\ & \operatorname{Si} \beta(r_{14} + 2l) - \operatorname{Si} \beta(r_{14} - 2l)] \cos 2\beta l + \\ & 30[2 \operatorname{Ci} \beta(r_{04} - l) - 2 \operatorname{Ci} \beta(r_{04} + l) + \\ & \operatorname{Ci} \beta(r_{14} + 2l) - \operatorname{Ci} \beta(r_{14} - 2l)] \sin 2\beta l. \end{aligned} \qquad (25)$$

When $\beta\rho \ll 1$ and $\rho \ll l$, the above equations may be simplified by using the following approximations:

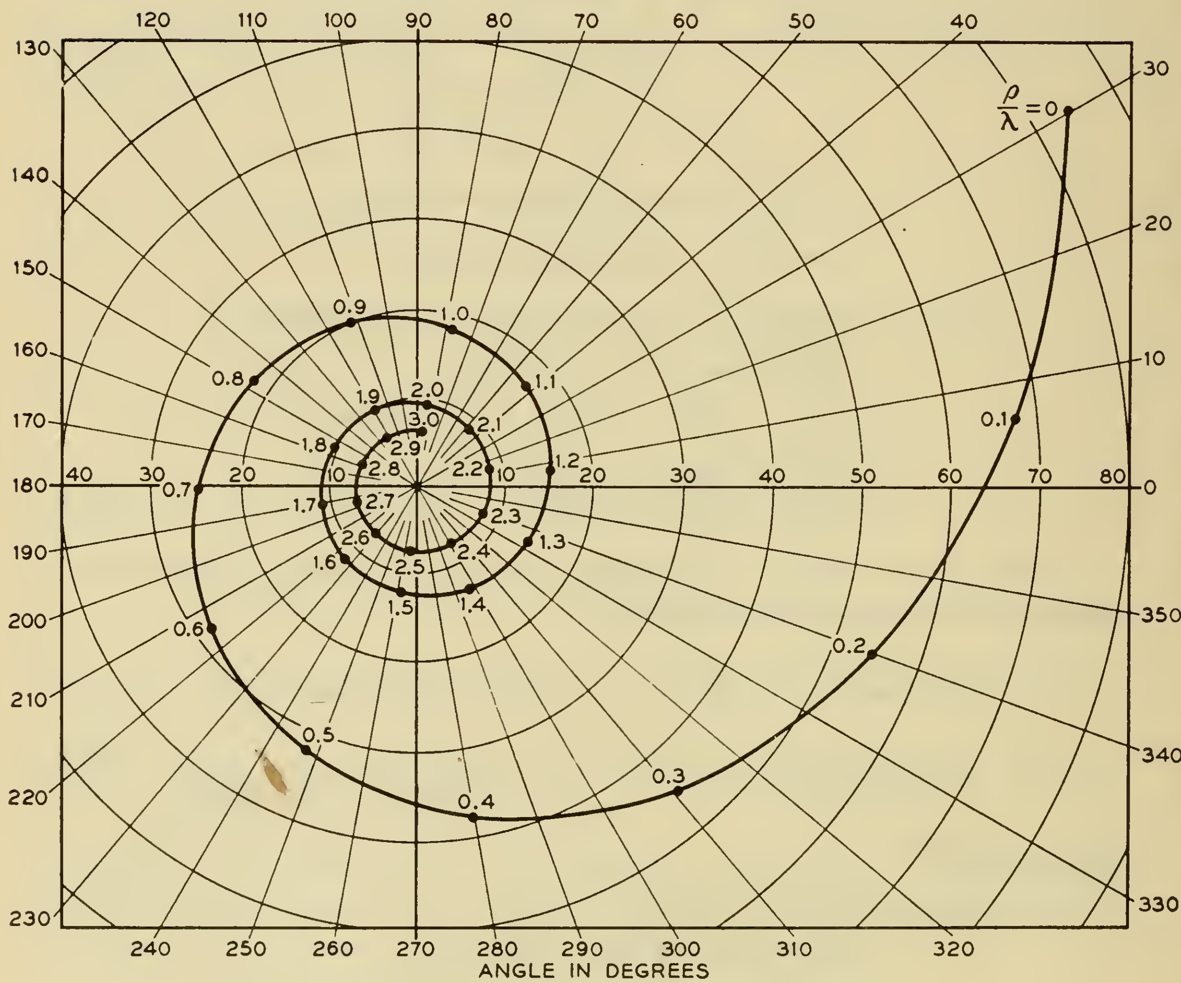
$$\sqrt{l^2 + \rho^2} \simeq l + \frac{\rho^2}{2l}, \qquad \sqrt{4l^2 + \rho^2} \simeq 2l + \frac{\rho^2}{4l},$$
$$\operatorname{Si} \beta\rho \simeq 0, \qquad \operatorname{Ci} \beta\rho = C + \log \beta\rho. \qquad (26)$$

Thus, we find

$$\begin{aligned} R_{12}{}^a = {} & 60(C + \log 2\beta l - \operatorname{Ci} 2\beta l) + \\ & 30 (\operatorname{Si} 4\beta l - 2 \operatorname{Si} 2\beta l) \sin 2\beta l + \\ & 30(C + \log \beta l - 2 \operatorname{Ci} 2\beta l + \operatorname{Ci} 4\beta l) \cos 2\beta l \\ = {} & 60 \operatorname{Cin} 2\beta l + 30 (\operatorname{Si} 4\beta l - 2 \operatorname{Si} 2\beta l) \sin 2\beta l + \\ & 30(2 \operatorname{Cin} 2\beta l - \operatorname{Cin} 4\beta l) \cos 2\beta l, \end{aligned} \qquad (27)$$



(a)



(b)

FIG. 13.4 The mutual impedance between two parallel half-wave antennas: in (a) the solid curve represents the mutual resistance and the broken curve the mutual reactance; in (b) the amplitude and phase are shown as a polar diagram.

$$X_{12}{}^a = 60\,\mathrm{Si}\,2\beta l + 30(2\,\mathrm{Si}\,2\beta l - \mathrm{Si}\,4\beta l)\cos 2\beta l -$$
$$30\left(\log\frac{l\lambda}{\rho^2} - C - \log 2\pi - \mathrm{Ci}\,4\beta l + 2\,\mathrm{Ci}\,2\beta l\right)\sin 2\beta l$$
$$= 60\,\mathrm{Si}\,2\beta l + 30(2\,\mathrm{Si}\,2\beta l - \mathrm{Si}\,4\beta l)\cos 2\beta l +$$
$$30\left(2\,\mathrm{Cin}\,2\beta l - \mathrm{Cin}\,4\beta l - 2\log\frac{l}{\rho}\right)\sin 2\beta l, \tag{28}$$

where $C = 0.577\cdots$ is Euler's constant.

The superscript a in the above equations anticipates the result of Section 13.6, namely, that $Z_{12}{}^a$ is the mutual impedance with reference to current antinodes, that is, with reference to the maximum current amplitudes. We shall make a a subscript, when there are no other subscripts.

Equation 23 gives the exact value for the mutual impedance when both antennas are infinitely thin. In this limiting case the current in the first antenna induces a certain voltage across the terminals of the second antenna but no current along it, since the inductance per unit length is infinite. Since there is no induced current, there is no reaction on the first antenna and no secondary action on the second antenna.

13.4 Mutual impedance of infinitely thin half-wave antennas

If $l = \lambda/4$, we have

$$R_{12} = 60\,\mathrm{Ci}\,\beta\rho - 30\,\mathrm{Ci}(u+\pi) - 30\,\mathrm{Ci}(u-\pi),$$
$$X_{12} = 30\,\mathrm{Si}(u+\pi) + 30\,\mathrm{Si}(u-\pi) - 60\,\mathrm{Si}\,\beta\rho, \tag{29}$$
$$u = \sqrt{\left(\frac{2\pi\rho}{\lambda}\right)^2 + \pi^2}.$$

The solid curve in Fig. 13.4a represents the mutual resistance and the broken curve the mutual reactance. Figure 13.4b shows the amplitude and phase for various values of ρ/λ.

The mutual impedance between two quarter-wave antennas above a perfect ground (Fig. 13.5) is one half of the mutual impedance between two half-wave antennas in free space.

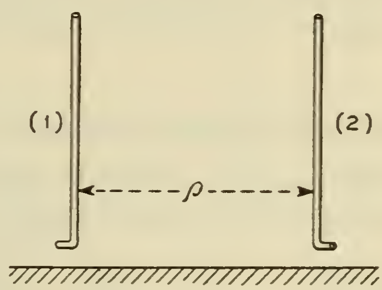


FIG. 13.5 Two parallel vertical antennas.

As ρ approaches zero, the mutual impedance between two half-wave antennas approaches a definite limit. Since the input impedance is the average of the mutual impedance around the antenna (See Section 12.4), we find that the input impedance of the

half-wave antenna is simply the limit of Z_{12} as $\rho \to 0$. Thus,

$$Z_i = 30 \operatorname{Cin} 2\pi + 30j \operatorname{Si} 2\pi = 73.13 + 42.54j. \tag{30}$$

As with all the formulas of this section, equation 30 refers to an infinitely thin half-wave antenna. It is an approximation for thin antennas.

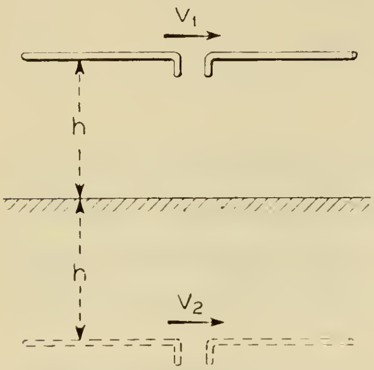


FIG. 13.6 A horizontal antenna.

13.5 Impedance of horizontal half-wave antennas above a perfect ground

In the case of a horizontal antenna above a perfect ground (Fig. 13.6), we can replace the ground by an image antenna. In the image $V_2 = -V_1$, $I_2 = -I_1$, and the impedance is

$$Z = \frac{V_1}{I_1} = Z_{11} - Z_{12}. \tag{31}$$

Figure 13.7 shows Z for a horizontal half-wave antenna as a function of h.

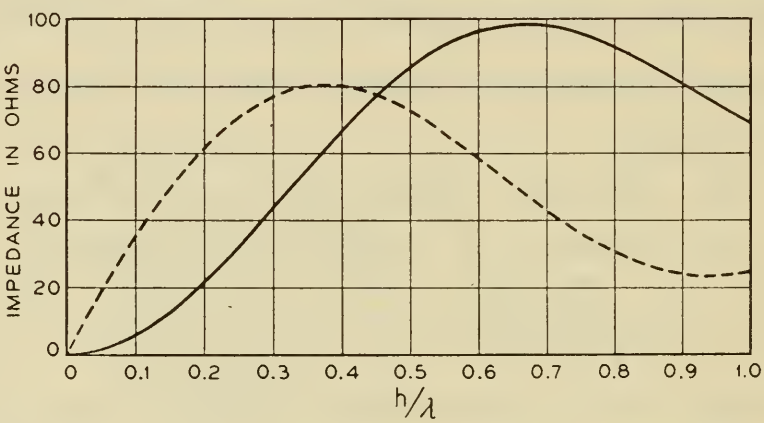


FIG. 13.7 The input impedance of a horizontal half-wave antenna over a perfect ground: the solid curve represents the input resistance and the broken curve the input reactance.

13.6 Mutual radiation of parallel antennas

The mutual complex power of two parallel infinitely thin antennas (Fig. 13.3) can be obtained from equations 23 and 12–23. Noting that

$$I_{i,1} = I_1 \sin \beta l, \qquad I_{i,2} = I_2 \sin \beta l, \tag{32}$$

where I_1 and I_2 are the maximum amplitudes, we have

$$2\Psi_{12} = \tfrac{1}{2} Z_{12}{}^a (I_1 I_2{}^* + I_1{}^* I_2) = Z_{12}{}^a \operatorname{re}(I_1 I_2{}^*). \tag{33}$$

This result can also be obtained directly from equation 12–19.

If I_1 and I_2 are in phase, we have

$$2\Psi_{12} = Z_{12}{}^a I_1 I_2{}^*. \tag{34}$$

13.7 Radiation from a single antenna

In accordance with equation 12–21, the complex radiated power for an antenna is the average of the mutual power round the antenna. For a thin antenna, the expression 27 for the mutual radiation resistance with reference to the maximum current amplitude is seen to be independent of the radius of the antenna. In the mutual reactance the only term depending on the radius is $\log(l/\rho)$; its average value around the cylinder is* $\log(l/a)$. Hence,

$$\Psi = \tfrac{1}{2} Z_a I_0{}^2, \tag{35}$$

where Z_a is obtained from $Z_{12}{}^a$ by substituting a for ρ, and I_0 is the maximum amplitude of the antenna current.

13.8 Half-wave vertical antennas above ground

A half-wave vertical antenna just over the ground and its image (Fig. 13.8) form a full-wave antenna. Hence, its impedance is half the impedance obtained from equations 27 and 28 by substituting $\beta l = \pi$; thus, in the limit, as the radius vanishes,

$$Z = 99.54 + j62.72. \tag{36}$$

This expression is for an infinitely thin half-wave antenna. At resonance an antenna of finite radius will be shorter, but the shape of the current distribution will be substantially the same; hence, the impedance of a self-resonant antenna just above a perfect ground is approximately

$$Z = 99.5. \tag{37}$$

The ground currents near the base are small since the magnetic field there is small. At greater distances some power will be absorbed when the ground is not perfectly conducting, but this power has already left the antenna. Hence, the effect of finite conductivity on the impedance of the half-wave vertical antenna should be small.

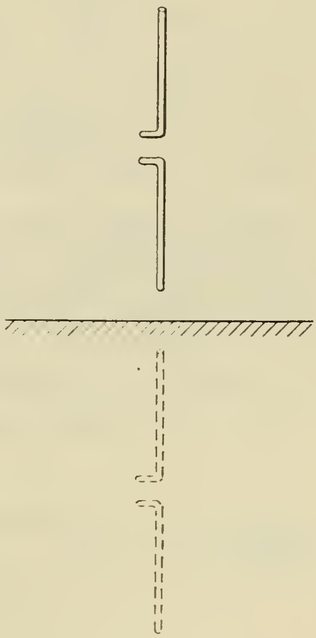


Fig. 13.8 A vertical antenna just above ground.

* See equation 8–88.

13.9 Asymptotic formula for the impressed voltage

Equation 18 gives the exact value of the electric intensity produced by a sinusoidal current filament (equation 15) in the direction parallel to the filament. If the current is distributed uniformly around a cylinder of radius a, the intensity is given (exactly) by the average value of equation 18 around the cylinder. In order to produce the distribution of current given by equation 15 on the surface of a cylinder of *finite* radius, we should, therefore, impress a continuously distributed field equal and opposite to the average value of equation 18,

$$E_z^i = 60jI_0 \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{e^{-j\beta r_1}}{2r_1} + \frac{e^{-j\beta r_2}}{2r_2} - \frac{e^{-j\beta r_0}}{r_0} \cos \beta l \right) d\varphi, \tag{38}$$

$$\rho = 2a \sin \frac{\varphi}{2} \cdot$$

Part of this impressed field is almost entirely concentrated in the center; it is given by the real part of the last term in the integrand,

$$E_{z,p}^i = -60jI_0 \cos \beta l \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos \beta r_0}{r_0}\, d\varphi, \tag{39}$$

and we shall call it the *principal part* of the impressed field. As a approaches zero, the voltage arising from this part becomes concentrated nearer and nearer the center. There are also strong components near the ends of the antenna; but these are not very effective in producing the current, because of the high impedance of the antenna as seen from a point very near the end. Thus, we have a direct verification of the conclusion already reached in several different ways to the effect that, as the radius of the transmitting antenna approaches zero, the current in it becomes more nearly sinusoidal. The impressed voltage producing this current is, asymptotically,

$$V \sim \int_{-l}^{l} E_{z,p}^i \, dz = -j\hat{K} I_0 \cos \beta l, \tag{40}$$

where

$$\hat{K} = 60 \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-l}^{l} \frac{\cos \beta r_0}{r_0}\, dz. \tag{41}$$

This integral is a special case of equation 8–89 with $z = 0$, $z_1 = -l$, $z_2 = l$; hence,

$$\hat{K} = 120 \left(\log \frac{2l}{a} - \operatorname{Cin} \beta l \right) = 120 \left(\log \frac{\lambda}{2\pi a} + 0.116 + \operatorname{Ci} \beta l \right) \cdot \tag{42}$$

An asymptotic expression for the input impedance is thus

$$Z_i = \frac{V}{I(0)} = -j\hat{K} \cot \beta l. \tag{43}$$

It is seen to be identical with the input impedance of a transmission line of length l, open at the far end, whose characteristic impedance is $\hat{K}$. That it is a pure reactance is not surprising, since we have sought and obtained the limiting expression as a approaches zero and, hence, $\hat{K}$ approaches infinity. In these circumstances the energy stored around the antenna approaches infinity while the radiated power remains constant. In practice, however, $\hat{K}$ is never excessively large and is often deliberately made small; hence, we shall need a better approximation than equation 43. We shall obtain it in the following three sections.

Equation 43 can also be obtained from equations 8–90 if we use the values of L and C at the input end $z = 0$. If, instead, we use the average values of L and C, we obtain a similar equation 8–117,

$$Z_i = -2jZ_0 \cot \beta l, \tag{44}$$

where

$$2Z_0 = 120\left(\log \frac{\lambda}{2\pi a} + 0.116 + \operatorname{Ci} 2\beta l - \frac{\sin 2\beta l}{2\beta l}\right) \cdot \tag{45}$$

The difference,

$$\hat{K} - 2Z_0 = 120\left(\operatorname{Ci} \beta l - \operatorname{Ci} 2\beta l + \frac{\sin 2\beta l}{2\beta l}\right), \quad (46$$

approaches 37 ohms when l/λ approaches zero; as l/λ approaches infinity, the difference approaches zero. In any case this difference becomes smaller in comparison with either value as the radius of the antenna diminishes.

13.10 Asymptotic formula for the input impedance of a symmetric antenna

In accordance with equation 12–32, the input impedance is the average value of the mutual impedance around the antenna. Hence, its asymptotic expression may be obtained from equation 23,

$$Z_i = \frac{Z_a}{\sin^2 \beta l}, \tag{47}$$

where Z_a is given by equations 27 and 28 with $\rho = a$. Thus, we find

$$Z_i = -j\hat{K} \cot \beta l + \frac{\hat{R} + j\hat{X}}{\sin^2 \beta l}, \tag{48}$$

where

$$\hat{R} = 60 \text{ Cin } 2\beta l + 30(\text{Si } 4\beta l - 2 \text{ Si } 2\beta l) \sin 2\beta l + 30(2 \text{ Cin } 2\beta l - \text{Cin } 4\beta l) \cos 2\beta l, \tag{49}$$

$$\hat{X} = 60 \text{ Si } 2\beta l + 30(2 \text{ Si } 2\beta l - \text{Si } 4\beta l) \cos 2\beta l + 30(2 \text{ Cin } 2\beta l - \text{Cin } 4\beta l - 2 \text{ Cin } \beta l + 2 \log 2) \sin 2\beta l. \tag{50}$$

We may also write

$$Z_i = -2jZ_0 \cot \beta l + \frac{\hat{R} + j[\hat{X} + (Z_0 - \frac{1}{2}\hat{K}) \sin 2\beta l]}{\sin^2 \beta l}. \tag{51}$$

Expressions 48 and 51 are identical except in form and give a higher-order approximation to Z_i than the expressions in the preceding section.

13.11 Asymptotic formula for the input admittance

The formulas of the preceding section are exact for antennas of zero radius; but, for actual antennas, no matter how thin, they are approximate. The principal source of error is in the denominator and is due to the difference between the actual input current and the asymptotic input current. Obviously, this error will be proportionally larger, the smaller is $\sin \beta l$. Hence, these formulas deteriorate progressively as l approaches a multiple of $\lambda/2$. We shall now obtain a companion formula for the input admittance, which will be more accurate in the vicinity of $l = n\lambda/2$ and will deteriorate progressively as we approach a resonant condition. In the next section we shall combine these formulas into one for the entire range.

In a transmitting antenna the electromotive force is impressed in a highly localized region. To obtain the sinusoidal current we had to impress $E_z{}^i = -E_z$, given by equation 38. This gave us a voltage $V = -j\hat{K}I_0 \cos \beta l$, largely but not completely localized. Let us now subject the antenna to an additional impressed field equal to $-E_z{}^i = E_z$ along the antenna and a concentrated emf at $z = 0$. The former will wipe out the original distributed field. In order to obtain the input impedance, we need only the change in the input current produced by the additional impressed field; this change is

$$I_1 = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{-l}^{l} E_z\, Y(z; 0)\, dz - j\hat{K}I_0 \cos \beta l\, Y(0; 0), \tag{52}$$

where the transfer admittance $Y(z; 0)$ is the current at $z = 0$ due to a unit voltage at* $z = z$. By the reciprocity theorem, the transfer admit-

* The transfer admittance $Y(0;\, 0)$ is, of course, the input admittance. In these formulas it is to be interpreted as $\lim Y(0;\, \frac{1}{2}s)$ as the length s of the gap approaches zero, while always remaining larger than the diameter of the antenna.

tance is a symmetric function, $Y(z;\ 0) = Y(0;\ z)$. Now the approximate value of the current $Y(0;\ z)$ produced by a unit voltage at $z = 0$ is obtained if we divide equation 15 by equation 40; thus,

$$Y(0;z) = Y(z;0) = -\frac{\sin\beta(l - |z|)}{j\hat{K}\cos\beta l}. \tag{53}$$

Substituting in equation 52, we have

$$I_1 = -\frac{1}{j\hat{K}\cos\beta l}\,\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-l}^{l} E_z \sin\beta(l - |z|)\,dz + I_0\sin\beta l. \tag{54}$$

Since*

$$\begin{aligned} -\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-l}^{l} E_z \sin\beta(l - |z|)\,dz &= Z_i I_0 \sin^2\beta l \\ &= I_0(-j\hat{K}\sin\beta l\cos\beta l + \hat{R} + j\hat{X}), \end{aligned} \tag{55}$$

we have

$$\begin{aligned} I_1 &= I_0\frac{\hat{R} + j\hat{X} - j\hat{K}\sin\beta l\cos\beta l}{j\hat{K}\cos\beta l} + I_0\sin\beta l \\ &= \frac{\hat{R} + j\hat{X}}{j\hat{K}\cos\beta l}\,I_0. \end{aligned} \tag{56}$$

This increment in the current is produced by annihilation of the distributed field along the antenna; by adding it to the original approximation to the input current, we obtain the second approximation,

$$I_i = I_0\sin\beta l + \frac{\hat{R} + j\hat{X}}{j\hat{K}\cos\beta l}\,I_0. \tag{57}$$

Dividing by the impressed voltage (equation 40), we have an asymptotic formula for the input admittance,

$$Y_i = j\hat{K}^{-1}\tan\beta l + \frac{\hat{R} + j\hat{X}}{\hat{K}^2\cos^2\beta l}. \tag{58}$$

This equation is complementary to equation 48 since it is more accurate in the vicinity of $l = \lambda/2$ and deteriorates as l approaches $\lambda/4$.

We have now carried our analysis as far as it is practicable. The next step is to bridge the gap between equations 48 and 58 by means of an "analytic interpolation."

* See equations 18, 21, 23, 47, and 48.

13.12 General formula for the input impedance of a thin symmetric antenna

Presently we shall show that equation 48 represents the input impedance of a uniform transmission line terminated into an impedance,

$$\hat{Z}_t = \frac{\hat{K}^2}{\hat{Z}}, \qquad \hat{Z} = \hat{R} + j\hat{X}, \tag{59}$$

as $\hat{K}$ approaches infinity. Similarly, equation 58 represents the input admittance. Hence, the input impedance of the uniform line,

$$Z_i = \hat{K}\frac{\hat{Z}_t \cos \beta l + j\hat{K} \sin \beta l}{\hat{K} \cos \beta l + j\hat{Z}_t \sin \beta l} = \hat{K}\frac{\hat{Z} \sin \beta l - j\hat{K} \cos \beta l}{\hat{K} \sin \beta l - j\hat{Z} \cos \beta l}, \tag{60}$$

combines two expressions, 48 and 58, into one; and, thus, it may be taken as an approximation to the input impedance of a cylindrical antenna for the complete range of frequencies.

It is easy to show that equation 48 represents the first two terms of an asymptotic expansion for equation 60. We note that as $\hat{K} \sin \beta l$ approaches infinity,

$$\begin{aligned} Z_i &= \hat{K}\frac{\hat{Z} \sin \beta l - j\hat{K} \cos \beta l}{\hat{K} \sin \beta l}\left(1 - j\frac{\hat{Z}}{\hat{K}} \cot \beta l\right)^{-1} \\ &= (\hat{Z} - j\hat{K} \cot \beta l)\left(1 + j\frac{\hat{Z}}{\hat{K}} \cot \beta l - \frac{\hat{Z}^2}{\hat{K}^2} \cot^2 \beta l - \cdots\right) \\ &= \hat{Z} - j\hat{K} \cot \beta l + \hat{Z} \cot^2 \beta l + j\frac{\hat{Z}^2}{\hat{K}} \cot \beta l\,(1 + \cot^2 \beta l) - \cdots \\ &= -j\hat{K} \cot \beta l + \frac{\hat{Z}}{\sin^2 \beta l} + O\left(\frac{1}{\hat{K}}\right), \end{aligned} \tag{61}$$

where $O(1/K)$ represents terms of the order of $1/\bar{K}$ and higher.

Similarly, if we take the reciprocal of equation 60, assume that $\hat{K} \cos \beta l$ approaches infinity, and use the above method, we shall find that Y_i approaches asymptotically the expression given by equation 58.

There is a simple interpretation of $\hat{Z}$. From equation 48 we obtain the asymptotic formula for the complex power input into the antenna,

$$\begin{aligned} \Psi &= \tfrac{1}{2}Z_iI_iI_i^* = \tfrac{1}{2}Z_iI_0I_0^* \sin^2 \beta l \\ &= \tfrac{1}{2}[-j\hat{K} \sin \beta l \cos \beta l + \hat{R} + j\hat{X}]I_0I_0^*, \end{aligned} \tag{62}$$

where I_0 is the current at the antinode. Therefore,

$$\hat{Z} = \hat{R} + j\hat{X} = \frac{2\Psi}{I_0I_0^*} + j\hat{K} \sin \beta l \cos \beta l. \tag{63}$$

The first term on the right is twice the complex power output of an infinitely thin antenna (when the current is strictly sinusoidal) per unit maximum current at the antinode; $\hat{K}$ is obtained from the first term of the asymptotic expansion for the input voltage, which is calculated by integrating the principal part of the electric intensity produced by a sinusoidally distributed current. Hence, these two expressions, both obtainable from the sinusoidal approximation to the antenna current, give $\hat{Z}$ and, consequently, the next approximation to the input impedance.

In the case of a loop of length $2l$, we have a similar formula for the input impedance,

$$Z_i = \hat{K}\,\frac{\hat{Z}\cos\beta l + j\hat{K}\sin\beta l}{\hat{K}\cos\beta l + j\hat{Z}\sin\beta l}, \tag{64}$$

where

$$\hat{Z} = \frac{2\Psi}{I_0 I_0^*} - j\hat{K}\sin\beta l\cos\beta l. \tag{65}$$

In all these formulas we might equally well use the average value $2Z_0$ instead of $\hat{K}$ if we make the appropriate change in $\hat{X}$. For the dipole antenna the latter is given by $(Z_0 - \frac{1}{2}\hat{K})\sin 2\beta l$, to be added to $\hat{X}$ as in equation 51. For the loop antenna this term must be subtracted from $\hat{X}$. It is possible that the use of $2Z_0$ will give somewhat better results; but this could only be verified by further analysis or by a comparison with experiment.

One further observation might be made which will enable us to improve equation 60 without essentially complicating it by adding higher-order correction terms. When considering the various factors affecting antenna current in Chapter 8, we concluded that there is an end effect due to a somewhat larger capacitance near the ends of the antenna and, for antennas of larger radius, another effect due to the capacitance of the flat ends. This end effect effectively lengthens the antenna and thus alters the values of both $\hat{X}$ and $\hat{R}$. To incorporate this effect we shall change the form of equation 60 by multiplying both numerator and denominator of the fraction by j.

$$Z_i = \hat{K}\,\frac{\hat{K}\cos\beta l - \hat{X}\sin\beta l + j\hat{R}\sin\beta l}{\hat{R}\cos\beta l + j(\hat{K}\sin\beta l + \hat{X}\cos\beta l)}, \tag{66}$$

and introducing

$$\Theta = \tan^{-1}\frac{\hat{X}}{\hat{K}}\cdot \tag{67}$$

Since

$$\hat{X} = \hat{K}\tan\Theta, \tag{68}$$

equation 66 becomes

$$Z_i = \hat{K} \frac{\hat{K} \cos(\beta l + \Theta) + j\hat{R} \sin \beta l \cos \Theta}{\hat{R} \cos \beta l \cos \Theta + j\hat{K} \sin(\beta l + \Theta)}. \tag{69}$$

Inasmuch as $\hat{X}$ is small compared with $\hat{K}$, we have approximately

$$\Theta = \frac{\hat{X}}{\hat{K}}, \tag{70}$$

and the effective lengthening of the antenna due to a surplus capacitance near the ends of the antenna is

$$\frac{\Theta}{\beta} = \frac{\hat{X}}{\beta \hat{K}}. \tag{71}$$

To this we add the lengthening (equation 8–126) due to the capacitance of the flat ends,

$$\delta_{\text{cap}} = \frac{aZ_0}{30\pi} \simeq \frac{a\hat{K}}{60\pi}. \tag{72}$$

Thus the total effective lengthening of the antenna due to the end effect is

$$\delta = \frac{\hat{X}(l)}{\beta \hat{K}} + \frac{a\hat{K}}{60\pi} = \frac{\lambda \hat{X}(l)}{2\pi \hat{K}} + \frac{a\hat{K}}{60\pi}. \tag{73}$$

We now use the effective length of the antenna $l + \delta$ instead of the actual length l in the formula for the input impedance,

$$Z_i = \hat{K} \frac{\hat{K} \cos \beta(l + \delta) + j\hat{R} \sin \beta(l + \delta)}{\hat{R} \cos \beta(l + \delta) + j\hat{K} \sin \beta(l + \delta)}. \tag{74}$$

In this equation $\hat{R}$ *is also calculated for the effective length* $l + \delta$. If the lengthening is excessive, as when the antenna is capacitively loaded, $\hat{R}$ may have to be recalculated for the current distribution with a "blunt nose" at $z = l$.

Let us now recall that, in deriving equation 74, we assumed that the length of the antenna gap approaches zero as the diameter approaches zero while always remaining larger than the diameter. Hence, we have automatically excluded the capacitance of the input region, the base capacitance between the bottom of the cylinder shown in Fig. 12.5, for instance, and the ground plane, as well as the near-base capacitance between the lower portion of the cylinder itself and the ground plane. As explained in Section 12.10, the capacitance of the input region is substantially in parallel with the rest of the antenna and thus can easily be taken into account. Likewise, if there is an insulator at the base

of the antenna, we should add its impedance in parallel with the impedance given by equation 74.

Instead of $\hat{K}$ defined by equation 42, we may use $2Z_0$ defined by equation 45, provided that at the same time we replace $\hat{X}$ by $\hat{X} + (Z_0 - \frac{1}{2}\hat{K}) \sin 2\beta l$. This follows from equation 51. The difference between the two formulas diminishes as the characteristic impedance $\hat{K}$ (or $2Z_0$) increases.

13.13 Mode theory of antennas

All the equations in the first 11 sections of this chapter have been obtained rigorously; but the transition from the asymptotic forms (equations 48 and 58 to 60) must be considered as an "analytic interpolation." The final result is not very surprising. We have seen that the distribution of the zeros and poles of the impedance of a thin antenna approaches that of a transmission line; hence, we should expect to be able to express this impedance in some such form as equation 60 with a properly chosen parameter $\hat{K}$ and auxiliary functions $\hat{R}$ and $\hat{X}$. But our present method did not yield the final equation 60 directly; it gave us only the asymptotic forms (equations 48 and 58). If it were not for computational complexities, we could continue successive approximations.

A similar formula may be obtained by solving Maxwell's equations subject to the boundary conditions at the surface of the antenna.* This method exhibits so clearly the physical similarity between antennas and waveguides that it may properly be described as the "waveguide theory of antennas" or the "mode theory of antennas." It is beyond the scope of the present volume to consider this theory in detail, and we shall merely summarize some of the important conclusions.

Let us consider a perfectly conducting double cone, such as the one shown in Fig. 4.8, and assume that it extends to infinity. Suppose that a certain voltage V_0 is impressed between the apices of the cones. On account of symmetry Maxwell's equations reduce to the simpler form given by equations 4–6, 4–7, and 4–8. They involve the radial component E_r of the electric intensity, the meridian component E_θ, and the azimuthal component H_φ of the magnetic intensity. On the conical boundaries E_r must vanish. In Section 4.6 we obtained one solution of the equations in which the radial component vanishes between the cones as well as on their boundaries. The electric lines given by this solution are circular arcs extending from one cone to the other. The TEM wave represented by this solution is very similar to waves on

* S. A. Schelkunoff, Theory of antennas of arbitrary size and shape, *IRE Proc.*, **29**, September 1941, pp. 493–521.

parallel wires. In Section 4.12 we found that Maxwell's equations possess other solutions which represent the higher-order modes of propagation; however, these solutions require infinite voltages at the apex whenever the cone is infinite in length. Thus, a finite voltage between the apices of an infinite double cone excites only the TEM waves.

Suppose now that the cones are of finite length l (along the generators of the cones, Fig. 13.9). If we add a spherical conducting surface of radius l (Fig. 13.9*a*), the waves originating at A, B are totally and uniformly reflected. If the impedance of the reflecting surface is uni-

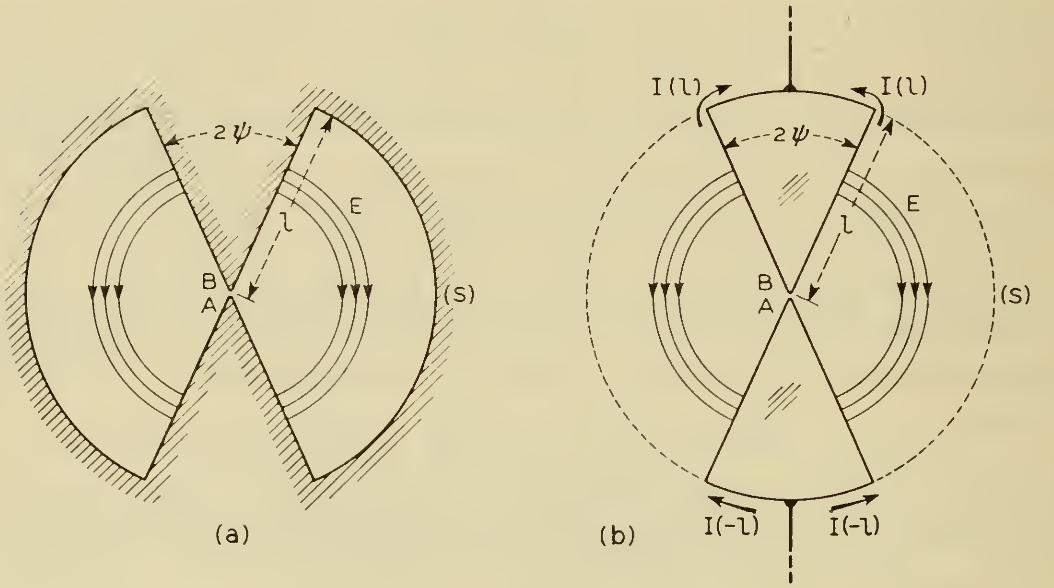


FIG. 13.9 (*a*) A biconical cavity resonator, and (*b*) a biconical antenna.

form, even though different from zero, the reflection is also uniform, even though not total. The reflected wave is similar to the incident wave but is moving in the opposite direction.

Consider now a double cone of finite length l in free space. A spherical wave emerging from the center A, B has no means of knowing that the cone is finite until it reaches the surface of discontinuity S (Fig. 13.9*b*). Beyond this surface the wave must move on without the aid of conductors. Outside S the wave cannot be transverse electromagnetic, since such a wave requires conductors on which the electric lines may terminate. We may also think of the free space outside S as a biconical transmission line with cones of zero radius. The characteristic impedance of TEM waves is then infinite; hence, they are effectively eliminated from consideration. Among the modes of propagation peculiar to free space, there is one that can be excited by a current element. In this mode E_θ varies as $\sin\theta$, and the electric lines

look as shown in Figs. 4.22 and 4.23. On the other hand, in the TEM mode of the original wave inside the boundary sphere S, E_θ varies as $1/\sin\theta$, and on the surface of the spherical caps of the cones E_θ should vanish. This means that, on reaching the surface S, the TEM wave will excite many other modes of propagation in such a way that their combined E_θ will conform to the impressed E_θ. All of these waves have a radial component of E; and, since the field must be continuous across S, the free-space waves will react back on the antenna region inside S and excite higher modes of propagation, such as the one shown in Fig. 4.27.

With each of these higher-order waves, there will be associated a certain current in the cones. The total current at distance r from A, B may thus be expressed as the sum,

$$I(r) = I_0(r) + I_1(r) + I_2(r) + \cdots, \tag{75}$$

of currents associated with the various waves in the antenna region. The first term represents the current associated with the TEM wave. All the higher-order waves we shall lump together under the name "complementary wave." Thus, we shall write equation 75 as

$$I(r) = I_0(r) + \bar{I}(r). \tag{76}$$

The complementary current vanishes at $r = 0$,

$$\bar{I}(0) = 0, \tag{77}$$

while the total voltage along a typical meridian vanishes for all higher-order waves, so that

$$V(r) = V_0(r). \tag{78}$$

Thus, the input admittance,

$$Y_i = \frac{I(0)}{V(0)} = \frac{I_0(0)}{V_0(0)}, \tag{79}$$

is given solely by the current and voltage associated with the TEM wave.

At $r = l$, we have

$$I(l) = I_0(l) + \bar{I}(l), \tag{80}$$

where $I(l)$ is the total current reaching and going over the edge of the cone into its cap (Fig. 13.10b), or into its interior surface when the cone is hollow (Fig. 13.10a). The principal current is

$$I_0(l) = I(l) - \bar{I}(l). \tag{81}$$

Thus, when we consider the propagation of principal waves, it appears that at $r = l$ we have an effective admittance,

$$Y(l) = \frac{I(l)}{V_0(l)} - \frac{\bar{I}(l)}{V_0(l)}. \tag{82}$$

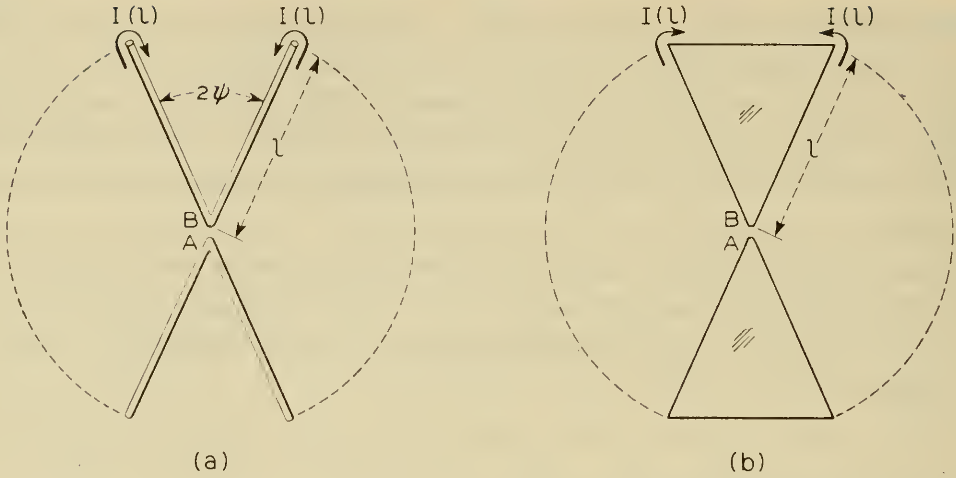


FIG. 13.10 (*a*) A hollow biconical antenna, and (*b*) a biconical antenna with flat "caps."

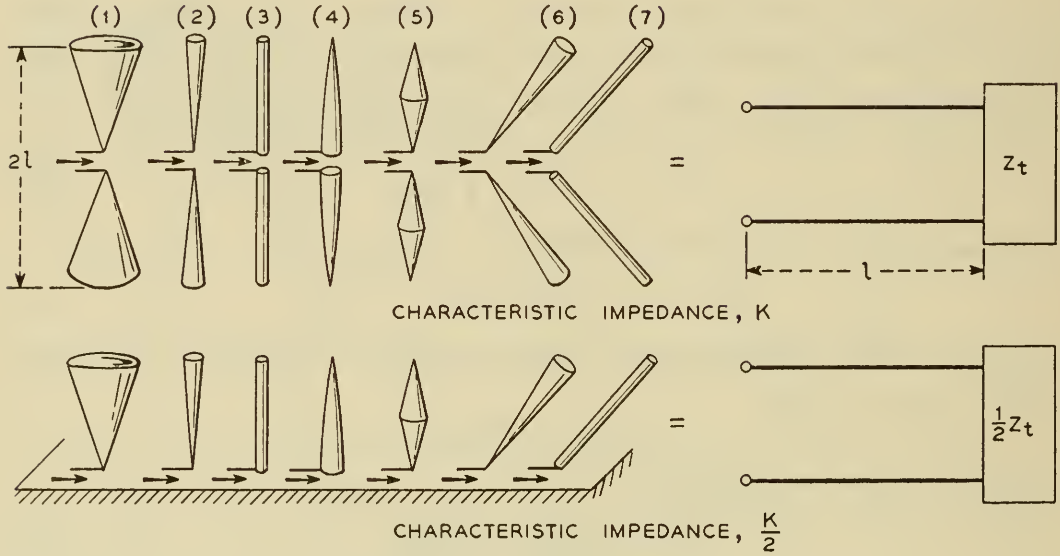


FIG. 13.11 The input impedance of a symmetric antenna equals the input impedance of a transmission line (in general, tapered) terminated into a proper impedance.

This *terminal admittance* is a parallel combination of the admittance of the cap,

$$Y_c = \frac{I(l)}{V_0(l)}, \tag{83}$$

and the *complementary admittance*,

$$\bar{Y} = -\frac{\bar{I}(l)}{V_0(l)}, \tag{84}$$

representing the effect of a sudden termination of the cones on the field in the antenna region inside S. Since the propagation of the voltage

and current associated with the principal wave in a biconical antenna is identical with that in a uniform transmission line whose distributed parameters are given by equations 4–34, we have the following theorem: *The input impedance of a symmetric biconical antenna of any angle equals the input impedance of a uniform transmission line whose length equals the length of one antenna arm.* The effective impedance at the end of the antenna is composed of two impedances in parallel. One of these represents the impedance of the cap (or the interior surface of the cone, if the cone is hollow). The other represents the impedance of higher-order waves formed by the sudden termination of the conductors. For antennas of other shapes (Fig. 13.11), we have a similar theorem; the main difference being the nonuniformity of the equivalent line.

13.14 Characteristic impedance of an antenna

The input impedance of an antenna contains one important parameter analogous to the characteristic impedance of a transmission line. The value of this parameter, however, depends on the particular method of analysis and, as in the case of integral equations, even on how the analysis is carried out. Theoretically these differences are compensated by the corresponding differences in associated functions involved in the input impedance formula; but the compensation would be complete only in exact formulas. Because of analytic difficulties it has been impossible to obtain exact formulas;* in approximate formulas the compensation may not be complete, and there may be a corresponding difference in the final results. Nevertheless, there are several formulas obtained by different methods which are in substantial agreement with each other and with experiments. In view of the foregoing remarks we shall summarize the various values for the characteristic impedance of an antenna and shall refer to the input impedance formulas in which they are to be used.

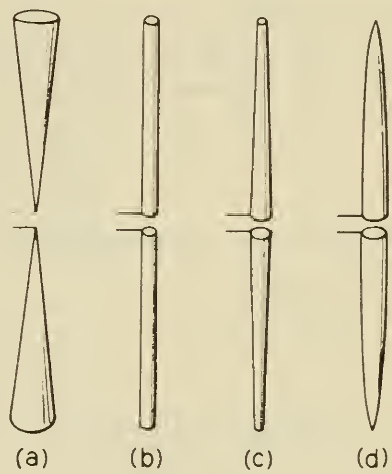


Fig. 13.12 (*a*) A biconical antenna, (*b*) a cylindrical antenna, (*c*) a linearly tapered antenna, and (*d*) a spheroidal antenna.

In the mode theory the characteristic impedance of an antenna is the impedance of the principal or TEM waves. In the case of a biconical antenna (Fig. 13.12*a*), the impedance K is constant along it.

* Except in the case of spheroidal conductors.

From equation 4–27 we have

$$K = 120 \log \cot \tfrac{1}{2}\psi, \tag{85}$$

where ψ is the cone angle as defined in Fig. 4.8: that is, the angle between the axis of the cone and its generators. If ψ is small,

$$K = 120 \log \frac{2l}{a}, \tag{86}$$

where a is the maximum radius of the cone.

For any other shape the distributed parameters L and C are variable, and the *nominal characteristic impedance* is defined as the ratio $\sqrt{L/C}$. For thin antennas this is given by equation 4–37,

$$K(z) = 120 \log \frac{2z}{\rho}, \tag{87}$$

except when $z < \rho$ (see Section 12.10, input regions). In this case the transmission line equivalent to the antenna is nonuniform, and two cases should be considered. In the first (and the more important) case, the length l does not exceed $3\lambda/4$, and the solution is expressed in terms of the *average characteristic impedance*,

$$K_a = \frac{1}{l}\int_0^l K(z)\,dz = \frac{120}{l}\int_0^l \log \frac{2z}{\rho(z)}\,dz, \tag{88}$$

and certain supplementary functions of l. For cylindrical antennas $\rho = a$, and

$$K_a = 120 \log \frac{2l}{a} - 120. \tag{89}$$

For linearly tapered antennas (Fig. 13.12c), we have

$$K_a = 120 \log \frac{2l}{a_b} + \frac{120 a_t}{a_b - a_t} \log \frac{a_t}{a_b}, \tag{90}$$

where a_t and a_b are the radii at the top and bottom of the upper arm. For spheroidal antennas (Fig. 13.12d),

$$K_a = 120 \log \frac{l}{a}, \tag{91}$$

where a is the radius at the input terminal. *For antennas of any shape of the longitudinal cross section,* K_a *is given by equation 89 if* a *is the logarithmic mean radius defined by*

$$\log a = \frac{1}{l}\int_0^l \log \rho(z)\,dz, \tag{92}$$

where $\rho(z)$ is the radius at a typical point.

When the angle between the antenna arms is ϑ instead of π (Fig. 13.13), the average characteristic impedance is

$$K_a(l, \vartheta) = K_a(l, \pi) + 120 \log \sin \tfrac{1}{2}\vartheta. \tag{93}$$

The average characteristic impedances for biconical, spheroidal, and cylindrical antennas are shown in Fig. 13.14. The curve for cylindrical antennas may be used for antennas of other shapes if a is interpreted as the logarithmic mean radius given by equation 92.

The following are a few representative values of K_a for cylindrical antennas:

$$\begin{aligned} l/2a &= 10,\ 50,\ 100,\ 200,\ 300,\ 600,\ 1000,\ 10000, \\ K_a &= 323,\ 516,\ 599,\ 682,\ 731,\ 814,\ 875,\ 1152. \end{aligned} \tag{94}$$

Practically, it would be difficult to obtain characteristic impedances higher than 1300 ohms.

The average characteristic impedance of cage structures may be determined from the effective radius given by equation 4–44.

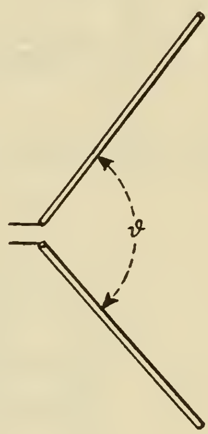


FIG. 13.13
A V antenna.

The average characteristic impedance as defined above is to be used in the input impedance formulas given by the mode theory in the next section when l does not exceed $3\lambda/4$. If l is large, it is more convenient to divide the equivalent line in Fig. 13.11 into two sections, one of length $\lambda/2$ near the input end and the other of length $l - (\lambda/2)$. In the second section, K varies slowly, and, in the first approximation, the reflections due to the variation of K may be neglected; even the second approximation is easy to calculate.*

The average impedance of long rhombic antennas should equal approximately the input impedance of infinitely long diverging wires. This can be calculated as suggested in the preceding paragraph, and for wires of uniform cross section is†

$$Z_{i,\infty} = 120\left(\log \frac{\lambda}{2\pi a} - 0.60 + \log \sin \frac{\vartheta}{2}\right) - j170, \tag{95}$$

where ϑ is the angle between the wires. Figure 13.15 shows the real component of this impedance for the case $\vartheta = \pi$. In obtaining equation 95, it was assumed that the distance between the input terminals is

* From the mathematical point of view we have to solve equations 4–33 with variable coefficients L and C (since $G = 0$). See *Applied Mathematics*, Chapter 11, Section 13, and Problem 8 on p. 219.

† *Electromagnetic Waves*, pp. 292–293.

not small compared with the radius, or else that the input ends are tapered. If the radius is constant but the antenna gap is infinitesimal,

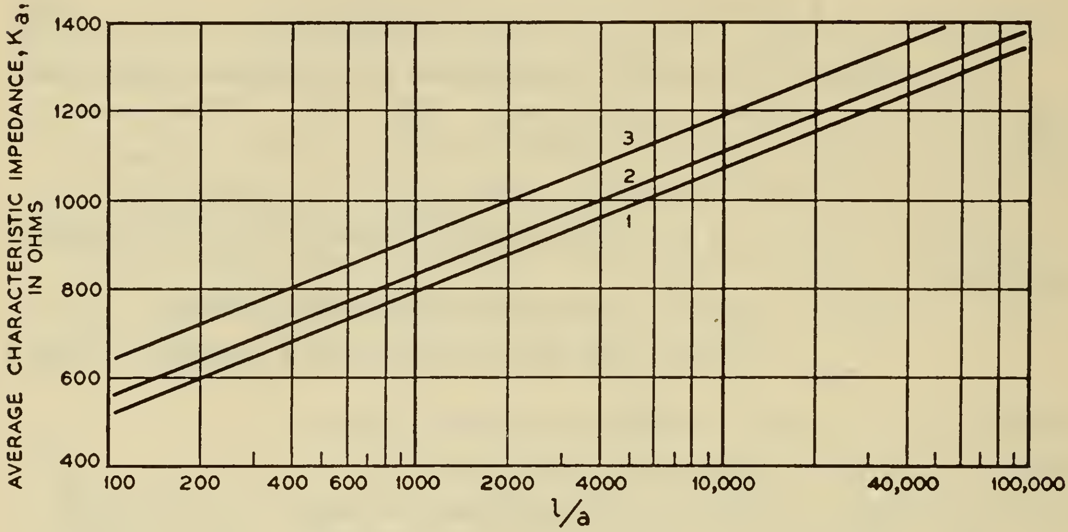


FIG. 13.14 The average characteristic impedance of antennas: (1) cylindrical, (2) spheroidal, and (3) biconical.

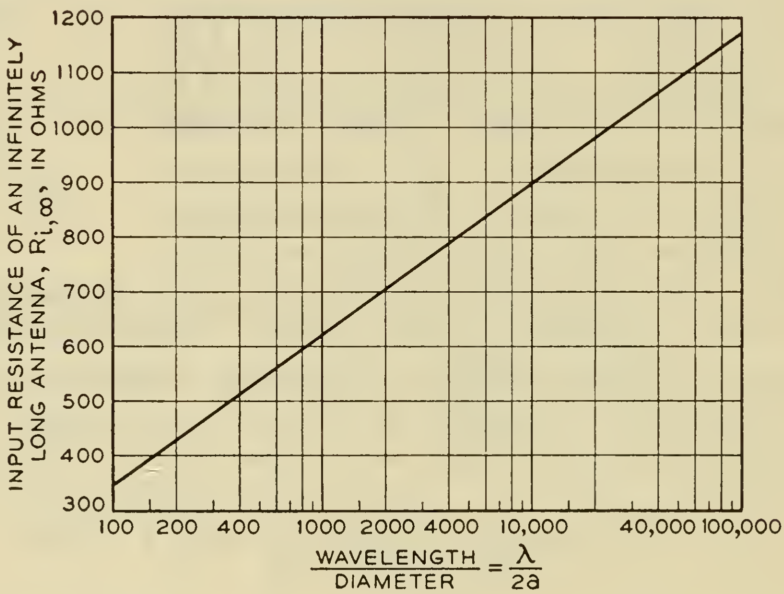
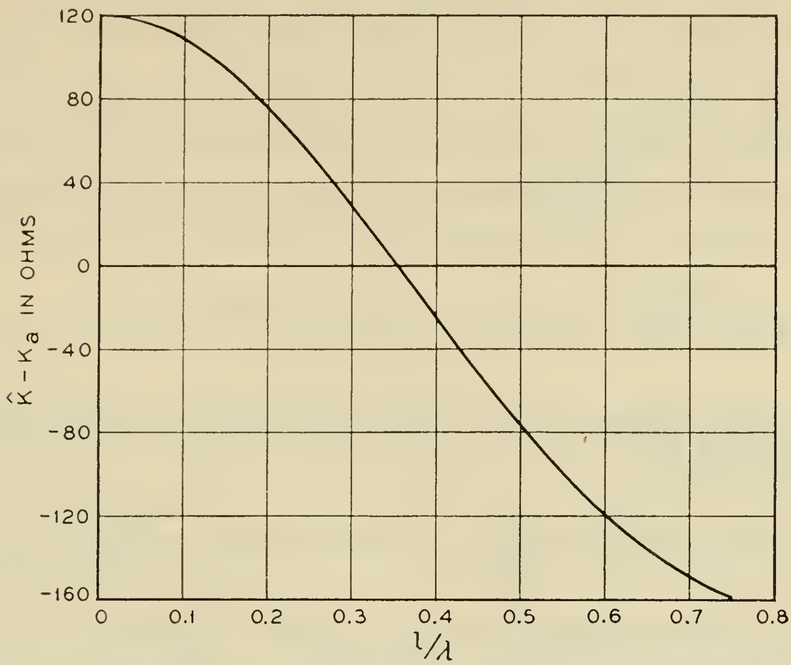


FIG. 13.15 The input resistance of an infinitely long thin antenna.

then the input conductance $C_{i,\infty}$ is the reciprocal of the above input resistance while the susceptance is infinite.

When the details of the mode theory of antennas are considered, it is found that there is an infinite sequence of characteristic impedances, one for each mode of propagation in the antenna region and one for each



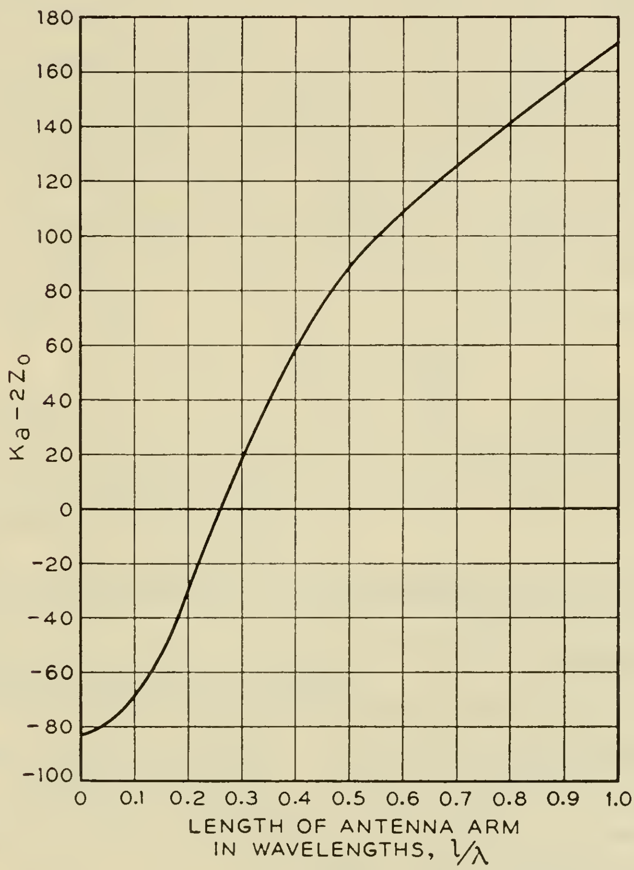


FIG. 13.16 Differences between various (cylindrical) antenna parameters: (*a*) $\hat{K} - K_a$; (*b*) $K_a - 2Z_o$.

mode in free space. Only one of these impedances, namely, K_a, appears explicitly in the formula for the input impedance; the others appear implicitly in certain complementary functions associated with K_a.

In Section 13.12 we obtained equation 60 for the input impedance when the end effect is neglected, and then equation 74 in which the end effect is included. The characteristic impedance in these formulas is

$$\hat{K} = 120\left(\log\frac{2l}{a} - \operatorname{Cin}\beta l\right) = 120\left(\log\frac{\lambda}{2\pi a} + 0.116 + \operatorname{Ci}\beta l\right). \quad (96)$$

There we also suggested that to the same order of approximation we could replace $\hat{K}$ by

$$2Z_0 = 120\left(\log\frac{\lambda}{2\pi a} + 0.116 + \operatorname{Ci}2\beta l - \frac{\sin 2\beta l}{2\beta l}\right), \quad (97)$$

provided we also replace $\hat{X}$ by $X + \frac{1}{2}(2Z_0 - \hat{K})\sin 2\beta l$. These two formulas will not give identical results; but the order of approximation is

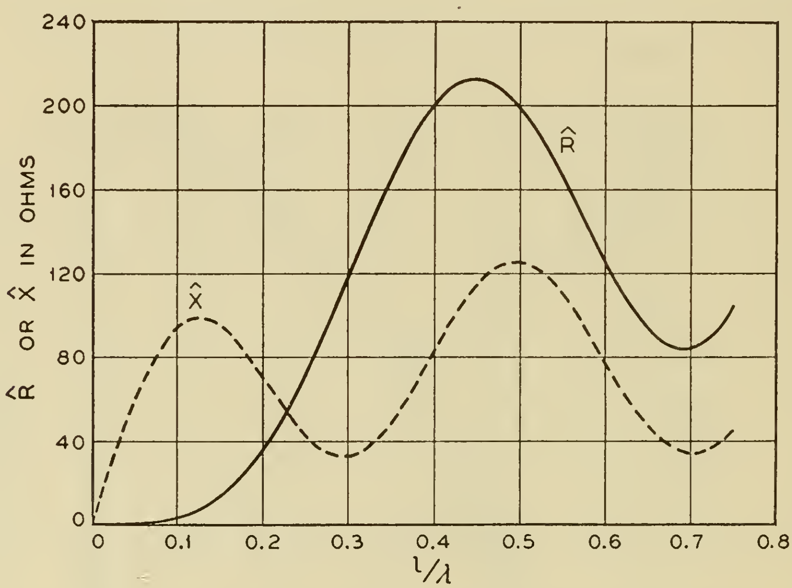


FIG. 13.17 The impedance $\hat{Z}$ for use in equations 60 and 74.

the same. In this elementary method of analysis, the definitions of $\hat{K}$ and $2Z_0$ depend on the assumption that a approaches zero — an assumption not needed in the mode theory. This assumption makes these parameters indefinite to the extent of an additive constant. Theoretically this indefiniteness is arbitrary; practically, common sense restricts it within certain relatively narrow limits.

It is impossible to represent either $\hat{K}$ or $2Z_0$ by a single curve since

these quantities depend on two parameters; but we can plot the differences,

$$
\begin{aligned}
\hat{K} - K_a &= 120(1 - \mathrm{Cin}\,\beta l) = 120(1 - C - \log \beta l + \mathrm{Ci}\,\beta l), \\
2Z_0 - K_a &= 120\left(1 + \log 2 - \mathrm{Cin}\,2\beta l - \frac{\sin 2\beta l}{2\beta l}\right) \\
&= 120\left(\mathrm{Ci}\,2\beta l - \log \beta l + 1 - C - \frac{\sin 2\beta l}{2\beta l}\right).
\end{aligned} \tag{98}
$$

These differences are shown in Fig. 13.16.

The associated impedance $\hat{Z} = \hat{R} + j\hat{X}$ appearing in equations 60 and 74 is given by equations 49 and 50 and shown in Fig. 13.17.

13.15 Input impedance of thin symmetric antennas according to the mode theory

Figure 13.11 represents a schematic diagram for the input impedance of a thin, symmetrically fed antenna. The line section represents the principal effects of the mean radius of the antenna and of the departure of the actual radius at various points from the mean radius. The terminating impedance Z_t represents the effect of radiation and of the local storage of energy near the ends of the antenna. As explained in the preceding section, the important parameter is not the arithmetic mean radius but the geometric or logarithmic mean radius given by equation 92; in other words, the important parameter is the arithmetic average (equation 88) of the nominal characteristic impedance,

$$
K[z, \rho(z)] = 120 \log \frac{2z}{\rho(z)}, \tag{99}
$$

where $\rho(z)$ is the radius of the antenna at distance z from the mid-point between the input terminals. If the arms of the antenna have different radii, $\rho(z)$ is their geometric mean.

The effect of the departure of the actual radius from the mean radius, that is, the effect of the difference between the nominal characteristic impedance and the average characteristic impedance is represented by the following two functions:

$$
\begin{aligned}
M(\beta l) &= \beta \int_0^l [K_a - K(z, \rho)] \sin 2\beta z \, dz, \\
N(\beta l) &= \beta \int_0^l [K_a - K(z, \rho)] \cos 2\beta z \, dz.
\end{aligned} \tag{100}
$$

Thus, for cylindrical antennas,

$$M(\beta l) = 60(\mathrm{Cin}\, 2\beta l - 1 + \cos 2\beta l),$$
$$N(\beta l) = 60(\mathrm{Si}\, 2\beta l - \sin 2\beta l), \qquad K_a = 120\left(\log \frac{2l}{a} - 1\right)\cdot \tag{101}$$

For antennas of diamond cross section (two cones arranged base to base),

$$M(\beta l) = 60(1 + \cos 2\beta l)\,\mathrm{Cin}\, 2\beta l - 60 \sin 2\beta l\, \mathrm{Si}\, 2\beta l,$$
$$N(\beta l) = 60(1 - \cos 2\beta l)\,\mathrm{Si}\, 2\beta l - 60 \sin 2\beta l\, \mathrm{Cin}\, 2\beta l, \tag{102}$$
$$K_a = 120 \log \frac{2l}{a}\cdot$$

For conical antennas, $M = N = 0$.

The principal term of the terminal impedance depends on the average characteristic impedance K_a. The simplest way to express this dependence is to write the terminal admittance in the following form,

$$Y_t = \frac{Z_a(\beta l)}{K_a^2} + j\omega C_t = \frac{Z_a(\beta l) + j\omega C_t K_a^2}{K_a^2}, \tag{103}$$

where in the first approximation Z_a depends only on l/λ, and C_t is the capacitance between the flat outer ends of the antenna arms. An approximate value of the admittance associated with this capacitance is

$$j\omega C_t = \frac{ja_t}{30\lambda}, \tag{104}$$

where a_t is the top radius.

It is to be noted that

$$Z_a(\beta l) = \lim K_a^2 Y_t, \quad \text{as} \quad K_a \to \infty\,. \tag{105}$$

This function *is the same for all antenna shapes*, and it may be described as the limiting value of the terminal impedance inverted by a quarter-wave transformer. It is also the radiation impedance of an infinitely thin biconical antenna referred to the current antinode. Since the shape does not affect the real part of this impedance, $R_a(\beta l)$ equals $R_{12}{}^a$ given by equation 27, computed for the cylindrical antenna. Hence, R_a also equals $\hat{R}$ appearing in equations 60 and 74; but $X_a \neq \hat{X}$. The complete expressions are

$$\begin{aligned} R_a(\beta l) &= 60\,\mathrm{Cin}\, 2\beta l + 30(2\,\mathrm{Cin}\, 2\beta l - \mathrm{Cin}\, 4\beta l) \cos 2\beta l + \\ &\qquad 30(\mathrm{Si}\, 4\beta l - 2\,\mathrm{Si}\, 2\beta l) \sin 2\beta l, \\ X_a(\beta l) &= 60\,\mathrm{Si}\, 2\beta l - 30(\mathrm{Cin}\, 4\beta l - \log 4) \sin 2\beta l - \\ &\qquad 30\,\mathrm{Si}\, 4\beta l \cos 2\beta l. \end{aligned} \tag{106}$$

These functions are shown in Fig. 13.18.

If the angle between the antenna arms is ψ instead of π, then,*

$$\begin{aligned} R_a &= 60 \text{ Cin } 2\beta lk + 30[2 \text{ Cin } 2\beta l - \text{Cin } 2\beta l(1-k) - \\ &\qquad \text{Cin } 2\beta l(1+k)] \cos 2\beta l + \\ &\qquad 30[-2 \text{ Si } 2\beta l + \text{Si } 2\beta l(1-k) + \\ &\qquad \text{Si } 2\beta l(1+k)] \sin 2\beta l, \\ X_a &= 60 \text{ Si } 2\beta lk + 30[\text{Si } 2\beta l(1-k) - \text{Si } 2\beta l(1+k)] \cos 2\beta l + \\ &\qquad 30[2 \log (1+k) + \text{Cin } 2\beta l(1-k) - \\ &\qquad \text{Cin } 2\beta l(1+k)] \sin 2\beta l, \qquad k = \sin \frac{\psi}{2}. \end{aligned} \tag{107}$$

The M and N functions are given by equations 100 with the following changes: (1) The lower and upper limits of integration must be the

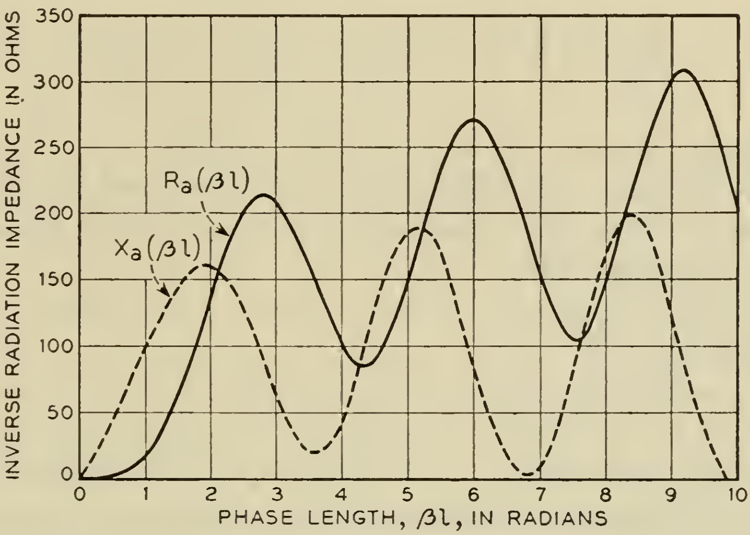


FIG. 13.18 The impedance Z_a for use in equation 108. It is the terminal impedance inverted by a quarter-wave transformer.

distances r_1 and r_2 of the beginning and the end of an antenna arm from the point of intersection of the arm; (2) z must be replaced by $r - r_1$. If ψ is not too small, the M and N functions are substantially independent of ψ.

Aside from the shunt admittance of the input region, which may or may not be significant, the input impedance of the antenna is

$$Z_i = K_a \frac{(K_a - M) \cos \beta l + j(Z_a + j\omega C_t K_a^2 - jN) \sin \beta l}{(Z_a + j\omega C_t K_a^2 + jN) \cos \beta l + j(K_a + M) \sin \beta l} \cdot \tag{108}$$

* For these formulas the authors are indebted to Miss Marion C. Gray.

Near resonance, $\beta l \simeq \pi/2$, and

$$R_i = \frac{K_a R_a}{K_a + M},$$
$$X_i = \frac{K_a(K_a - M)}{K_a + M}\left[\beta l - \frac{\pi}{2} + \frac{X_a - N + \omega C_t K_a^2}{K_a - M}\right]. \tag{109}$$

Near antiresonance, $\beta l \simeq \pi$, and

$$G_i = \frac{R_a}{K_a(K_a - M)},$$
$$B_i = \frac{K_a + M}{K_a(K_a - M)}\left[(\beta l - \pi) + \frac{X_a + N + \omega C_t K_a^2}{K_a + M}\right]. \tag{110}$$

Expressions 106 and 107 are the limits of the exact expressions,

$$R_a + \frac{R_a^{(1)}}{K_a} + \frac{R_a^{(2)}}{K_a^2} + \cdots, \qquad X_a + \frac{X_a^{(1)}}{K_a} + \frac{X_a^{(2)}}{K_a^2} + \cdots, \tag{111}$$

as K_a becomes infinite. These limits are calculable from the sinusoidal current distribution. The effect of the " blunt-nosed " distribution of current near the ends is involved in the terms of order $1/K_a$ and is thus neglected in equation 108.

13.16 Zeros and poles

The natural oscillation constants of an antenna with its input terminals short-circuited are obtained by equating the numerator in equation 108 to zero. The natural frequency is complex, and, in order to bring the damping into evidence, it is convenient to replace $j\omega$ by $\xi + j\omega$. For large K_a, the first term in the numerator of equation 108 is large unless $\cos \beta l$ is nearly equal to zero, whereas the second term is not very large. In the first approximation, therefore, $\xi = 0$ and $\omega\sqrt{\mu\varepsilon}l = (m + \frac{1}{2})\pi$. Using this approximation in the coefficients of $\cos \beta l$ and $\sin \beta l$, we obtain the next approximation,

$$(\xi + j\omega)\sqrt{\mu\varepsilon}l = -\frac{R_a}{K_a - M} + j\left[\left(m + \frac{1}{2}\right)\pi - \frac{X_a - N + \omega C_t K_a^2}{K_a - M}\right],$$
$$m = 0, 1, 2, \cdots. \tag{112}$$

Similarly, when the antenna input terminals are not so close that there is a substantial direct capacitance between them, in addition to the input impedance given by equation 108, the natural oscillation constants with the terminals floating are obtained by equating the

denominator in equation 108 to zero. Thus,

$$(\xi + j\omega)\sqrt{\mu\varepsilon}\, l = -\frac{R_a}{K_a + M} + j\left[m\pi - \frac{X_a + N + \omega C_t K_a^2}{K_a + M}\right], \qquad m = 1, 2, 3 \cdots. \tag{113}$$

13.17 Resonant and antiresonant frequencies

When K_a is very large, the resonant and antiresonant frequencies may be obtained either by equating to zero the reactive terms of the input impedance and admittance in such expressions as 109 and 110, or by noting that these frequencies are substantially equal to the natural frequencies and using equations 112 and 113. Thus, the resonant lengths of the antenna arms are given by

$$\frac{l}{\lambda} = \frac{2m + 1}{4} - \frac{X_a - N + \omega C_t K_a^2}{2\pi(K_a - M)}, \qquad m = 0, 1, 2, \cdots. \tag{114}$$

Similarly, the antiresonant lengths are given by

$$\frac{l}{\lambda} = \frac{m}{2} - \frac{X_a + N + \omega C_t K_a^2}{2\pi(K_a + M)}, \qquad m = 1, 2, 3, \cdots. \tag{115}$$

More accurate values of l/λ are obtained from the reactance curves. Experimentally, the resonant values agree well with those obtained from equation 114, even for moderately thick antennas; but at antiresonance the difference is large because X_a increases rapidly as $\pi - \beta l$ decreases. For a comparison between theory and experiment, see Section 13.24.

13.18 Quality factor

From equations 112 and 113 we obtain the quality factor Q. Thus, near a resonance,

$$Q = -\frac{\omega}{2\xi} = \frac{(m + \frac{1}{2})\pi(K_a - M) - X_a + N - \omega C_t K_a^2}{2R_a}, \qquad m = 0, 1, 2, \cdots, \tag{116}$$

and, near an antiresonance,

$$Q = -\frac{\omega}{2\xi} = \frac{m\pi(K_a + M) - X_a - N - \omega C_t K_a^2}{2R_a}, \qquad m = 1, 2, \cdots. \tag{117}$$

For the first resonance and the first antiresonance in cylindrical antennas the above equations become, respectively,

$$Q = \frac{K_a - 6}{93}, \qquad Q = \frac{K_a + 106}{127}, \tag{118}$$

as long as C_t is negligible; otherwise, we should subtract the quantity $aK_a{}^2/60\lambda R_a$.

13.19 Antiresonant impedance

As K_a approaches infinity, the antiresonant impedance of a cylindrical antenna is given by

$$R_{i,\max} = \frac{K_a(K_a - 146)}{199.1}. \tag{119}$$

More accurate values of the antiresonant impedance are obtained from resistance curves (Figs. 13.19 and 13.21). To illustrate the difference: if $K_a = 800$, equation 119 gives 2628 while the impedance curve gives 2500; if $K_a = 500$, the equation gives 889 and the curve 940. In terms of $\lambda/2a$ this equation becomes approximately

$$R_{i,\max} = \frac{[276 \log_{10}(\lambda/2a) - 110]^2}{199.1}, \tag{120}$$

provided we neglect the difference between l and $\lambda/2$. If we take this difference into account, then,

$$\frac{l}{a} = \frac{\lambda}{2a}\left(1 - \frac{\pi - \beta l}{\pi}\right), \tag{121}$$

and

$$R_{i,\max} = \frac{[276 \log_{10}(\lambda/2a) - 110 - 120(\lambda - 2l)/\lambda]^2}{199.1}. \tag{122}$$

13.20 Input impedance of cylindrical and biconical antennas

Figures 13.19 and 13.20 show the resistance and reactance of cylindrical antennas if the top capacitance C_t is neglected. Even hollow cylindrical antennas have some top capacitance due to the charge on the rims and on the *inner surface* of the cylinder near the open ends. Theoretically the top capacitance is zero only if the open ends are closed with perfect magnetic conductors.

Figures 13.21 and 13.22 show the resistance and reactance of cylindrical antennas when the top capacitance is included. The value of this capacitance is assumed equal to the static capacitance between the two exposed faces of the circular disks at the ends of the antenna. Further, the effect of this capacitance on R_a and X_a has not been included.

Figures 13.23 and 13.24 show the resistance and reactance of biconical antennas when the top capacitance is neglected. A comparison with the corresponding figures for the cylindrical antennas shows that the maximum values of the input resistance are higher for the biconical antennas.

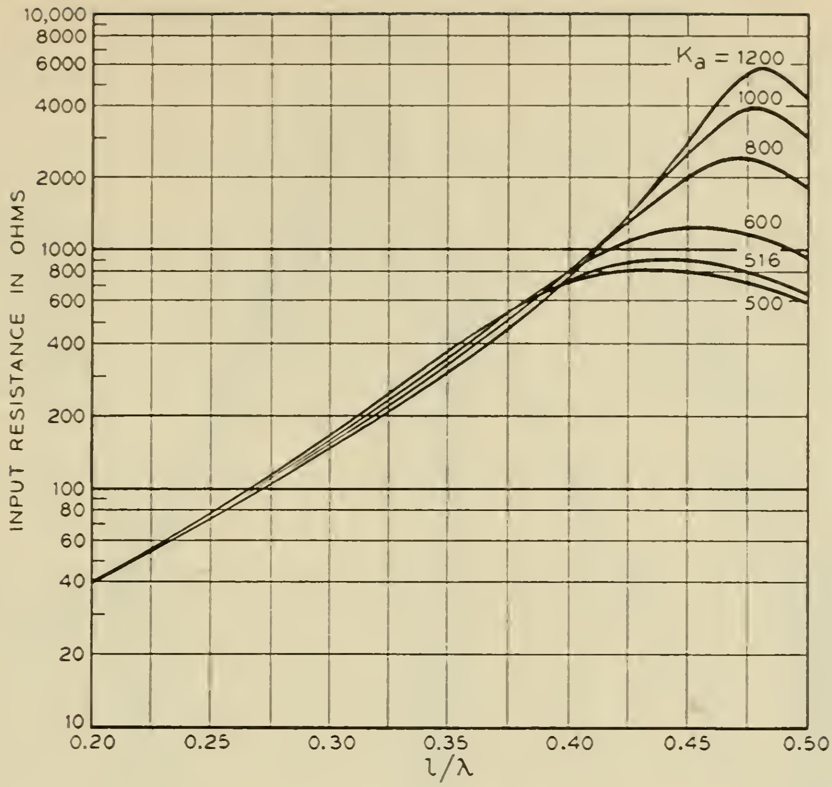


FIG. 13.19 The input resistance of cylindrical antennas in free space when the top capacitance is neglected.

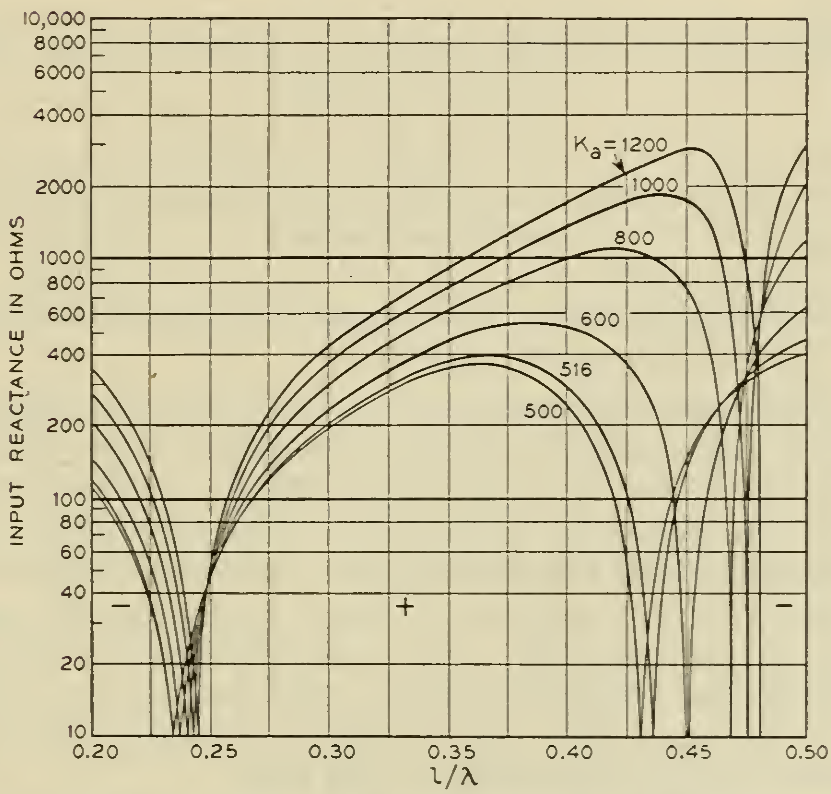


FIG. 13.20 The input reactance of cylindrical antennas in free space when the top capacitance is neglected.

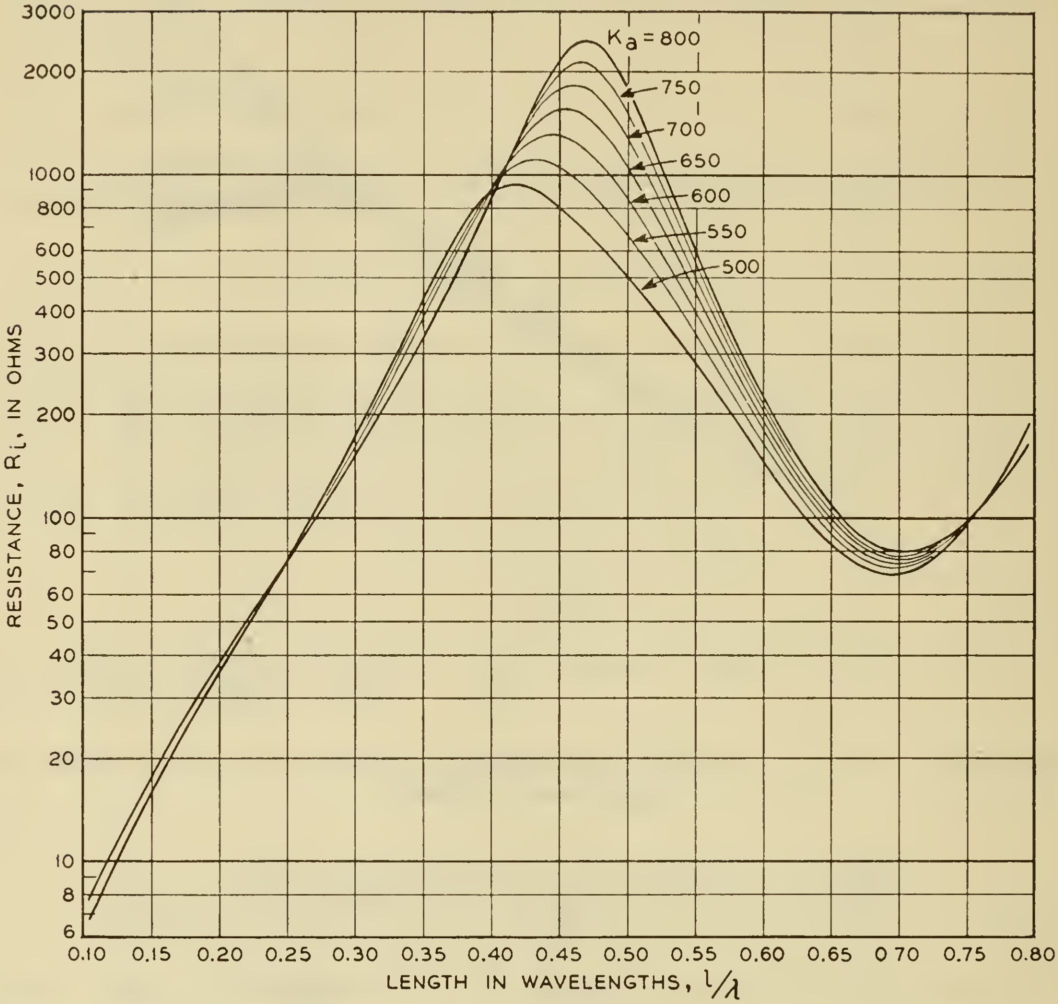


FIG. 13.21 The input resistance of cylindrical antennas in free space when the top capacitance is included.

13.21 Comparison between theoretical and experimental values of the antiresonant impedance

Figure 13.25 presents a comparison between theoretical and experimental values of the antiresonant impedance for different values of $l/2a$. The theoretical curves were obtained from the impedance formula 108. The upper curve is for the dipole antenna in free space; the lower curve is for a monopole above a perfect ground. The lower branches on each curve neglect the top capacitance C_t altogether; for the upper branches, $C_t = 2\varepsilon a$ and $j\omega C_t = ja/30\lambda$. The capacitance between the tops of the cylinders was obtained as follows. The capacitance of a disk of radius a in free space is $8\varepsilon a$; the capacitance of one face is $4\varepsilon a$; and the capacitance of two exposed faces in series is $2\varepsilon a$.

Experimental data were obtained by various investigators at dif-

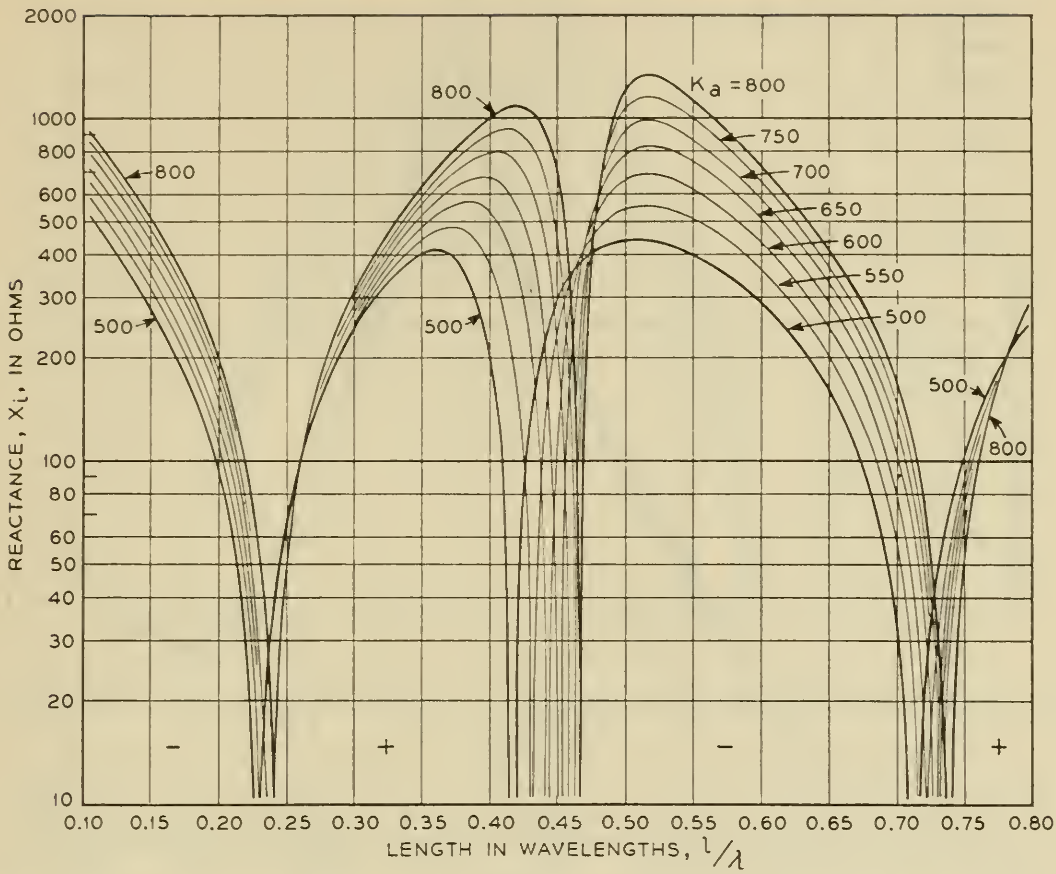


FIG. 13.22 The input reactance of cylindrical antennas in free space when the top capacitance is included.

ferent times and under different circumstances. The earliest data here presented were obtained by C. B. Feldman in 1934. He measured the maximum resistance of a vertical monopole above ground using $\lambda \simeq 18$ meters. The gap between the lower end of the antenna and the ground was 0.06λ. In Fig. 13.25 his experimental points are marked with the letter F. The next series of measurements* was made by A. C. Beck in 1936. He employed squirrel-cage antennas with tapered input ends.† Some measurements were made on balanced dipoles at the height of 60 ft above ground. Other measurements were made on the unbalanced monopoles. Several wavelengths were used in the range between 14 and 28 meters. His measurements are marked with the

* Heretofore unpublished.

† See also: G. Rösseler, F. Vilbig, and K. Vogt, Über das elektrische Verhalten von Vertikalantennen in Abhängigkeit von ihrem Durchmesser, *TFT*, **28**, May 1939, pp. 170–178.

F. E. Lutkin, R. H. J. Cary, and G. N. Harding, Wide-band aerials and transmission lines for 20–85 Mc/s, *IEE Jour.*, **93**, IIIA, 1946, pp. 552–558.

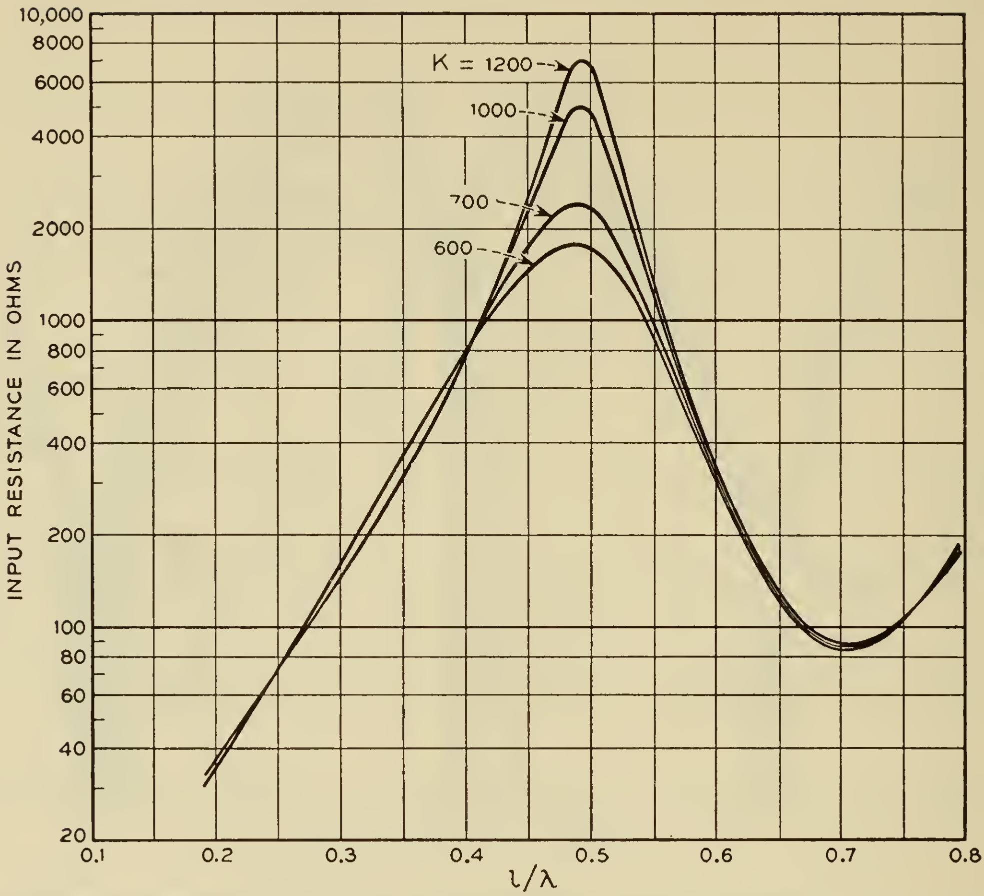


Fig. 13.23 The input resistance of biconical antennas when the top capacitance is neglected.

letter *B*. Both sets of measurements had been made before the present theory was available. This is true also of the point marked *MS* which was obtained by Morrison and Smith.*

Extensive impedance measurements were reported by Brown and Woodward in April 1945. They used $\lambda = 5$ meters. Their results are marked *BW*. We have omitted the antennas with the two largest diameters because the measured values included substantial base admittance (see Section 13.23). The point corresponding to $l/2a = 1750$ is seen to be considerably below the theoretical curve. This point, however, should be seriously questioned. The maximum resistance of a sharply antiresonant circuit is nearly equal to the maximum reactance swing. Other poles near the imaginary axis may either raise or lower the reactance curve in this vicinity; but they do not affect the reactance swing. Now the reactance swing as measured by Brown and Woodward is 1750 ohms while the maximum resistance is 1280 ohms.

* J. F. Morrison and P. H. Smith, The shunt-excited antenna, *IRE Proc.*, **25**, June 1937, pp. 673–696.

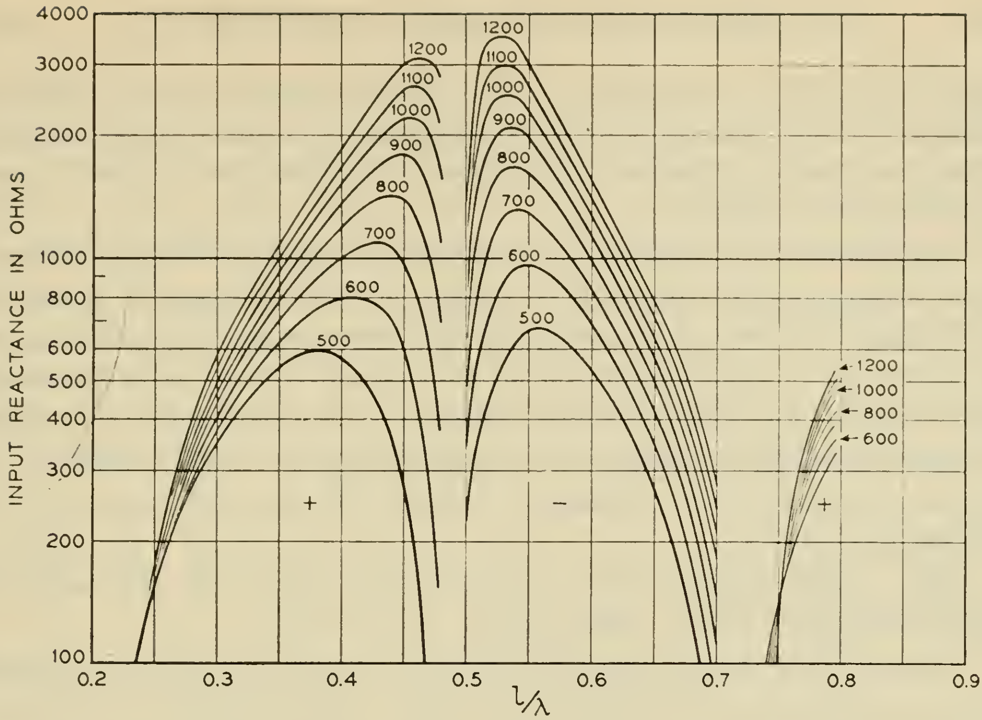


Fig. 13.24 The input reactance of biconical antennas when the top capacitance is neglected.

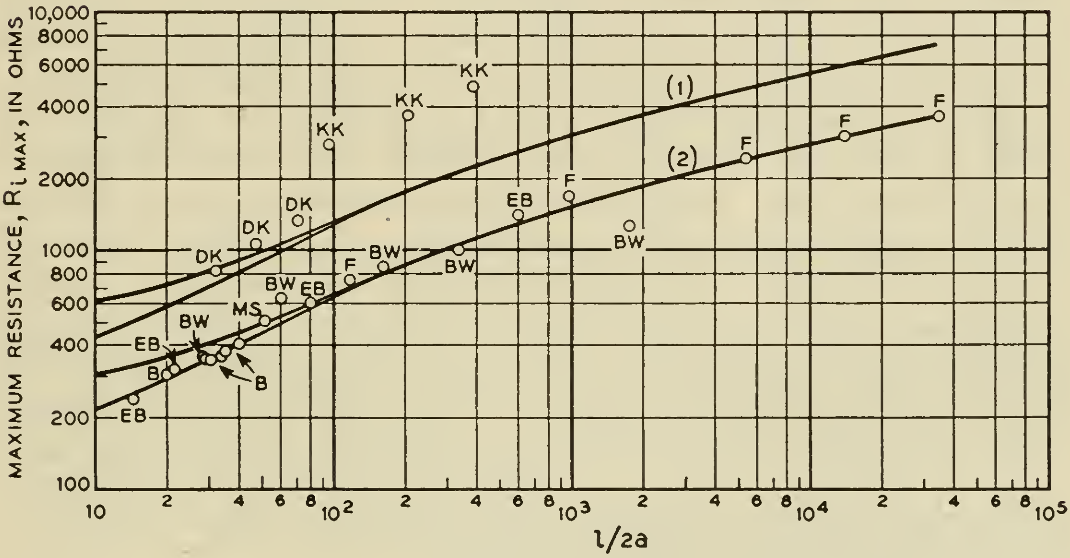


Fig. 13.25 The antiresonant impedance of a vertical antenna above a perfectly conducting ground. The upper curve includes the top capacitance; the lower curve excludes it. The antiresonant impedances of hollow antennas should lie between these curves. Various points are experimental. Points marked *F* were obtained by Feldman, those marked *BW* by Brown and Woodward, those marked *EB* by Edwards and Brandt, those marked *B* by Beck. The point marked *MS* was obtained by Morrison and Smith. All these points are for antennas based on a plane and correspond to the theoretical curve 2. Points marked *DK* and *KK* are for dipole antennas in free space and go with the theoretical curve 1. The former were obtained by D. D. King and the latter by Ronold King and D. D. King.

These values are inconsistent. The X–R diagram is also badly out of shape. If we take the reactance swing as more nearly representative of the true value of the maximum antenna resistance, we find a good agreement with theory.

Another set of measured impedance characteristics of balanced dipole antennas was reported in August 1945, by Ronold King and D. D. King.* The maximum values of the resistance from their curves are shown by the points marked KK. These values are considerably higher than those predicted by our theory. The next set of impedance characteristics was published by D. D. King† in October 1946. The maximum values of the resistance, marked DK, are much closer to the theoretical curve 1.

The points marked EB were obtained by C. F. Edwards and R. H. Brandt‡ in 1949. They used $\lambda = 7.62$ cm.

It should be noted that the theoretical curves have been obtained on the assumption that $\log(2l/a)$ is large. This quantity is not large in the leftmost sections of the curves shown in Fig. 13.25. In this range the experimental values tend to be closer to the theoretical curve in which the top capacitance is neglected. The upper curve shown in Fig. 13.26 is the graph of equation 120 for the dipole antenna; the lower curve is for the monopole backed by a perfectly conducting plane. It will be recalled that equation 120 was obtained from equation 119 which in turn was obtained from equation 108 on the assumption that K_a is very large. Several experimental points are shown to indicate that the simple equation 120 is satisfactory. However, if $\lambda/2a < 50$, care should be taken, when using this formula, that there is no excessive capacitance at the base of the antenna or near it which is in parallel with the rest of the antenna. For example, according to Brown and Woodward, the maximum resistance of an antenna with a diameter of 10° is 160 ohms. In this case, $\lambda/2a = 36$ and the theoretical value would be 260 ohms if there were no base and near-base capacitance. But the base capacitive reactance turns out to be 275 ohms; in addition, there is substantial near-base capacitance.

When we turn to Hallén's theory, we find that the values of the antiresonant resistance depend very substantially on the choice of the "expansion parameter" used in solving a certain integral equation by successive approximations. This parameter corresponds to the

* Ronold King and D. D. King, Microwave impedance measurements with application to antennas, II, *Jour. Appl. Phys.*, **16**, August 1945, pp. 445–453.

† D. D. King, The measured impedance of cylindrical dipoles, *Jour. Appl. Phys.*, **17**, October 1946, pp. 844–851.

‡ Heretofore unpublished.

average characteristic impedance K_a of the mode theory; but its numerical value cannot be fixed as definitely as that of K_a. The definition of the expansion parameter involves an assumption that the antenna radius is very small, and gives the value $2 \log(2l/a) + C$, where C is an essentially indefinite constant (or a slowly varying function whose values are comparable to a constant). Thus, a variety of series expansions for the antenna current may be obtained. Analytically they are identical; that is, one series can be transformed into another. But the numerical values of the antiresonant resistance obtained from the first few terms of the expansion for practical values of $2l/a$ are

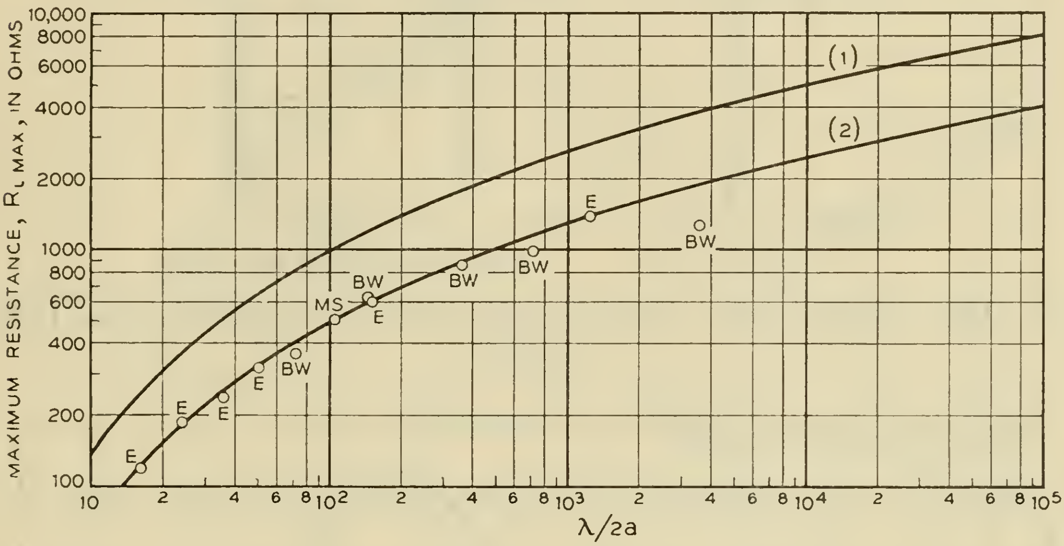


FIG. 13.26 The antiresonant impedance (solid curve) as given by the approximate formula 120 and some experimental points from Fig. 13.25.

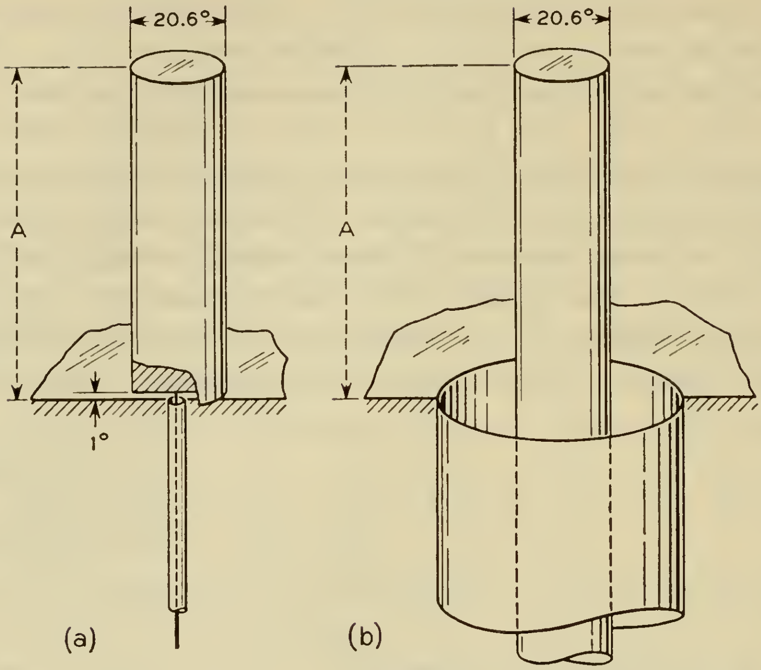
sensitive to the particular value of the constant C. This accounts for the differences between the values of the antiresonant resistance which have been reported in literature. The more recent values are in good agreement with those presented here.

In addition to the experimental data reported in this section, other measurements have been reported by Rösseler, Vilbig, and Vogt,* by Essen and Oliver,† by Smith and Holt Smith, and by Cochrane. For comparison between these data and theory the reader is referred to a book by R. A. Smith.‡

* G. Rösseler, F. Vilbig, and K. Vogt, Über das elektrische Verhalten von Vertikalantennen in Abhängigheit von ihrem Durchmesser, *TFT*, **28**, May 1939, pp. 170–178.

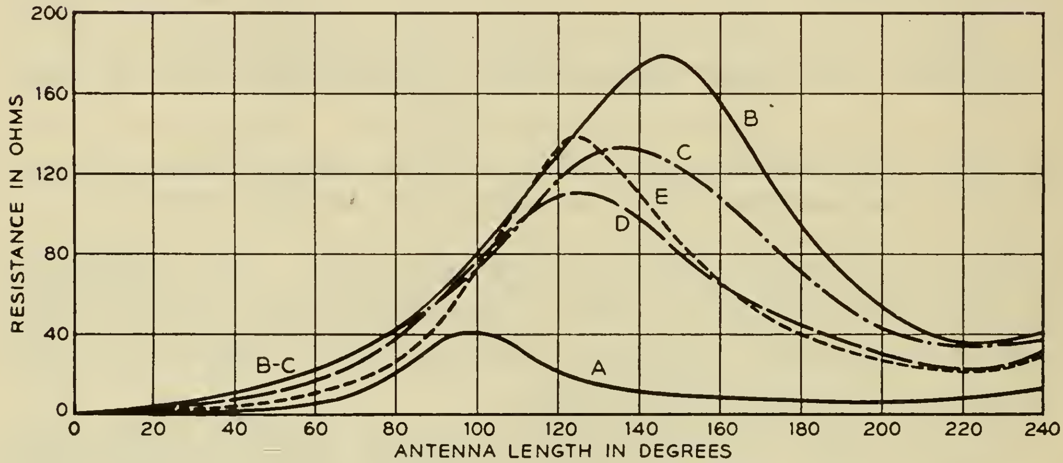
† L. Essen and M. H. Oliver, Aerial impedance measurements, *Wireless Eng.*, **22**, December 1945, pp. 589–593.

‡ R. A. Smith, *Aerials for Metre and Decimetre Wave-lengths*, University Press, Cambridge, 1949.



Brown and Woodward, Courtesy IRE

FIG. 13.27 Antennas whose impedance characteristics are presented in Figs. 13.28 and 13.29.

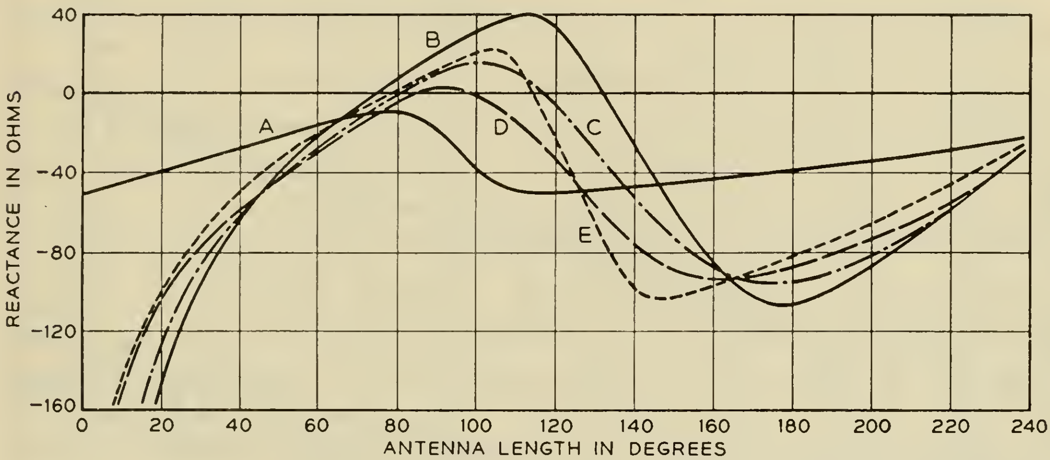


Brown and Woodward, Courtesy IRE

FIG. 13.28 Resistance as a function of antenna length A (in degrees). The diameter D is 20.6°. Curve A: the arrangement shown in Fig. 13.27a. Curve B: the arrangement of Fig. 13.27b, with the diameter of the outer conductor equal to 74°; the characteristic impedance of the transmission line is 77.0 ohms. Curve C: with the diameter of the outer conductor equal to 49.5°; the transmission line has a characteristic impedance of 52.5 ohms. Curve D: with the diameter of the outer conductor equal to 33°; the characteristic impedance is 28.3 ohms. Curve E: this curve was obtained for the antenna in Fig. 13.27(a) by tuning out the base reactance (but not the near-base reactance) with an inductive reactance.

13.22 Effects of base and near-base capacitance — experimental

To examine the effects of base and near-base capacitance on the antenna impedance, Brown and Woodward* used the arrangements shown in Fig. 13.27. Figures 13.28 and 13.29 present the measured resistance and reactance. Curves B, C, D refer to the arrangement shown in Fig. 13.27b. As the outer diameter of the coaxial pair decreases, the near-base capacitance increases and the maximum resistance decreases. Curve A refers to the arrangement shown in Fig. 13.27a where we have a large base capacitance in addition to the near-base capacitance. The



Brown and Woodward, Courtesy IRE

FIG. 13.29 Reactance curves corresponding to the resistance curves of Fig. 13.28.

maximum resistance is markedly depressed. Curve E refers to the same arrangement but with the base capacitance tuned out. It might be expected that, if the near-base capacitance were also tuned out, the maximum resistance would be still larger, and its position would move further to the right. If the base capacitance is excluded, the theoretical value of $R_{\max}$ is about 140 ohms.

The base capacitance and the corresponding admittance are

$$C_b = \frac{\varepsilon\pi a^2}{h}, \qquad j\omega C_b = \frac{j\pi a^2}{60\lambda h}, \tag{123}$$

where h is the distance between the base of the cylinder (Fig. 13.27a) and the ground plane. The near-base capacitance and admittance are approximately

$$C_{nb} = 4\varepsilon a \log\frac{a}{h}, \qquad j\omega C_{nb} = \frac{ja}{15\lambda}\log\frac{a}{h}\cdot \tag{124}$$

* George H. Brown and O. M. Woodward, Experimentally determined impedance characteristics of cylindrical antennas, *IRE, Proc.*, 33, April 1945, pp. 257–262.

Hence, the ratio of the admittances is

$$\frac{Y_{\mathrm{nb}}}{Y_{\mathrm{b}}} = \frac{4h}{\pi a} \log \frac{a}{h} \cdot \tag{125}$$

For the present arrangement, $h/a = 0.1$, and $Y_{\mathrm{nb}}/Y_{\mathrm{b}} = 0.29$. An inductive reactance equal to 65 ohms was needed to tune out the base reactance; hence, an inductive reactance of 50.4 ohms would be needed to tune out both the base and the near-base reactance.

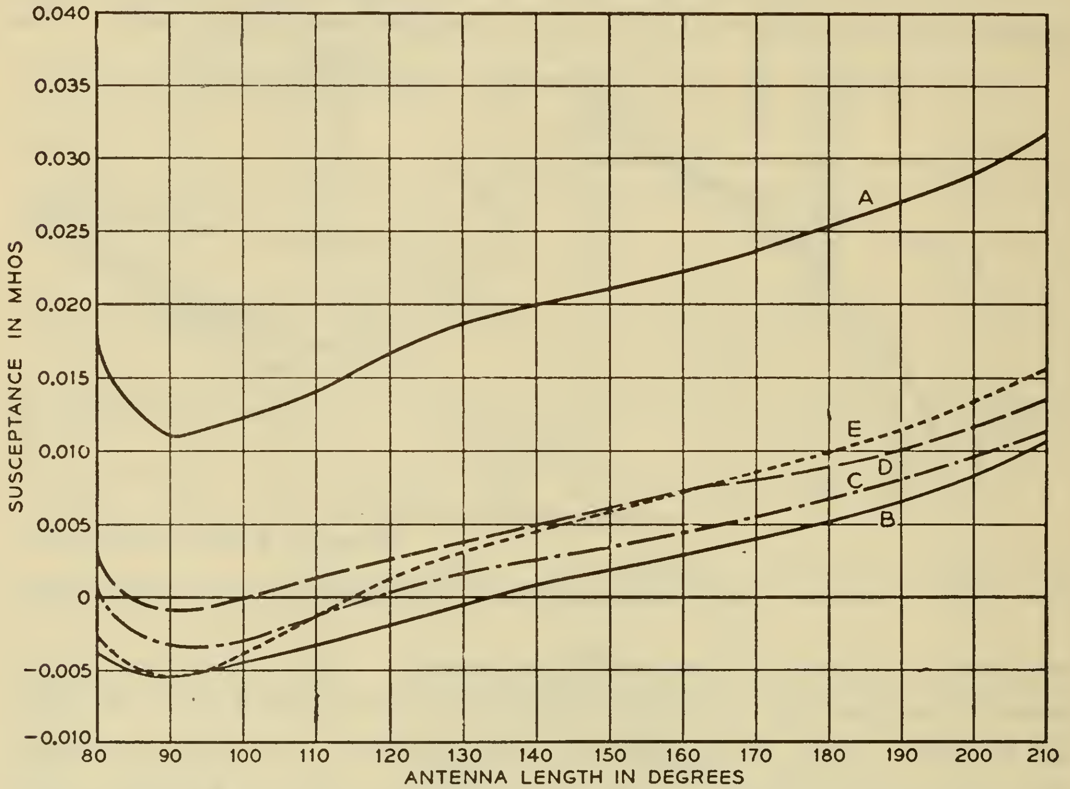


FIG. 13.30 The susceptance calculated from the data in Figs. 13.28 and 13.29.

Figure 13.30 shows the susceptance calculated from the data in Figs. 13.28 and 13.29. It shows clearly the difference between the capacitances of the input regions of various antennas.

Thus, Brown and Woodward's experiments confirm the theoretical conclusions of Section 12.10 with regard to the short-circuiting effect of the proximity between the antenna terminals. Even if we use the arrangement in Fig. 13.27*b*, with no overlap between conducting surfaces and, hence, no direct base capacitance, we still have the near-base capacitance, particularly for cylinders of large diameter. Theoretically this capacitance approaches infinity as the distance between the antenna terminals approaches zero, no matter how small the radius of the antenna as long as it is kept constant. Hence, if the gap is infinitesimal, so that

the antenna is driven by a hypothetical potential discontinuity across the driving point, the input impedance is automatically zero. Such idealized driving conditions are physically unrealizable; but they are valuable in theory because we can evaluate the distribution of the antenna current for an arbitrary distribution of the impressed field simply by integrating the current distribution for a unit voltage discontinuity across a typical driving point. The fact that the current at the driving point turns out to be infinite does not invalidate the method, since the integrals for the case when the voltage is distributed in a *finite* interval are convergent.

The idealized driving by a voltage discontinuity would have possessed even greater practical importance if this method of driving led to a finite input impedance, for then the impedance would have been substantially independent of the driving conditions, and the problem of its calculation would have been greatly simplified. As it is, we cannot avail ourselves of this simplification. At this point we cannot overlook a theoretical difficulty created by Hallén's method of successive approximations for the solution of an integral equation for the cylindrical antenna.* This method gives *a finite value* of the input impedance for a zero gap, which seemingly contradicts all other theoretical analyses. Another difficulty is created by a custom prevailing in circuit theory in which one speaks of a voltage at this or that point of a circuit or network and of the "driving-point" impedance. The implication is that the distance between the terminals of a generator or a passive element is not important and that presumably it can be infinitesimal without affecting the impedance.

The explanation of the first difficulty is that Hallén's method of successive approximations (in all its special applications, as far as the choice of the expansion parameter is concerned) involves a *tacit assumption* that the impressed voltage is distributed over a segment of *finite* length, even though in the original statement of the problem the voltage is supposed to be concentrated at a point. If we try to remove the tacit assumption and adhere strictly to the original assumption of driving by a potential discontinuity, Hallén's method of approximations breaks down immediately.

The second difficulty is removed as soon as we realize (see Section 2.4) that the custom of ignoring "gaps" in circuits is due to the fact that, for normal distances between the terminals of circuit elements and normal lengths of connecting leads, the stray capacitances are so small that at low frequencies their effects on the impedances across

* Erik Hallén, Theoretical investigations into the transmitting and receiving qualities of antennae, *Nova Acta*, Uppsala, 1938.

the terminals are negligible. Thus, the custom does not imply that these capacitances are unimportant under *any* conditions. In fact, in high-frequency circuits these effects are carefully scrutinized.

13.23 Resonant impedance — theory and experiment

The theoretical value of the resonant impedance of the infinitely thin, perfectly conducting half-wave dipole in free space is known exactly; it is $(\eta/4\pi)\text{Cin}\,2\pi = 73.129 \cdots$ ohms.* Figure 13.31 shows the resonant impedance as a function of the wavelength in diameters. The curve has been obtained from equation 108 with the end capacitance C_t

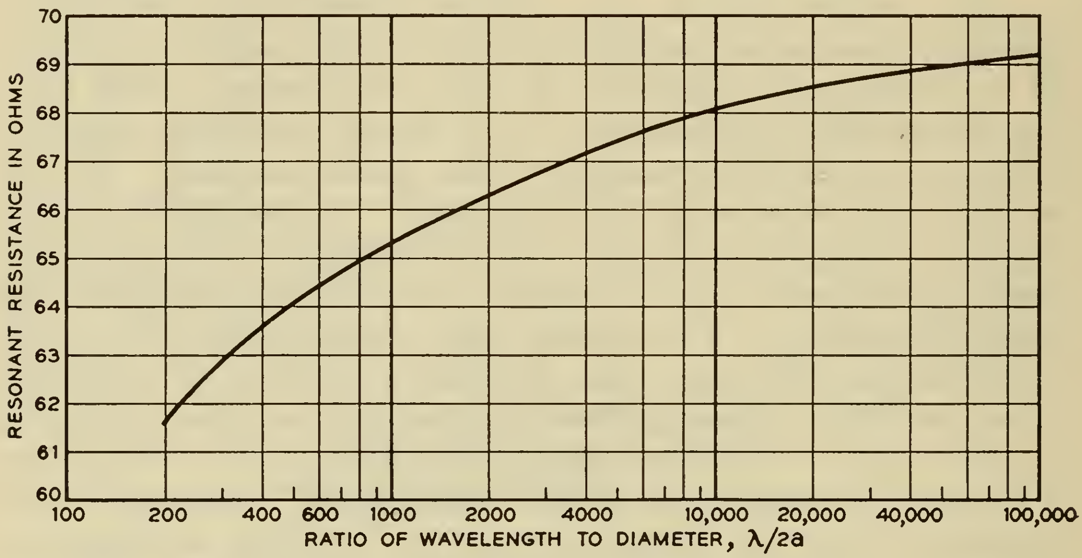


Fig. 13.31 The resonant resistance of a half-wave dipole according to the first approximation given by the mode theory.

and the higher-order terms in equation 111 neglected. If C_t is included, still lower values are obtained; for instance, when $\lambda/2a = 200$ we would find about 56 ohms instead of 61.6. In the vicinity of resonance, the input resistance is given by equation 109 and is substantially equal to $R_a(\beta l)$. The second term in equation 111 may easily affect R_a to the extent of several ohms.

The resonant impedances obtained by Hallén's method agree quite well with those in Fig. 13.31, provided the terms of order $1/K_a{}^2$ are included;† thus, if

$$\Omega = 2 \log \frac{2l}{a} = \frac{K_a}{60} + 2, \tag{126}$$

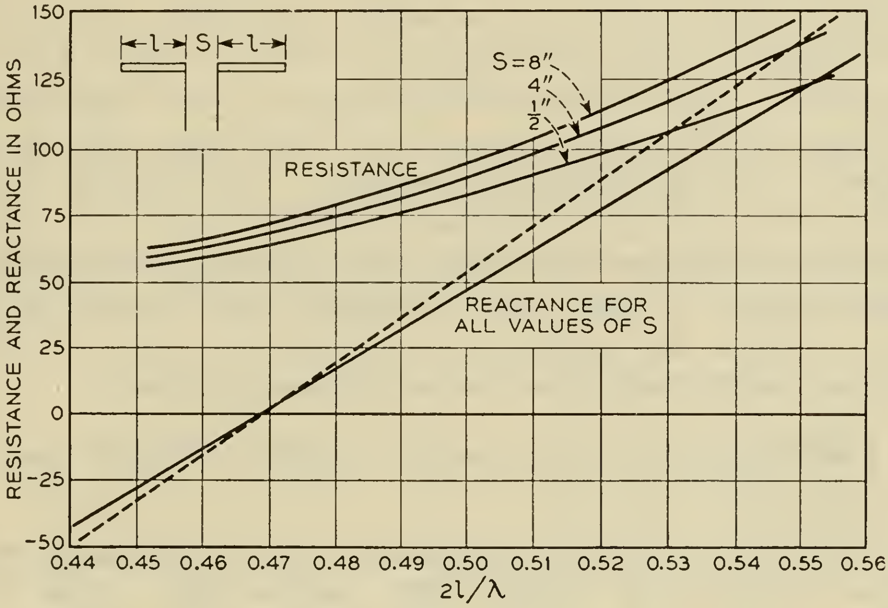
* This numerical value is based on the value 3×10^8 meters per second for the velocity of light and, hence, on 120π ohms for the intrinsic impedance of free space.

† C. J. Bouwkamp, Hallén's theory for a straight conducting wire, used as a transmitting or receiving aerial, *Physica*, 9, July 1942, pp. 609–631.

we have the following values:

$$\Omega = \quad 10, \quad 15, \quad 20,$$
$$R_{\text{res}} = 61.3, 66.3, 68.5 \text{ (Fig. 13.31)},$$
$$R_{\text{res}} = 60.4, 67.7, 70.5 \text{ (C. J. Bouwkamp)}.$$

The values are lower if only the terms of order $1/K_a$ are included; thus, for $\Omega = 15$, King and Blake* find 61.2 instead of 67.7 obtained by including an extra term. Values nearer 70 ohms were obtained by M. C. Gray and by King and Middleton using Hallén's method



R. A. Smith, Aerials for Metre and Decimetre Wave-lengths. University Press, Cambridge, 1949

Fig. 13.32 The impedance of dipoles with various center spacings. The solid curves show the values measured by Smith and Holt Smith ($\lambda = 6$ meters, $a = 0.625$ in., $\lambda/4a = 100$); the dotted curve shows measurements by Cochrane ($\lambda = 7.2$ meters, $a = 0.375$ in., $\lambda/4a = 200$).

but choosing expansion parameters differing from Ω. All these theoretical calculations are based on the assumption that the antenna gap is very small but not so small that the base capacitance is significant.

Thus, there is some uncertainty about the theoretical values of the resonant impedance. Experiments are mutually conflicting; some reports give 73 ohms or even larger values whereas others give smaller values. R. A. Smith† presents experimental evidence which shows

* Ronold King and F. G. Blake, Jr., The self-impedance of a symmetrical antenna; *IRE Proc.*, **30**, July 1942, pp. 335–349.

† R. A. Smith, *Aerials for Metre and Decimetre Wave-lengths*, University Press, Cambridge, 1949.

that, in the vicinity of resonance, the resistive part of the impedance increases with the length of the gap while the reactance remains the same. He shows curves (Fig. 13.32) for $\lambda = 6$ meters, $a = 0.625$ in., $\lambda/4a = 100$, and for three different spacings between the terminals of a balanced dipole, $s = 0.5$ in., 4 in., 8 in. For the longest gap, the resonant resistance is very nearly 73 ohms. For the shortest gap (which is comparable to the radius so that the near-base capacitance is still negligible), the resonant resistance is 61 ohms, compared with 61.6 ohms in Fig. 13.31. This is the kind of gap that is assumed in the mode theory of antennas.

Edwards and Brandt have measured consistently 36 ohms for unipoles backed by large metal sheets and fed by coaxial lines (see Section 13.21). Data have been obtained for $\lambda = 3$ in. In proportion to the length of the unipole, the gaps have been larger than the 8-in. gap referred to in the preceding paragraph. Smith reports that, in the vicinity of resonance, the impedance depends not only on the spacing at the center of the dipole but also on the method of connecting the transmission lines. It may then be expected that there will be some difference between the resistances of quarter-wave unipoles fed by coaxial lines and the half-values for the half-wave dipoles.

13.24 Resonant and antiresonant lengths — theory and experiment

Figures 13.33 and 13.34 show the resonant and antiresonant lengths of dipoles obtained from equation 108. In the upper curves 1, the end capacitance C_t is neglected; in the lower curves 2, it is taken into consideration.

Experimental values are scattered about these curves. Beck measured *antiresonant* lengths of two cage antennas for which $\lambda/2a = 60$ and obtained two values, 0.78 and 0.81. The first of these is slightly below curve 2, and the second is somewhat below curve 1. For another pair of cage antennas $\lambda/2a = 70$, and two values of $2l/\lambda$ are 0.80 and 0.835. The first of these falls exactly on curve 2, whereas the second is slightly above curve 1. For still another cage antenna for which $\lambda/2a = 80$, Beck obtained $2l/\lambda = 0.84$, which is slightly below curve 1.

For tubular conductors with $\lambda/2a = 200$, Brown and Woodward obtained $2l/\lambda = 0.85$ at $\lambda = 5$ meters; Feldman obtained 0.88 at $\lambda = 18$ meters. Edwards and Brandt obtained 0.90 at $\lambda = 3$ in. for a solid conductor with the same value of $\lambda/2a$. Feldman's value is near curve 2, whereas the Edwards-Brandt value is near curve 1. For an antenna with $\lambda/2a = 1000$, Feldman's value for $2l/\lambda$ is 0.92 while

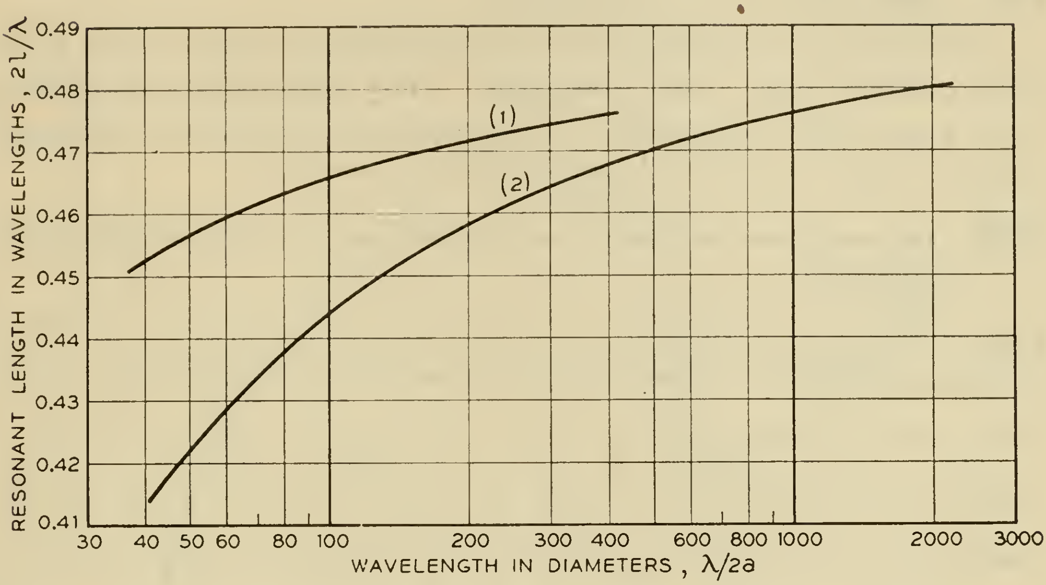


FIG. 13.33 Resonant length of a cylindrical antenna according to the mode theory: (1) the end capacitance C_t is neglected; (2) the end capacitance is included.

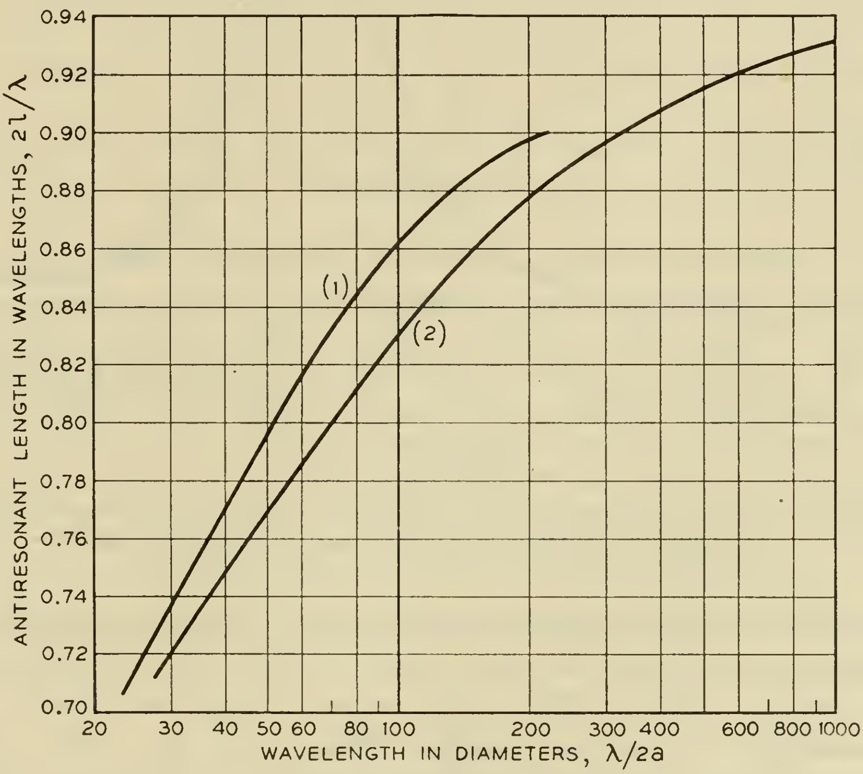


FIG. 13.34 Antiresonant length of a cylindrical antenna according to the mode theory: (1) the end capacitance is neglected; (2) the end capacitance is included.

curve 2 gives 0.93. Essen and Oliver obtain 0.92 for an antenna with $\lambda/2a = 400$ while curve 2 gives 0.908.

For half-wave antennas Brown and Woodward obtain $2l/\lambda = 0.467$ 0.473, 0.476 for $\lambda/2a = 400, 800, 2000$. The corresponding values from curve 2 are 0.467, 0.474, 0.48. For an antenna with $\lambda/2a = 100$ Smith and Holt Smith find 0.467 which is close to curve 1.

13.25 Dependence of the input impedance on the shape of the antenna

Figures 13.35 and 13.36 show the input resistance and reactance of three antennas of different shapes (from the mode theory of antennas). The

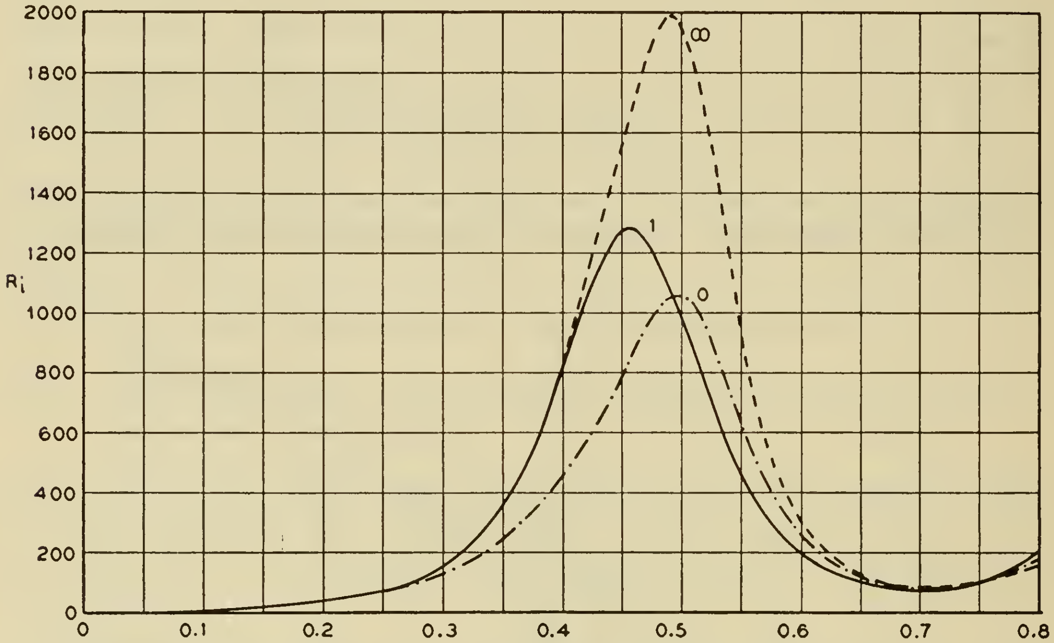


Fig. 13.35 The input resistance of three dipoles of different shape having the same average characteristic impedance, 600 ohms (for cylinder $l/2a \simeq 100$): The curve having the greatest maximum is for the cone, the next is for the cylinder, and the lowest is for the inverted cone with its base at the input terminals. The numbers marking the curves are the ratios of the diameters at the top and the bottom. The end capacitance is included for the cone and cylinder.

average characteristic impedances of these antennas equal 600 ohms. The numbers opposite the curves are for the ratio of the top radius to the bottom radius. Thus, the curve having the highest resistance maximum is for the biconical dipole, the one with the next highest maximum is for the cylindrical dipole, and the one with the lowest maximum is for two cones arranged base to base.

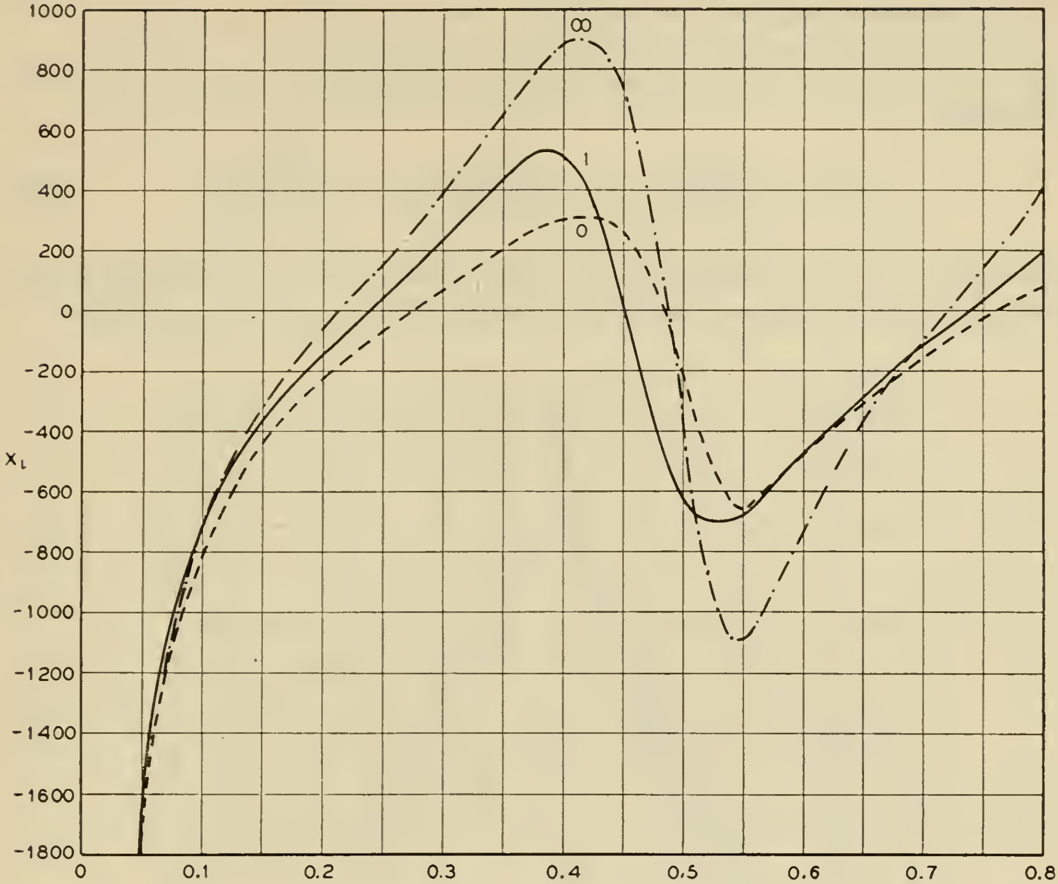


FIG. 13.36 The input reactance of the antennas described in the caption to the preceding figure.

13.26 The input resistance of infinitely thin antennas

The dotted line in Fig. 13.37 represents the exact input resistance of an infinitely thin perfectly conducting symmetric dipole. The equation for this resistance is

$$R_i = \frac{R_a(\beta l)}{\sin^2 \beta l} \cdot \tag{127}$$

The current distribution in *any* infinitely thin perfectly conducting antenna is exactly sinusoidal. Hence, we can determine the exact radiated power and the exact input current. Then the exact input resistance is

$$R_i = \frac{2P}{I_i^2} \cdot \tag{128}$$

If I_1 and I_2 are the maximum amplitudes of the currents in the arms of an asymmetrically fed straight dipole, the radiated power is of the form

$$P = \tfrac{1}{2}R_{11}I_1^2 + R_{12}I_1I_2 + \tfrac{1}{2}R_{22}I_2^2. \tag{129}$$

If l_1, l_2 are the lengths of the arms,

$$I_i = I_1 \sin \beta l_1 = I_2 \sin \beta l_2; \tag{130}$$

therefore,

$$R_i = \frac{R_{11}}{\sin^2 \beta l_1} + \frac{2R_{12}}{\sin \beta l_1 \sin \beta l_2} + \frac{R_{22}}{\sin^2 \beta l_2}. \tag{131}$$

Similarly, we can determine the input resistance of an infinitely thin loop, circular or square, fed at the corner of the square or elsewhere.

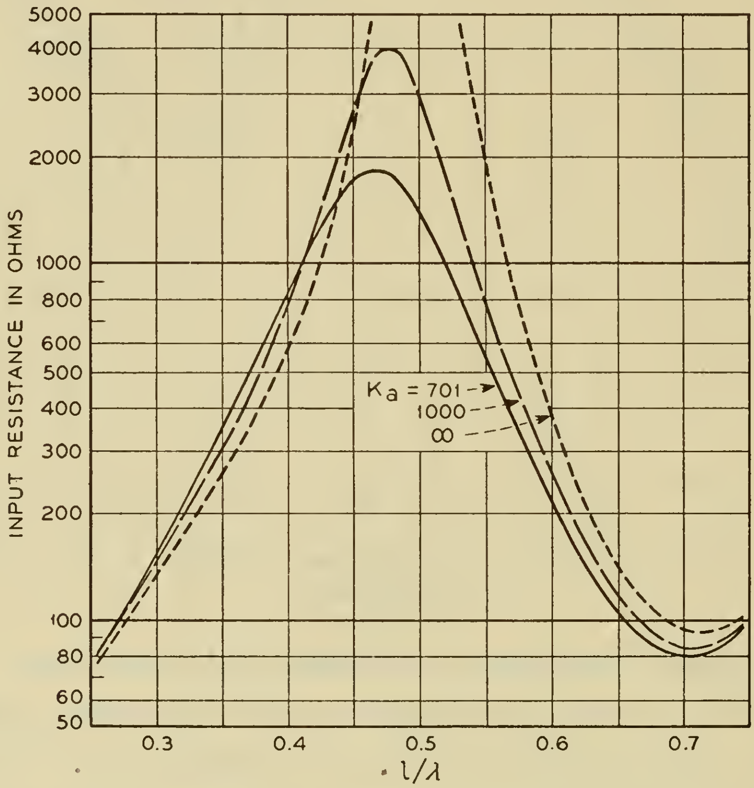


FIG. 13.37 The input resistance of three cylindrical antennas. The dotted curve is exact.

13.27 Comparison between the elementary theory and the mode theory of antennas

Figure 13.38 shows the input resistance of a very thin cylindrical antenna according to the mode theory (the solid curve) and according to the elementary theory (equation 64).

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1. E. C. Jordan, *Electromagnetic Waves and Radiating Systems*, Prentice-Hall, New York, 1950, Chapter 13.

2. S. A. Schelkunoff, *Advanced Antenna Theory,* John Wiley, New York, to be published. This book presents the mode theory of antennas, Hallén's theory, and the theory of spheroidal antennas. It also contains pertinent references to the original papers.

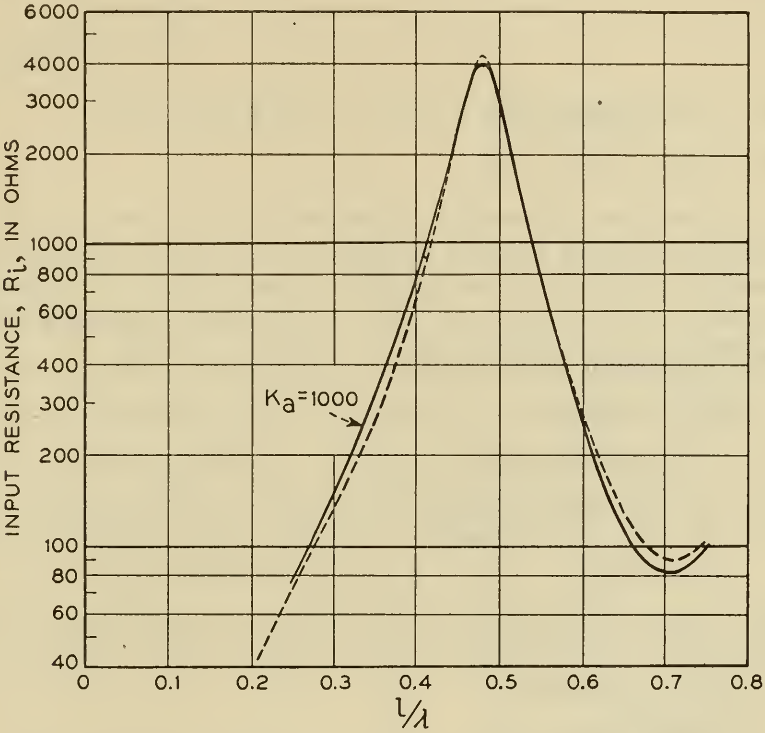


FIG. 13.38 The input resistance of a very thin cylindrical dipole: the solid curve according to the mode theory, the dotted curve according to the elementary theory (equation 64).

3. A. S. Meier and W. P. Summers, Measured impedance of vertical antennas over finite ground planes, *IRE Proc.*, 37, June 1949, pp. 609–616.

PROBLEMS

13.1–1 Consider two parallel capacitor antennas perpendicular to the line joining their centers. Let their input impedances in free space be, respectively, z_1 and z_2, and let the terminals of the second antenna be short-circuited. Obtain directly from field considerations: (*a*) V_1 when I_1 has a given fixed value, (*b*) I_1 when V_1 has a given fixed value. Show that the ratio V_1/I_1 is the same, irrespective of the method of its calculation, and that it is actually equal to $Z_{11} - (Z_{12}^2/Z_{22})$.

Ans. (*a*) $$V_1 = \left[Z_1 - \frac{(s_1Ts_2)^2}{Z_2}\right]I_1,$$

(*b*) $$I_1 = \frac{V_1}{Z_1}\left[1 + \frac{(s_1Ts_2)^2}{Z_1Z_2} + \frac{(s_1Ts_2)^4}{(Z_1Z_2)^2} + \cdots\right].$$

13.2–1 Show that the electric intensity parallel to the current filament given by equation 8–106 is

$$E_z = 30jA\left[\frac{\exp(-j\beta r_\xi)}{r_\xi}\sin 2\beta l - \frac{\exp(-j\beta r_l)}{r_l}\sin\beta(l+\xi) - \frac{\exp(-j\beta r_{-l})}{r_{-l}}\sin\beta(l-\xi)\right].$$

13.2–2 Derive equation 19.

13.2–3 Derive equation 20.

13.2–4 Obtain the asymptotic expressions for the current in and the electric intensity parallel to a linear antenna of length $l + \frac{1}{2}\lambda$ when a generator and a resistance are inserted at distances from the ends equal to $\frac{1}{4}\lambda$. The resistance is such that the current wave between the generator and the resistance is progressive. Assume that the antenna extends from $z = -\frac{1}{4}\lambda$ to $z = l + \frac{1}{4}\lambda$.

Ans.
$$\begin{aligned} I(z) &= I_0 \cos\beta z, && -\tfrac{1}{4}\lambda \le z \le 0; \\ &= I_0 \exp(-j\beta z), && 0 \le z \le l; \\ &= I_0 \exp(-j\beta l)\cos\beta(z-l), && l \le z \le l + \tfrac{1}{4}\lambda. \end{aligned}$$

$$E_z = 30jI_0\left(-\frac{e^{-j\beta r_1}}{r_1} + j\frac{e^{-j\beta r_0}}{r_0} - je^{-j\beta l}\frac{e^{-j\beta r_2}}{r_2} - e^{-j\beta l}\frac{e^{-j\beta r_3}}{r_3}\right),$$

where r_1, r_0, r_2, r_3 are the distances from $z = -\frac{1}{4}\lambda$, $z = 0$, $z = l$, $z = l + \frac{1}{4}\lambda$.

13.3–1 Derive equation 24.

13.3–2 Derive equation 25.

13.7–1 Obtain the radiation resistance (with reference to the maximum current amplitude) of a progressive current wave on a filament of length l. Use equation 12–17.

Ans. See equation 6–65.

13.7–2 Obtain the radiation resistance (with reference to the current antinode) of a symmetrical antenna loaded at both ends. Let the current distribution be $I(z) = I_0 \sin(\vartheta - \beta|z|)$ so that $I(0) = I_0 \sin\vartheta$ and $I(l) = I_0\sin(\vartheta - \beta l)$. Do it by two methods, one based on equation 12–13 and the other on equation 12–17.

Ans.
$$R_a = 30(\mathrm{Si}\, 4\beta l - 2\,\mathrm{Si}\, 2\beta l)\sin 2\vartheta + 30(\mathrm{Ci}\, 4\beta l - 2\,\mathrm{Ci}\, 2\beta l + \log\beta l + C)\cos 2\vartheta + 60\left[\left(\frac{\sin 2\beta l}{2\beta l} - 1\right)\sin^2(\vartheta - \beta l) - \mathrm{Ci}\, 2\beta l + \log\beta l + C + \log 2\right].$$

13.7–3 Obtain the radiation resistance with reference to the current antinode of the antenna described in Problem 13.2–4.

Ans.
$$\begin{aligned} R_a = {} & 15\,[\mathrm{Cin}(2\beta l + 2\pi) + 2\,\mathrm{Cin}(2\beta l + \pi) + \mathrm{Cin}\, 2\beta l] + \\ & 15\cos 2\beta l\,[\mathrm{Cin}(2\beta l + 2\pi) - 2\,\mathrm{Cin}(2\beta l + \pi) + \mathrm{Cin}\, 2\beta l] + \\ & 15\sin 2\beta l\,[-\mathrm{Si}(2\beta l + 2\pi) + 2\,\mathrm{Si}(2\beta l + \pi) - \mathrm{Si}\, 2\beta l]. \end{aligned}$$

14

RHOMBIC ANTENNAS

14.1 Rhombic antennas

*Rhombic antennas** consist of four straight wires arranged in the form of a rhombus (Fig. 14.1) or of four systems of wires similarly arranged. Rhombic antennas are usually terminated into an impedance such that the current waves in the wires are substantially traveling waves. The object is to throw the main beam of radiation in the forward y axis direction.

Rhombic antennas are essentially transmission lines deliberately designed to radiate power. They are end-fire arrays of current elements

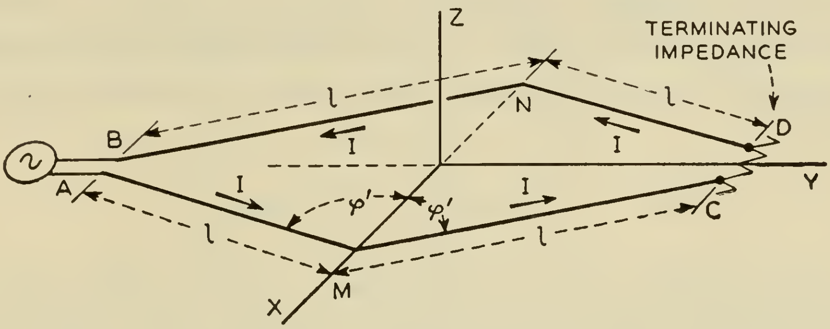


Fig. 14.1 A rhombic antenna.

inclined to the desired direction of radiation. The current elements must be inclined since they do not radiate along their axes.

Of course, considerable power is lost in the terminating impedance. This loss is the price paid for desirable features such as extreme simplicity coupled with high directivity.

14.2 Input Impedance

The input impedance of two infinitely long diverging wires is given by equation 13–95. From physical considerations it may be assumed that

*E. Bruce, Developments in short wave directive antennas *IRE Proc.*, **19,** August 1931, pp. 1406–1433.

this is also an approximate expression for the input impedance of the rhombic antenna, so terminated that reflections are reduced to a minimum. Thus, we have

$$Z_i = 120 \log \frac{\lambda}{2\pi a} - 72 + 120 \log \cos \varphi' - j170. \tag{1}$$

If, for example, $\varphi' = 70°$, $a = 0.041$ in., then:

$$\begin{aligned} \lambda &= 15, \qquad 26, \qquad 45 \text{ meters}, \\ Z_i &= 728 - j170, \quad 794 - j170, \quad 860 - j170 \text{ ohms}. \end{aligned} \tag{2}$$

Early experiments (1932) with a rhombic for which $\varphi' = 70°$, $l = 96$ meters, $a = 0.041$ in. showed that the best termination over the range of wavelengths from 15 to 45 meters was 820 ohms. When so terminated, the resistive component of the input impedance varied from about 660 ohms at 15 meters to about 830 ohms at 45 meters. Considering the difficulties of making such measurements at that time, and the fact that 820 ohms was a compromise termination, the experimental values check the calculated values satisfactorily.

14.3 Current distribution

Even without radiation, the current along the rhombic antenna will vary unless the wires are conical. The nominal characteristic impedance of the rhombic (for the principal waves) may be obtained from equation 4–42:

$$K(r) = 120 \log \frac{2r}{a} + 120 \log \cos \varphi', \tag{3}$$

where r is the distance from the generator. It can be shown that when $K(r)$ varies slowly, the current varies inversely as the square root of $K(r)$. This condition is satisfied between points about a quarter wavelength from the generator and from the load. Thus, there is a gradual fall and rise in the amplitude of the current. On this we must superimpose the attenuation due to radiation. Experimental curves showing the current distribution in rhombic antennas are given by W. N. Christiansen.*

The smaller the radius of the wires, the narrower the channels around the wires in which most of the energy leaving the generator travels. Thin wires are better guides for energy around the corners M, N of the rhombic. At the same time a greater proportion of the emitted power is dissipated in the load, and the antenna is less efficient. Thus,

* Rhombic antenna arrays, *AWA Tech. Rev.*, 7, October 1947, pp. 361–383.

to increase the efficiency, we must increase the radius of the wires. A practical method is to use two divergent wires for each wire of the simple rhombic. Since the effective radius is the geometric mean of the radius of each wire and the distance between them, we obtain better returns than we would by increasing the radius of a single wire. The lower characteristic impedance of the rhombic also reduces the fluctuations in the input impedance caused by inevitable mismatches, especially those arising from the clashing wavefronts (Fig. 8.23).

14.4 Radiation intensity

The shape of the radiation pattern of a rhombic is not sensitive to the distribution of the amplitude of the current; the phase velocity of the current waves is the important factor, and this is equal to the velocity in free space. But the absolute value of the radiation intensity does depend on the attenuation of the current waves; so does the total radiated power. Since the shape of the pattern remains substantially the same, the directivity is not appreciably affected by the attenuation.

Assuming uniform progressive currents, we obtain the following expression for the radiation intensity in the direction determined by the angles θ and φ (Fig. 5.1) of the rhombic (Fig. 14.1) in free space,

$$\Phi = 240\pi \left(\frac{l}{\lambda}\right)^2 \cos^2\varphi' \,\frac{\sin^2 u_1}{u_1}\,\frac{\sin^2 u_2}{u_2}\, I^2,$$

$$u_1 = \frac{\pi l}{\lambda}\,[1 - \sin\theta\cos(\varphi - \varphi')], \tag{4}$$

$$u_2 = \frac{\pi l}{\lambda}\,[1 + \sin\theta\cos(\varphi + \varphi')].$$

For a general expression for the field distribution from a rhombic antenna carrying exponentially damped progressive waves, see E. G. Hoffmann, *Hochfreqtechn. u. Electroakus.;* **62**, July 1943, pp. 15–20.

14.5 Optimum angle

The interior angle φ' is generally chosen to make the radiation intensity in the direction of the y axis maximum. In this direction $\varphi = \theta = \pi/2$, and

$$\Phi = 240\pi \left(\frac{l}{\lambda}\right)^2 \cos^2\varphi' \,\frac{\sin^4\left[\dfrac{\pi l}{\lambda}(1 - \sin\varphi')\right]}{\left[\dfrac{\pi l}{\lambda}(1 - \sin\varphi')\right]^2}\, I^2. \tag{5}$$

Equating to zero the derivative with respect to φ', we find

$$\tan\left[\frac{\pi l}{\lambda}(1 - \sin\varphi')\right] = \frac{2\pi l}{\lambda}\cos^2\varphi'. \tag{6}$$

Solving, we obtain, for optimum rhombics,

$$\begin{aligned} \frac{l}{\lambda} &= 1.5, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8; \\ \varphi_{\text{opt}}' &= 45°.4, \quad 51°.5, \quad 58°.6, \quad 62°.9, \quad 67°.9, \quad 70°.9. \end{aligned} \tag{7}$$

14.6 Shape of the major lobe

In the vertical plane, $\varphi = \pi/2$, the radiation intensity is

$$\Phi = 240\pi\left(\frac{l}{\lambda}\right)^2 \cos^2\varphi' \frac{\sin^4 u}{u^2} I^2, \qquad u = \frac{\pi l}{\lambda}(1 - \sin\theta\sin\varphi'). \tag{8}$$

In the horizontal plane, $\theta = \pi/2$; in this case there is no essential simplification in equation 4.

In the vicinity of $\theta = \pi/2$, the parameter u in equation 8 is a slowly varying function of θ; hence, the major lobe in the vertical plane is blunt. This shape is characteristic of end-fire arrays and could have been expected since, in the vertical plane, the rhombic is essentially an end-fire array. In the plane of the rhombic, on the other hand, the direct and return branches form a broadside array, and we should expect a sharper pattern. The rhombic antenna is, therefore, especially useful in short-wave radio circuits in which the vertical angle of arrival varies considerably.

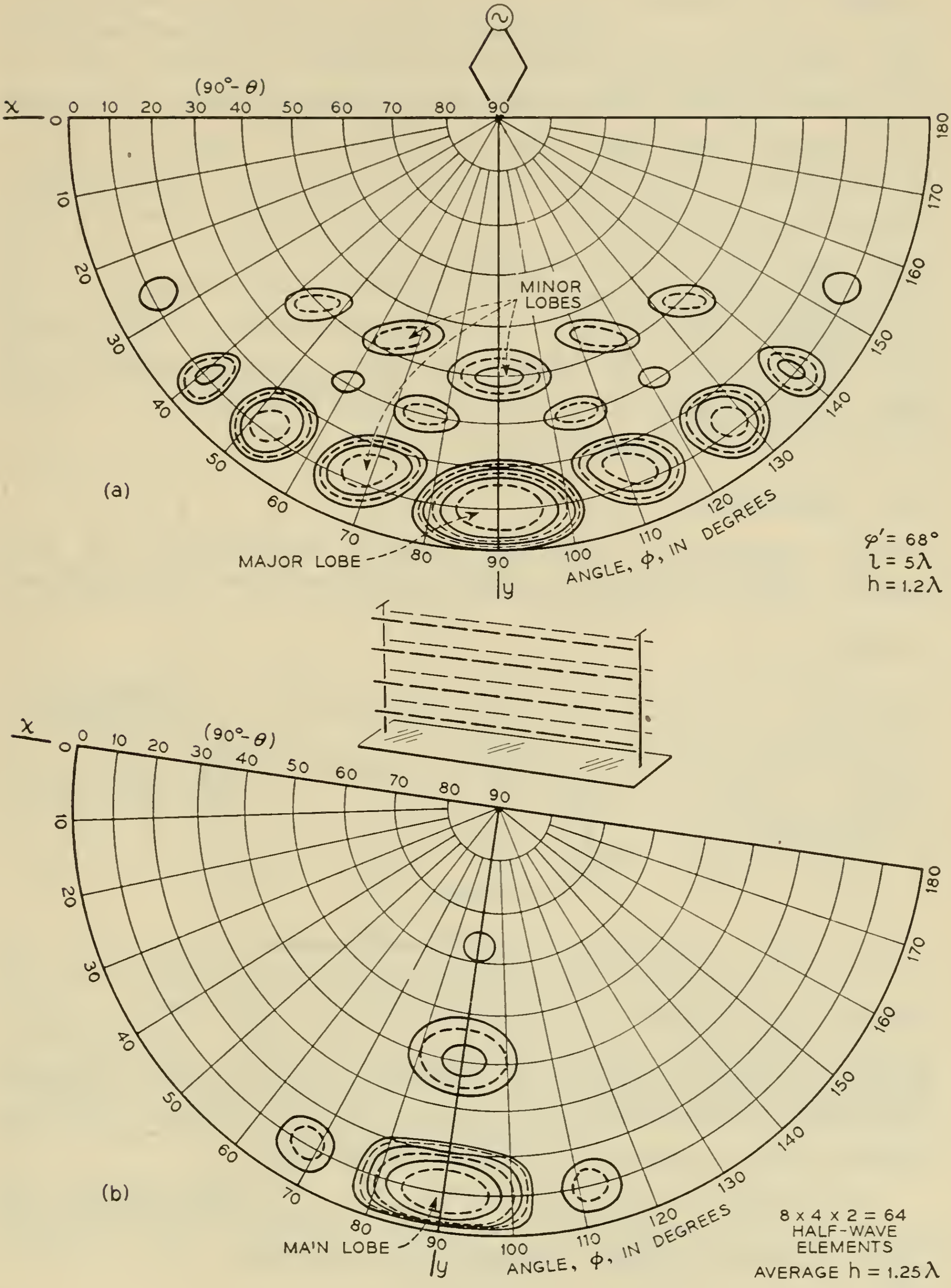
14.7 Minor lobes

The secondary lobes of the radiation pattern of the rhombic are large,* much larger than for ordinary end-fire and broadside arrays. A graphic picture that illustrates the major and minor lobes by means of intensity lines was prepared by W. N. Christiansen† (Fig. 14.2). This picture is for antennas above a perfect ground, so that the interference between the rhombic and its image is included. The contour lines are the lines of equal radiation intensity. The centers indicate the directions of maximum radiation within each lobe.

The minor lobes are large because, from the point of view of directivity, the current elements in a progressive wave along a wire are poorly

* Donald Foster, Radiation from rhombic antennas, *IRE Proc.*, **25**, October 1937, pp. 1327–1353.

† Directional patterns of rhombic antennae, *AWA Tech. Rev.*, **7**, September 1946, pp. 33–51.



Christiansen, Courtesy AWA Tech. Rev.

FIG. 14.2 A graphic picture of radiation (*a*) from a rhombic antenna and (*b*) from a pine-tree array. The contour lines are the lines of equal radiation intensity. The centers indicate directions of maximum radiation within each lobe.

arranged. For a single wire the space factor is maximum in the direction of the current wave; but in that direction the pattern of the element has a null. The space factor is given by equation 5–41 with $k = \beta$; and the electric intensity of the element is proportional to $\sin\theta$. For small θ the envelope of the space factor S varies as $1/\theta^2$; hence, the field of the progressive current wave varies as $1/\theta$, the pattern of the element counteracting the space factor. The θ's corresponding to the directions of the major lobe and the adjacent lobe are in the ratio of 1 to $\sqrt{3}$; hence, the radiation intensity of the single wire in the direction of the second lobe is only a third of the maximum intensity, or only 4.8 db down. In the case of the rhombic, the largest secondary lobe is 5.5 db below the major lobe; the smaller size of this lobe is due to the interference between the four arms of the rhombic.

14.8 The ground

Rhombic antennas are generally mounted horizontally over level ground. Figure 14.3 shows a typical rhombic antenna of the type used

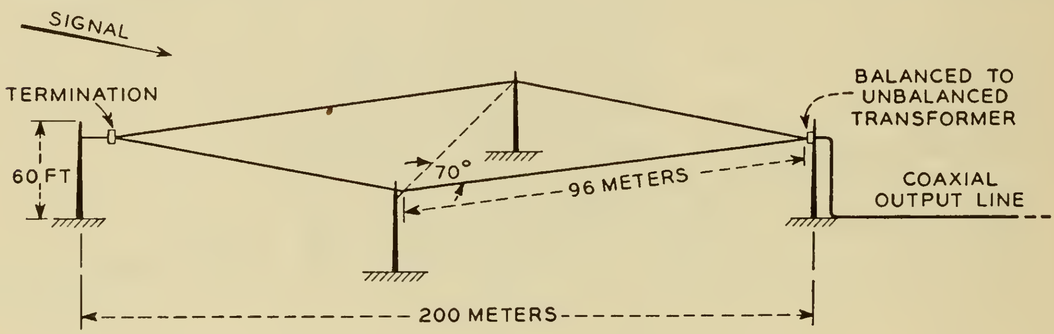


Fig. 14.3 A typical rhombic antenna.

in short-wave reception. Its dimensions are: $l = 96$ meters, $\varphi' = 70°$; and the height above the ground is $h = 17.6$ meters. The radiation patterns are shown in Fig. 14.4. Mechanical integration of the radiation pattern of this antenna revealed that in free space half the total radiated power is radiated through the minor lobes.

In obtaining the radiation intensity of a horizontal antenna, the ground may be considered as a perfect conductor. The image is negative, and the radiation intensity above ground is

$$\Phi_1 = 4 \sin^2 \left(\frac{2\pi h}{\lambda} \cos\theta \right) \Phi, \tag{9}$$

where Φ is the radiation intensity in free space; the remaining factor is the space factor of the antenna and its image.

The effect of the ground on the radiation resistance of a rhombic is

usually small. For $l = 6\lambda$ and $h = 1.1\lambda$, the radiation resistance is about 10 per cent above its value in free space.

Poor ground will increase the attenuation due to heat loss. In a single, long wire the series resistance due to the ground is about $10/h$

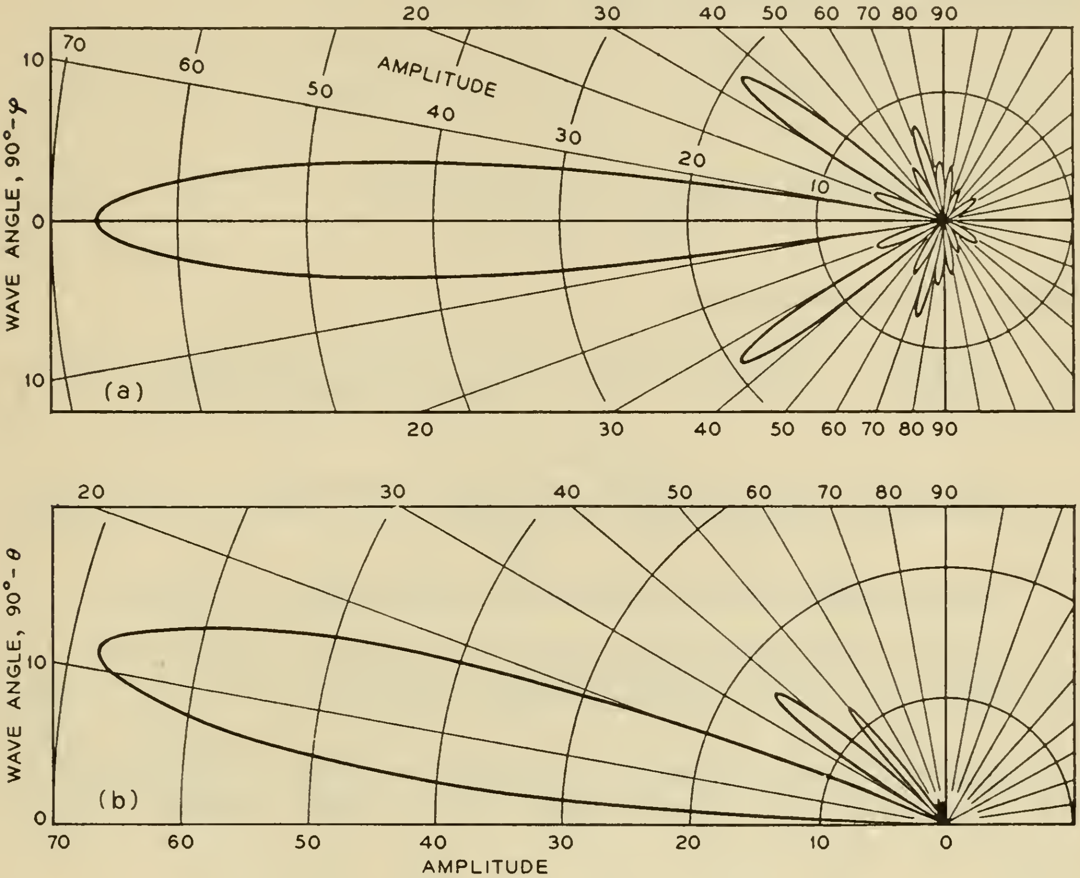


Fig. 14.4 Radiation patterns of the rhombic antenna shown in Fig. 14.3: (*a*) azimuthal pattern over a perfectly conducting ground when $\lambda = 16$ m, $l/\lambda = 6$, $\varphi' = 70°$, $h/\lambda = 1.1$, $\theta = 80°$; (*b*) vertical pattern.

ohms per meter at $\lambda = 18$ meters and "Holmdel ground" ($\varepsilon_r = 20$, $g = 0.015$). At $\lambda = 16$ meters and $l = 96$ meters the copper and ground-effect losses in the wires are about 0.5 db.

14.9 Power gain

If R_i is the input resistance of the rhombic, the power input is

$$P = \tfrac{1}{2} R_i I_0^2, \tag{10}$$

where I_0 is the amplitude of the input current. Hence, the average radiation intensity is

$$\Phi_{av} = \frac{P}{4\pi} = \frac{1}{8\pi} R_i I_0^2. \tag{11}$$

Assuming that the current waves are exponentially damped,

$$I(s) = I_0 e^{-\alpha s - j\beta s}, \tag{12}$$

where s is the distance from A, we obtain the electric intensity in the y direction of the rhombic antenna in free space,

$$E_x = -\frac{j120\pi I_0 \cos\varphi'}{\lambda r}[1 - e^{-\alpha l - j\beta(1-\sin\varphi')l}]\int_0^l e^{-\alpha s - j\beta(1-\sin\varphi')s}\,ds$$

$$= -\frac{j120\pi I_0 \cos\varphi'[1 - e^{-\alpha l - j\beta(1-\sin\varphi')l}]^2}{\lambda r\,[\alpha + j\beta(1 - \sin\varphi')]}. \tag{13}$$

Hence, in the y direction,

$$\Phi = \frac{r^2 E_x E_x^*}{240\pi} = \frac{60\pi I_0^2 \cos^2\varphi'[1 - 2e^{-\alpha l}\cos\beta(1 - \sin\varphi')l + e^{-2\alpha l}]^2}{\lambda^2\,[\alpha^2 + \beta^2(1 - \sin\varphi')^2]}$$

$$= \frac{240\pi I_0^2 e^{-2\alpha l}\cos^2\varphi'[\cosh\alpha l - \cos\beta(1 - \sin\varphi')l]^2}{\lambda^2[\alpha^2 + \beta^2(1 - \sin\varphi')^2]}. \tag{14}$$

The power gain of the rhombic in the y direction is therefore

$$g_P = \frac{\Phi}{\Phi_{\text{av}}} = \frac{1920\pi^2 e^{-2\alpha l}\cos^2\varphi'[\cosh\alpha l - \cos\beta(1 - \sin\varphi')l]^2}{R_i\lambda^2[\alpha^2 + \beta^2(1 - \sin\varphi')^2]}. \tag{15}$$

In practice, the attenuation constant α is small. Neglecting α^2 in the denominator and approximating $\cosh\alpha$ by unity, we find

$$g_P = \frac{1920 e^{-2\alpha l}\cos^2\varphi'\sin^4\dfrac{\pi l}{\lambda}(1 - \sin\varphi')}{R_i(1 - \sin\varphi')^2}. \tag{16}$$

For a half-wave antenna, the power gain (and its directivity) is

$$g = \frac{120}{R_1}, \qquad R_1 = 73. \tag{17}$$

Dividing equation 16 by equation 17, we obtain the power of the rhombic with respect to the half-wave antenna.

It is to be noted that the exponential factor,

$$e^{-2\alpha l} = \frac{I_T}{I_0}, \tag{18}$$

equals the ratio of the amplitudes of the current through the termination and the input current.

14.10 Directivity

The power radiated by a rhombic antenna may be obtained by either of the two methods described in Chapter 5, that is, either by integrating the radiation intensity over a unit sphere or by integrating the product $-\frac{1}{2}E_sI^*$ along the antenna. If the current is assumed to be sinusoidal, the second method is easier. In order to find E_s, it is necessary to calculate the components of E parallel to a current filament (equation 12–47) and perpendicular to it.* Using this method and assuming uniform progressive waves† Marion C. Gray obtained the following expressions for the radiated power and directivity, for optimum rhombics in free space,

$$P_{\text{rad}} = 120\,[\log(2\beta l \cos^2 \varphi') + 0.577\cdots]I_0^2, \tag{19}$$

$$g_D = \frac{8(\beta l - u')\sin^4 u'}{u'[\log(2\beta l \cos^2 \varphi') + 0.577]}, \qquad u' = \frac{\pi l}{\lambda}(1 - \sin \varphi'),$$
$$G_D = 10 \log_{10} g_D. \tag{20}$$

From this we obtain

$$\begin{aligned} l/\lambda &= 1.5, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8; \\ G_D &= 11.9, \quad 13.2, \quad 15.1, \quad 16.4, \quad 18.2, \quad 19.5. \end{aligned} \tag{21}$$

These values indicate that the power radiated because of the mutual interaction between the several arms of the rhombic is not negligible. Thus, adding 6 db (to allow for the four arms of the rhombic) to the directivity of a single wire carrying a progressive wave (see equation 6–68), we obtain

$$\begin{aligned} l/\lambda &= 1.5, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8; \\ G &= 13.4, \quad 14.1, \quad 15.3, \quad 16.2, \quad 17.5, \quad 18.4. \end{aligned} \tag{22}$$

The gain in the forward direction is reduced if $\varphi' \neq \varphi'_{\text{opt}}$. The reduction is obtained from the radiation intensity (equation 5) if we assume that the radiated power is approximately independent of φ'. Thus,

$$\frac{g(\varphi'_{\text{opt}})}{g(\varphi')} = \frac{\cos^2\varphi'_{\text{opt}}\left[\frac{\pi l}{\lambda}(1 - \sin\varphi')\right]^2 \sin^4\left[\frac{\pi l}{\lambda}(1 - \sin\varphi'_{\text{opt}})\right]}{\cos^2\varphi'\left[\frac{\pi l}{\lambda}(1 - \sin\varphi'_{\text{opt}})\right]^2 \sin^4\left[\frac{\pi l}{\lambda}(1 - \sin\varphi')\right]}. \tag{23}$$

Such a reduction in gain occurs when a fixed antenna is used over a large wavelength range, and it is illustrated in Fig. 14.5 for an antenna

* *Electromagnetic Waves*, p. 371, equation 9.25–12.

† The directivity varies very slowly with changes in attenuation (see Problem 6.1–11).

with $l = 96$ meters and $\varphi' = 68°$. In long-distance short-wave circuits, the interior angle φ' is made equal to its optimum value for the shortest wavelength in the range. The curve reveals a 4.5-db loss when the wavelength is twice as large. This loss occurs only for the signals arriving in the horizontal direction. Actually in short-wave circuits the

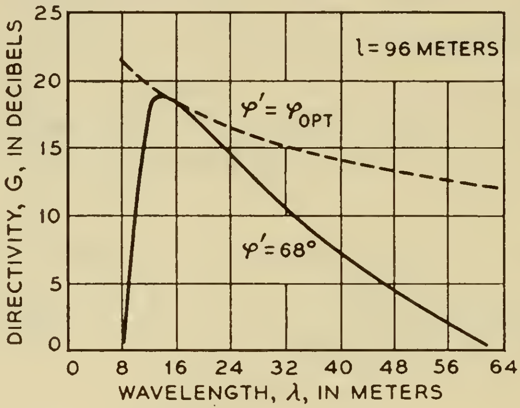


FIG. 14.5 Directivity of a fixed antenna (solid line) and directivity of an optimum antenna (dotted line) as a function of wavelength.

signal angle θ usually decreases as the wavelength is increased; hence, the actual loss is smaller. The following table shows that the optimum value of φ' increases as the signal angle θ is decreased,

	$l/\lambda =$	1.5	2	3	4	6	8	
For $\theta = 90°$	$\varphi'_{opt} =$	45.4	51.5	58.6	62.9	67.9	70.9	
For $\theta = 75°$	$\varphi'_{opt} =$	47.0	53.5	61.4	66.2	72.1	75.9	(24)
For $\theta = 60°$	$\varphi'_{opt} =$	52.1	60.0	69.8	75.7	83.0	85.3	

It is this property that makes it possible to use fixed rhombic antennas in short-wave circuits over a large range of wavelengths.

14.11 Loss in the terminating resistance

Let the loss T in the terminating resistance be expressed in decibels. Since the loss in the wires is negligible, T must equal the difference between the directivity and the power gain. Hence,

$$T = G_D - G_P, \qquad 10^{T/10} = \frac{g_D}{g_P}. \tag{25}$$

For *exponential attenuation* we also have

$$T = -10\log_{10}\left(1 - \frac{I_T^2}{I_0^2}\right) = -10\log_{10}(1 - e^{-4\alpha l}), \tag{26}$$
$$e^{-4\alpha l} = 1 - 10^{-T/10}.$$

Noting that the directivity g_D is approximately independent of the attenuation of the current waves, we can find R_i from equations 16, 20, 25, and 26,

$$R_i = \sqrt{10^{T/10}\,(10^{T/10} - 1)}\;240\left[\log\left(\frac{4\pi l}{\lambda}\cos^2\varphi'\right) + 0.577\right]\cdot \tag{27}$$

In the case of cylindrical wires, and particularly in the case of multiple cylindrical wires which are used to diminish the impedance level, the nominal characteristic impedance given by equation 3 varies rapidly near the generator and more slowly at greater distances. Hence, the current amplitude decreases much more rapidly within, say, the first half wavelength than subsequently. This more rapid variation in the current, over such a short distance, will not greatly affect the directivity and the power gain; but its effect on equation 26 will be large. Equation 27 will no longer apply, but a similar equation will. All that we have to do in equation 26 is replace I_0 by the current $I(\lambda/2)$ at distance $\lambda/2$ from the input terminals, since from there on the nominal characteristic impedance varies slowly, and the effect on the attenuation from this cause is small. Simultaneously, we should use the nominal characteristic impedance at this distance instead of R_i, since the input power is

$$P = \tfrac{1}{2}R_iI_0^2 = \tfrac{1}{2}K\left(\frac{\lambda}{2}\right)\left[I\left(\frac{\lambda}{2}\right)\right]^2\cdot \tag{28}$$

Thus, equation 27 becomes

$$K\left(\frac{\lambda}{2}\right) = \sqrt{10^{T/10}\,(10^{T/10} - 1)}\;240\left[\log\left(\frac{4\pi l}{\lambda}\cos^2\varphi'\right) + 0.577\right]\cdot \tag{29}$$

This is a more general equation than 27; for, if the nominal characteristic impedance does not vary, $K(\lambda/2)$ is equal to R_i. From equation 3 we find

$$K\left(\frac{\lambda}{2}\right) = 120\log\frac{\lambda}{a} + 120\log\cos\varphi'. \tag{30}$$

Figure 14.6 shows the relation (equation 29) between this impedance [or the input impedance when the nominal characteristic impedance $K(r)$ is substantially constant] and the loss in the termination.

In order to test the sensitivity of these values to the over-all current distribution, they were recomputed on the assumption that the current amplitude is constant in each arm of the rhombic and drops suddenly at the mid-points M and N (Fig. 14.1). For $l/\lambda = 6$, these results are shown in Fig. 14.6 by circles.*

* In computing the curves in Fig. 14.6, α^2 was retained in equation 15; its effect, however, was found to be small.

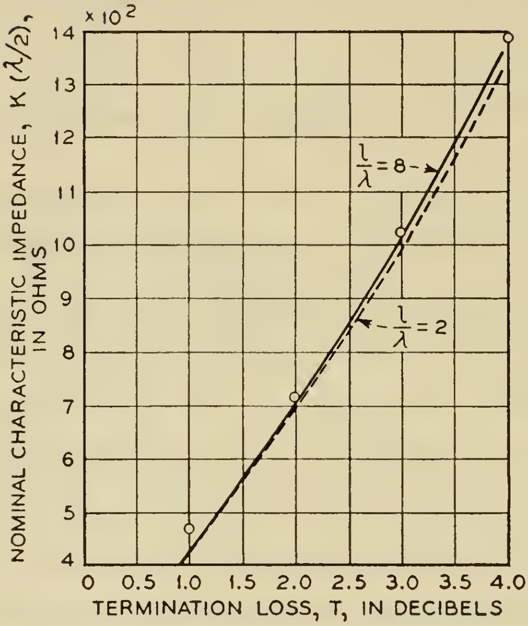


Fig. 14.6 The nominal characteristic impedance of a rhombic antenna, at a distance from the input terminals equal to λ/2, as a function of the termination loss.

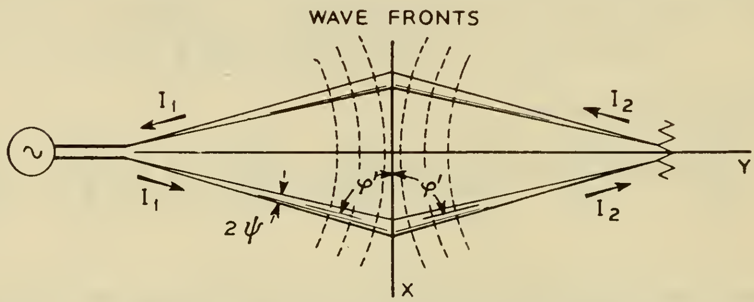


Fig. 14.7 A rhombic antenna constructed from conical conductors.

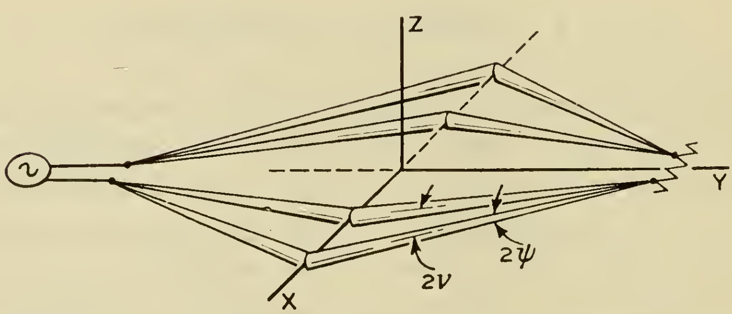


Fig. 14.8 A rhombic antenna constructed of twin conical conductors.

For the antenna mentioned at the end of Section 14.2, the measured termination loss at 16 meters was about 3 db. In Fig. 14.6 this corresponds to $K(\lambda/2) = 1010$ ohms. From equation 30, we find $K(\lambda/2) = 1030$ ohms.

The curves in Fig. 14.6 show the advantage of using multiple conductors. For example, for a rhombic made with conically tapered wires (Fig. 14.7) and having the following dimensions,

$$\varphi' = 68°, \qquad \psi = 10^{-4},$$

we find, from equation 4–42,

$$K = 120 \log \frac{2 \cos \varphi'}{\psi} = 1071 \text{ ohms}, \qquad T = 3.2 \text{ db}.$$

If *two* sets of conical conductors are used (Fig. 14.8), the antenna impedance is reduced (see equation 4–43) to

$$K = 120 \log \frac{2 \cos \varphi'}{\sqrt{2\psi\nu}} \cdot \tag{31}$$

If $\varphi' = 68°$, $\psi = 10^{-4}$, $\nu = 10^{-2}$, then,

$$K = 752 \text{ ohms}, \qquad T = 2.2 \text{ db},$$

and there is 1-db increase in signal gain over the single-wire rhombic. Experimental work with multiple-wire rhombics has indicated gains between 0.5 and 1.5 db over single-wire rhombics.

14.12 MUSA receiving system

The MUSA* receiving system employs sharp vertical plane directivity, capable of being steered to meet waves arriving at a receiving location at varying angles. It consists of an end-fire array of antennas, of fixed directional pattern, whose outputs are combined in phase for the desired angle. An experimental system, constructed at Holmdel, consisted of six rhombic antennas (Fig. 14.9). The antenna system extended three quarters of a mile toward England. The outputs were conducted by coaxial transmission lines to the phase shifters in the receiving building. The phase shifts, ϑ, 2ϑ, 3ϑ, 4ϑ, 5ϑ, are indicated schematically in Fig. 14.9. The phasing is actually accomplished by means of rotatable phase shifters operating at an intermediate frequency† (Fig. 14.10).

* H. T. Friis and C. B. Feldman, Multiple unit steerable antenna, *IRE Proc.*, **25**, July 1937, pp. 841–917.

† The technique used here to phase the output antennas takes advantage of the fact that, when two radio frequency voltages are separately heterodyned with the same beating oscillator, the different frequency outputs obtained have the same relative phases as the original radio-frequency voltages before the frequency change was accomplished. See H. T. Friis, A new directional receiving system, *IRE Proc.*, **13**, December 1925, pp. 685–707.

These phase shifters, one for each antenna except the first, are geared together in gear ratios 1, 2, 3, 4, 5. The direction of the main beam in the vertical plane may then be altered by rotating the assembly. Three sets of such phase shifters are paralleled, each set constituting a separately steerable branch. The one toward the right serves as an exploring or monitoring circuit for determining the angles at which waves are arriving. The remaining branches may then be set to receive at these angles. The patterns at the top of this figure show that the

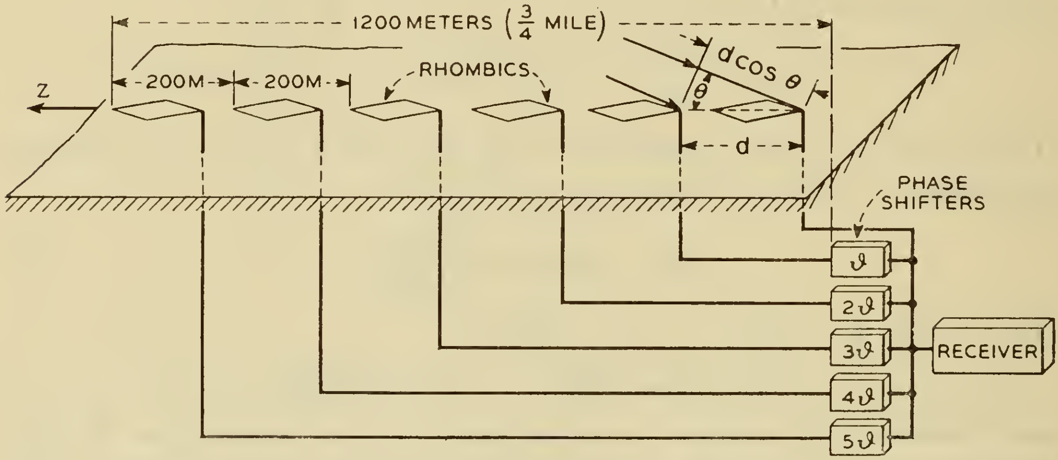


FIG. 14.9 An experimental system of six rhombic antennas.

branches A' and A'' are adjusted to receive waves arriving from directions making angles, respectively, 12° and 23° with the horizontal plane.

The space factor is

$$S = \frac{\sin\left\{\dfrac{N}{2}\left[\vartheta - 2\pi\dfrac{d}{\lambda}\left(\dfrac{c}{v} - \cos\theta\right)\right]\right\}}{\sin\left\{\dfrac{1}{2}\left[\vartheta - 2\pi\dfrac{d}{\lambda}\left(\dfrac{c}{v} - \cos\theta\right)\right]\right\}}, \tag{32}$$

where c/v is the ratio of the velocity of light to that in the transmission lines,* $N = 6$ is the number of rhombics, d is the spacing between rhombic output ends. A family of calculated directional patterns of the experimental MUSA is shown in Fig. 14.11. At the top of each column is shown the principal lobe of the vertical directional pattern of the unit rhombic antenna, calculated in the meridian plane. Beneath are shown six vertical patterns of the MUSA, which are obtained by multiplying the space factor by the unit antenna pattern. The upper

* The velocity in coaxial transmission lines is smaller than the velocity of waves in free space because of the insulators.

pattern corresponds to phasing for zero angle ϑ. The remaining ones are plotted for increments of 60° in ϑ. Note that the width of the major lobe, at the 3-db points, at $\lambda = 16$ meters, and $\theta = 16°$, is only 2.5°.

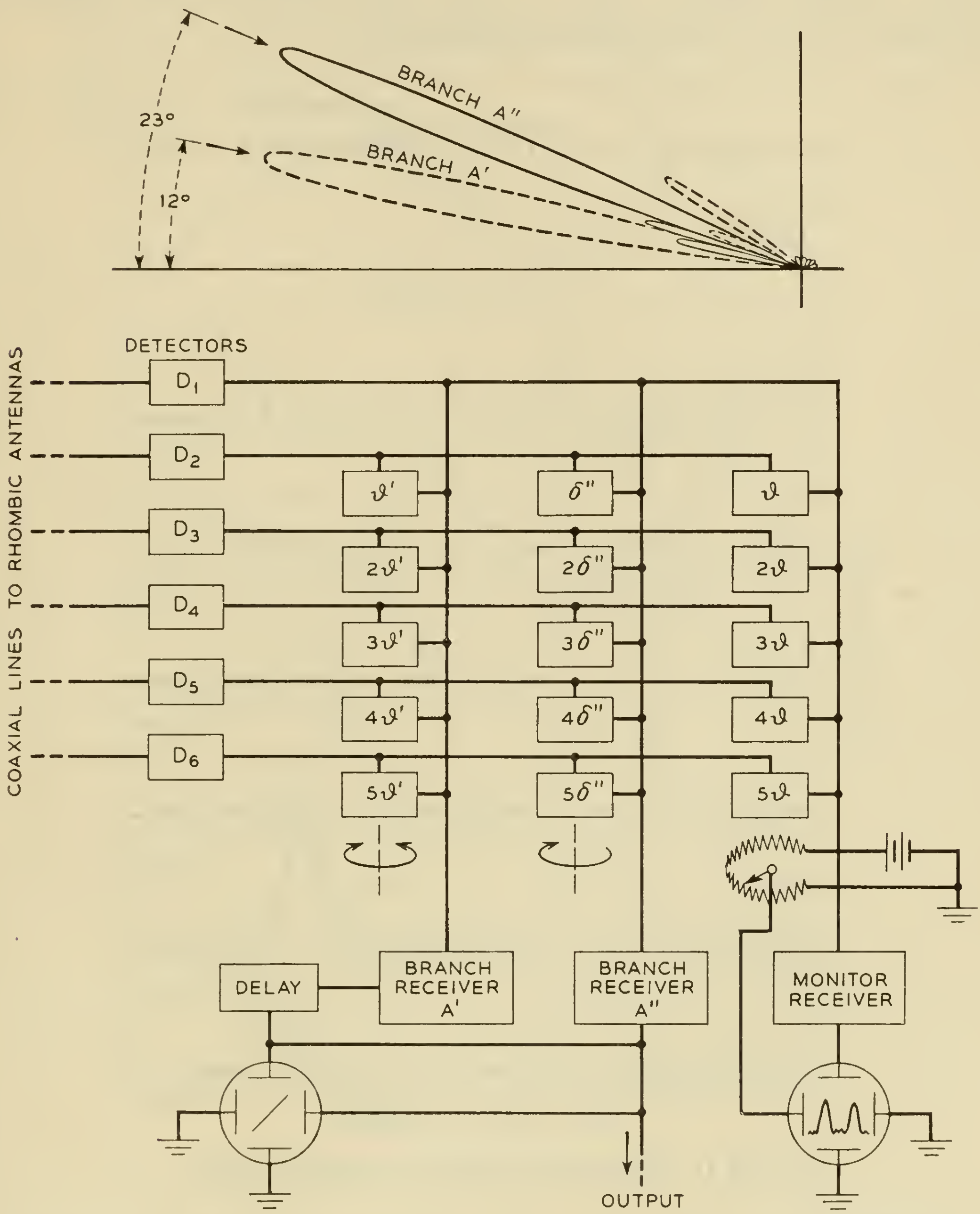


Fig. 14.10 A diagram of phasing equipment.

A commercial MUSA system which consists of a row of 16 rhombic antennas 2 miles long has been constructed at Manahawken, N. J.*

* F. A. Polkinghorn, A single-sideband MUSA receiving system for commercial operation on transatlantic radio telephone circuits, *IRE Proc.*, **28**, April 1940, pp. 157–170.

The vertical-plane directivity for this system is $16/6 = 2.67$ higher than for the experimental 6-antenna system; the lobe width at $\lambda = 16$ meters is now less than 1° when the steering angle θ is 16°. To be able to construct steerable antennas with 1° major lobes illustrates the status of the short-wave art just before World War II.

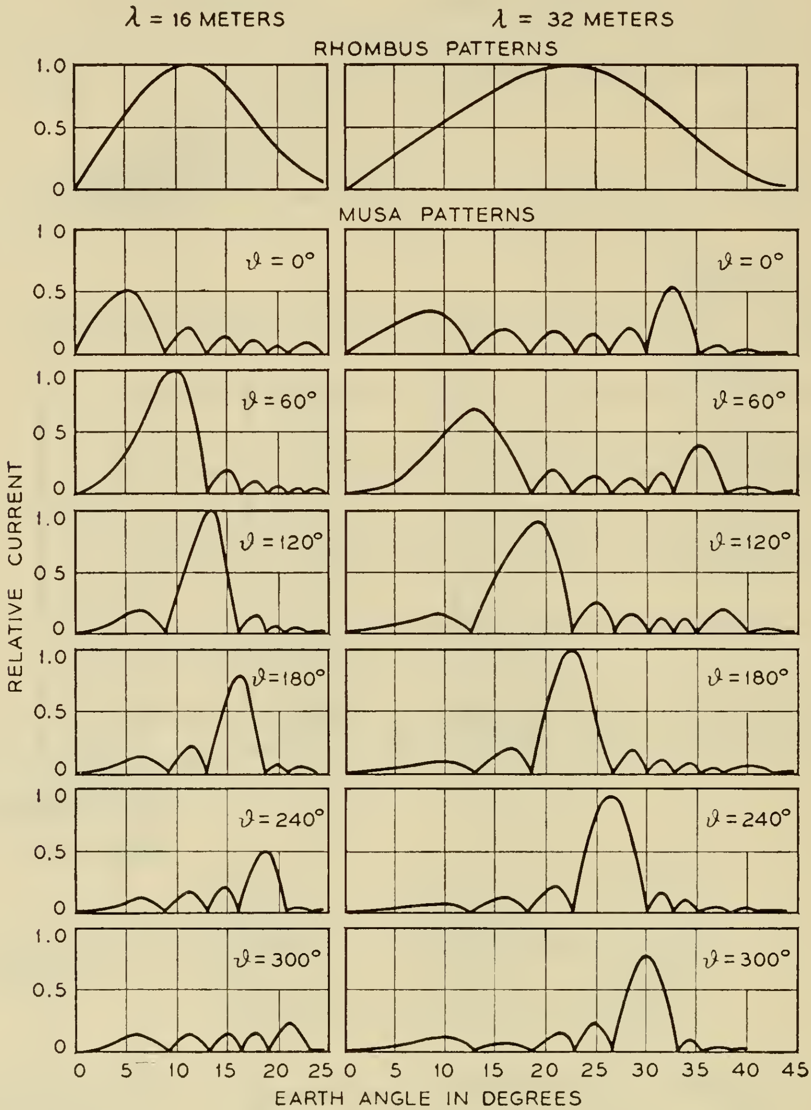


FIG. 14.11 Directional patterns of a MUSA system.

MUSAs have been used to unravel the complicated transmission phenomena of short waves, and for that purpose they have been a very valuable tool. For short-wave reception, they give considerable improvement in signal-to-noise ratio and reduction in selective fading. A long system is expensive, and, in addition, the benefits of increased lengths or higher selectivity diminish gradually. Results with the

2-mile long system indicate that the point of diminishing return has been reached.

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2. E. Bruce, A. C. Beck, and L. R. Lowry, Horizontal rhombic antennas, *IRE Proc.*, **23**, January 1935, pp. 24–46.
3. A. E. Harper, *Rhombic Antenna Design*, D. Van Nostrand, New York, 1941.

15

LINEAR ANTENNA SYSTEMS

15.1 Linear antenna systems

Heretofore we have considered relatively simple linear antennas. In studying their properties, we kept the source of power in the background since it does not affect their performance. When analyzing directive radiation, we assumed certain relationships between the amplitudes and phases of the currents in the elements of an array and ignored completely the question of actually obtaining these relationships. The problems arising from the necessity of connecting the antenna to the source of power (or to the load in the receiving case) are largely, although not exclusively, circuit and transmission line problems. To a great extent the problems of power distribution between the various elements of the array are also circuit and transmission line problems.

To illustrate the different kinds of problems that arise in the design of antenna systems consisting of linear antennas, we shall discuss a few typical examples. So many systems have been developed in the past that it is not feasible to consider them all within the limits of this book. Superficial descriptions of some of these systems may be found in radio handbooks; but for more detailed analysis the reader should consult the original papers.

15.2 Impedance matching

The principal elements in the system connecting the antenna to the transmitter are shown in Fig. 15.1. For maximum transfer of power, the transmitter must be " matched " to the antenna, and, if a long line is used between them, it is desirable that the line should be matched both to the antenna and to the transmitter. In this connection the term " impedance matching " implies that the impedance seen from the antenna terminals, for instance, toward the transmitter must equal the conjugate of the antenna impedance; that is, the resistive components

must be equal and the reactive components must be equal in magnitude but opposite in sign.

In order to design a matching circuit, we must know either the antenna impedance or the antenna admittance. The former is more

FIG. 15.1 A transmitter-antenna system.

convenient for matching circuits of the series type, and the latter is preferable for circuits of the parallel type. The series and shunt representations of the antenna (Fig. 15.2) are equivalent. In these repre-

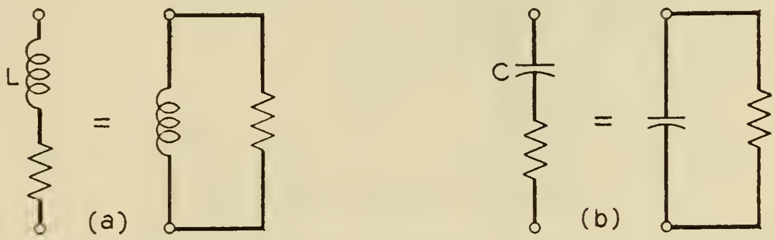


FIG. 15.2 Series and shunt representations of an antenna.

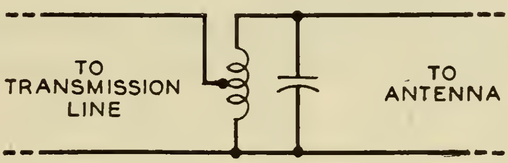


FIG. 15.3 Simple parallel circuit for matching transmitter to antenna.

sentations ωL and ωC denote merely positive and negative reactances and susceptances; their actual variation with frequency may be complex. A simple matching circuit of the parallel type is shown in Fig. 15.3. For other examples the reader should consult handbooks.

15.3 Traps

If a coaxial line is used to convey energy from the transmitter to the antenna, the outer surface of the outer conductor of the line becomes part of a parasitic radiating circuit. The voltage across the coaxial line is impressed not only between the input ends of the antenna arms but also between one of these ends and the outer surface of the outer conductor. To remedy this situation, a quarter-wave trap may be added as shown in Fig. 15.4*a*. This trap introduces a large impedance into the parasitic

radiating circuit and reduces its current. Figure 15.4*b* shows a similar trap for a coaxial antenna.*

If the waves are not too short, the coaxial feeder may be wound into a coil near the input terminals of the antenna. This coil introduces a large inductive reactance into the parasitic circuit.

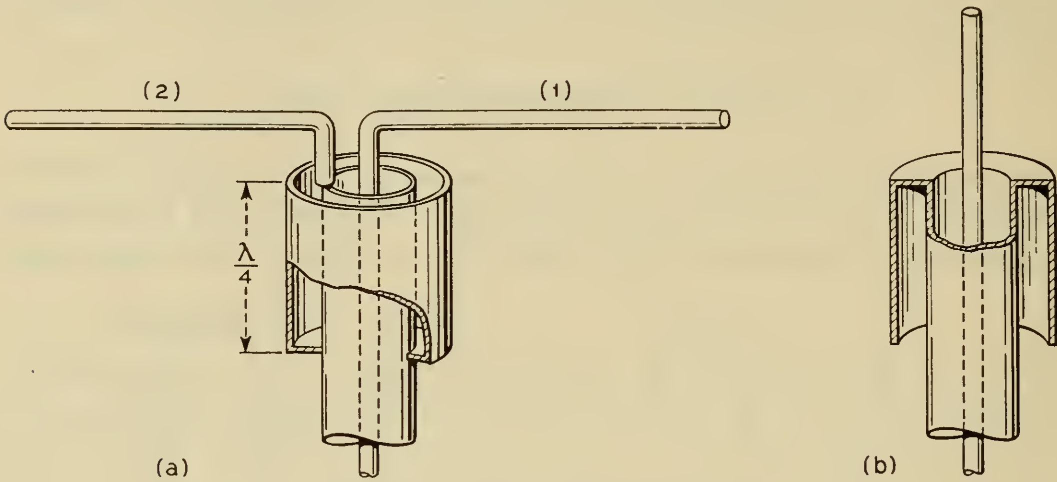


Fig. 15.4 Quarter-wave traps.

15.4 Transformers between balanced antennas and unbalanced transmission lines

Figure 15.5 shows a transformer for use between a balanced receiving antenna (such as a rhombic) and a coaxial output circuit. The secondary winding is split into two parts to obtain symmetry with respect to the balanced primary circuit. Adjacent terminals of the transformer are grounded to decrease undesirable capacitive coupling.

15.5 Feeder systems

Additional problems arise in the design of antenna arrays. In a linear broadside array of the point-to-point type, the currents in all the elements must be in phase. By feeding the elements with separate transmission lines from a common center, we can insure that the impressed voltages are equal. But the impedances of the various elements are affected by their coupling to other elements; hence, the impedances of the central elements will be different from the impedances of the elements near the ends of the array. We may have to take some loss due to the mismatch, or else insert matching circuits.

Instead of using separate transmission lines, we may use a single line, as shown in Fig. 15.6. Transpositions of the feeder conductors

* W. C. Tinus, Ultra-high-frequency antenna terminations, *Electronics*, **8**, August 1935, pp. 239–241.

insure equal phases of the impressed voltages. In the case of two antennas this method is perfect; otherwise, we have to consider impedance mismatches much more serious than those due to the differences in the

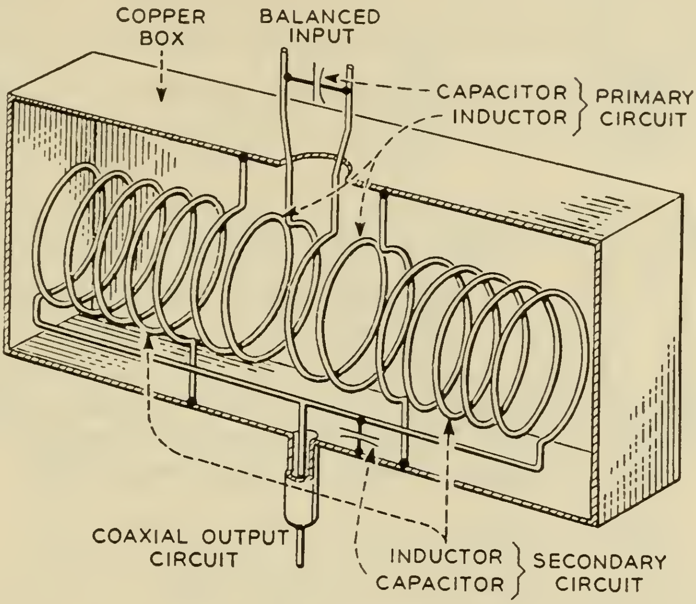


FIG. 15.5 A transformer for use between a balanced antenna and an unbalanced line.

position of the elements within the array. Suppose, for instance, that the impedance of each antenna is 600 ohms and the impedance of the line is also 600 ohms. The impedances are matched at the terminals

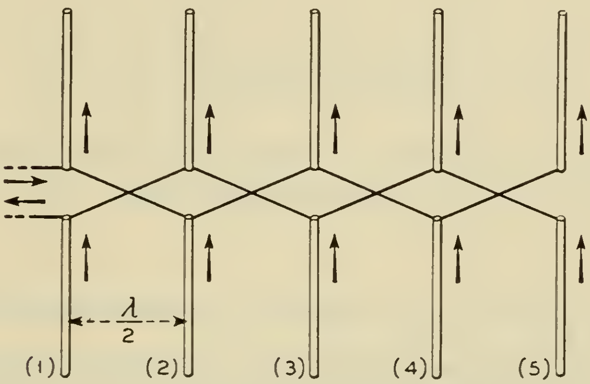


FIG. 15.6 Use of a transmission line for feeding the elements of an antenna array.

of the fifth antenna (Fig. 15.6); but the impedance at the terminals of the fourth antenna is only 300 ohms, and we shall have a mismatch unless the line impedance between the third and fourth antenna is reduced to 300 ohms. The line impedance between the second and third antennas must be 200 ohms, between the first and second 150 ohms, and

the impedance of the input line 120 ohms. If we wish to use a uniform line, we must insert transformers, or take the consequences of impedance mismatch.

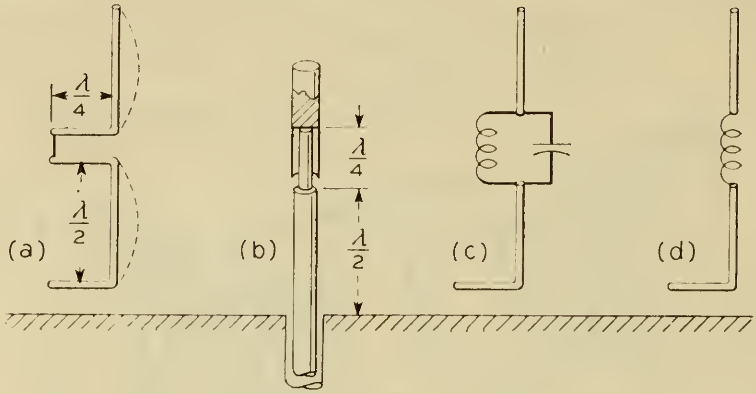


FIG. 15.7 Illustrating various methods used to obtain suitable phasing of vertical currents in half-wave antennas: (*a*) and (*b*) quarter-wave sections of transmission lines, (*c*) resonant circuit, (*d*) loosely wound helix.

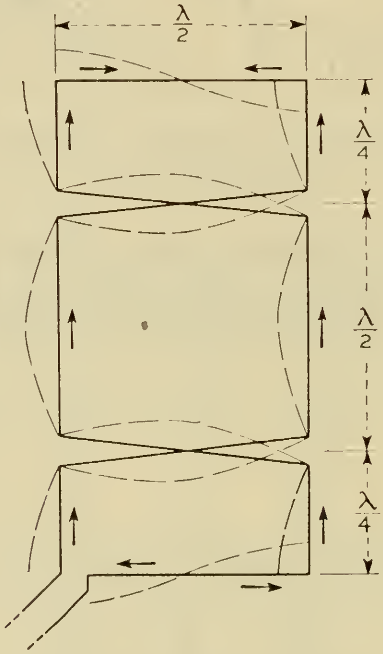


FIG. 15.8 Sterba array of half-wave and quarter-wave elements.

No transposition is needed in the case of an end-fire array of elements one-half wavelength apart; but we must still consider the problem of impedance matching.

Simple broadcast-type broadside arrays of half-wave antennas are shown* in Figs. 15.7*a* and *b*. Quarter-wave sections of transmission lines cause current reversals at half-wave points so that the vertical

* Antennas of the type shown in Fig. 15.7*a* are called *Franklin antennas*.

radiating currents are substantially in phase. Resonant circuits at half-wave points (Fig. 15.7*c*) might be used, but their design would be complicated by the interactions between the inductors and capacitors and the antenna. The velocity of waves along the winding of a not too closely wound helix is nearly equal to that in free space; hence, a wire about one-half wavelength in length wound into such a helix (Fig. 15.7*d*) would also introduce the required current reversal and keep the vertical currents in phase.

The *Sterba array* of half-wave and quarter-wave elements* (Fig. 15.8) has a valuable feature: A steady current may be easily passed through the closed loop for de-icing purposes.

Schematic diagrams of the *Walmsley* (England) and *Chireaux-Mesny* (France) arrays may be found in handbooks.† As in the Sterba

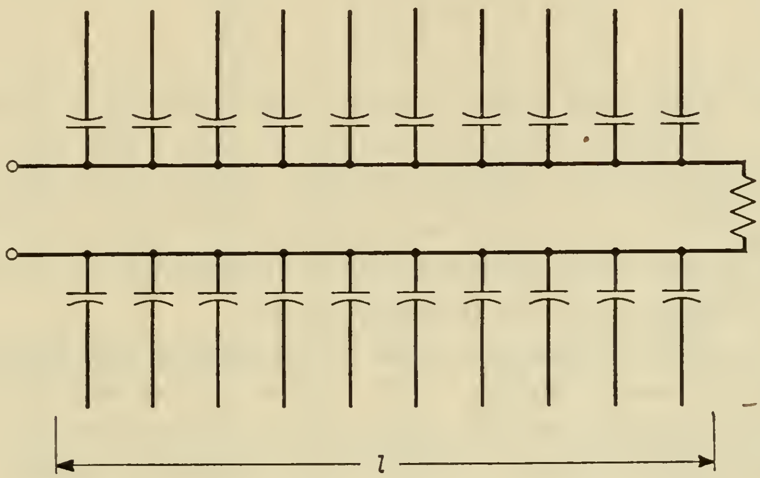


FIG. 15.9 The fishbone receiving antenna.

array, the principal aim is to design a system in which the fields are strengthened in some particular direction when the size of the array is increased.

15.6 Fishbone antennas

The *fishbone receiving antenna* is a long terminated transmission line fed by untuned dipoles, the distance between each dipole being less than a

* E. J. Sterba, Theoretical and practical aspects of directional transmitting systems, *IRE Proc.*, **19**, July 1931, pp. 1184–1215.

† Harold Pender and Knox McIlwain, *Electrical Engineers' Handbook*, John Wiley, New York, 1951.

Frederick E. Terman, *Radio Engineer's Handbook*, McGraw-Hill, New York, 1948.

quarter wavelength.* The dipoles are loosely coupled to the transmission line by means of small capacitors (Fig. 15.9). Thus, each dipole adds a small capacitive and resistive loading to the line.

Assuming uniform loading and equal (in magnitude) currents in the dipoles, we find that the space factor for an antenna of length l extended along the z axis is

$$S = \frac{\sin\left[\frac{\vartheta'}{2} + \pi\frac{l}{\lambda}(1 - \cos\theta)\right]}{\frac{\vartheta'}{2} + \pi\frac{l}{\lambda}(1 - \cos\theta)}, \tag{1}$$

where

$$\vartheta' = 2\pi l\left(\frac{1}{\lambda_{\text{line}}} - \frac{1}{\lambda}\right). \tag{2}$$

Figure 15.10 shows typical patterns when the wave velocity along the line equals that in free space, so that $\vartheta' = 0$. The blunt shape of the major lobe is often a desirable feature. Figure 15.11 illustrates the effect of the excess ϑ' of the phase lag along the line over that in free space.† The major lobe becomes sharper, and, for $\vartheta' = \pi$, the pattern is similar to the pattern of a broadside array.

Figure 15.12 exhibits the effect of exponential attenuation of the currents in successive dipoles. The shape of the major lobe is not affected appreciably even by a large attenuation; but the nulls are eliminated, and the general shape of the radiation pattern becomes smoother. Evidently the directivity is not much affected by the attenuation.

For the case of zero attenuation, the directivity may be expressed as

$$g = \frac{4l}{\lambda}AB, \tag{3}$$

where the factor A is the function of ϑ' shown in Fig. 15.13, and B is a correction factor for the directivity of each dipole (Fig. 15.14). For long antennas B is nearly equal to unity; A is unity when $\vartheta' = 0$, and the wave velocity along the line equals that in free space.

In practice, it is difficult to design a highly efficient fishbone an-

* H. H. Beverage and H. O. Peterson, Diversity receiving system of RCA Communications, Inc., for radio telegraphy, *IRE Proc.*, **19**, April 1931, pp. 531–561.

† An analysis of the directivity of such arrays may be found in W. W. Hansen and J. R. Woodyard, A new principle in directional antenna design, *IRE Proc.*, **26**, March 1938, pp. 333–345.

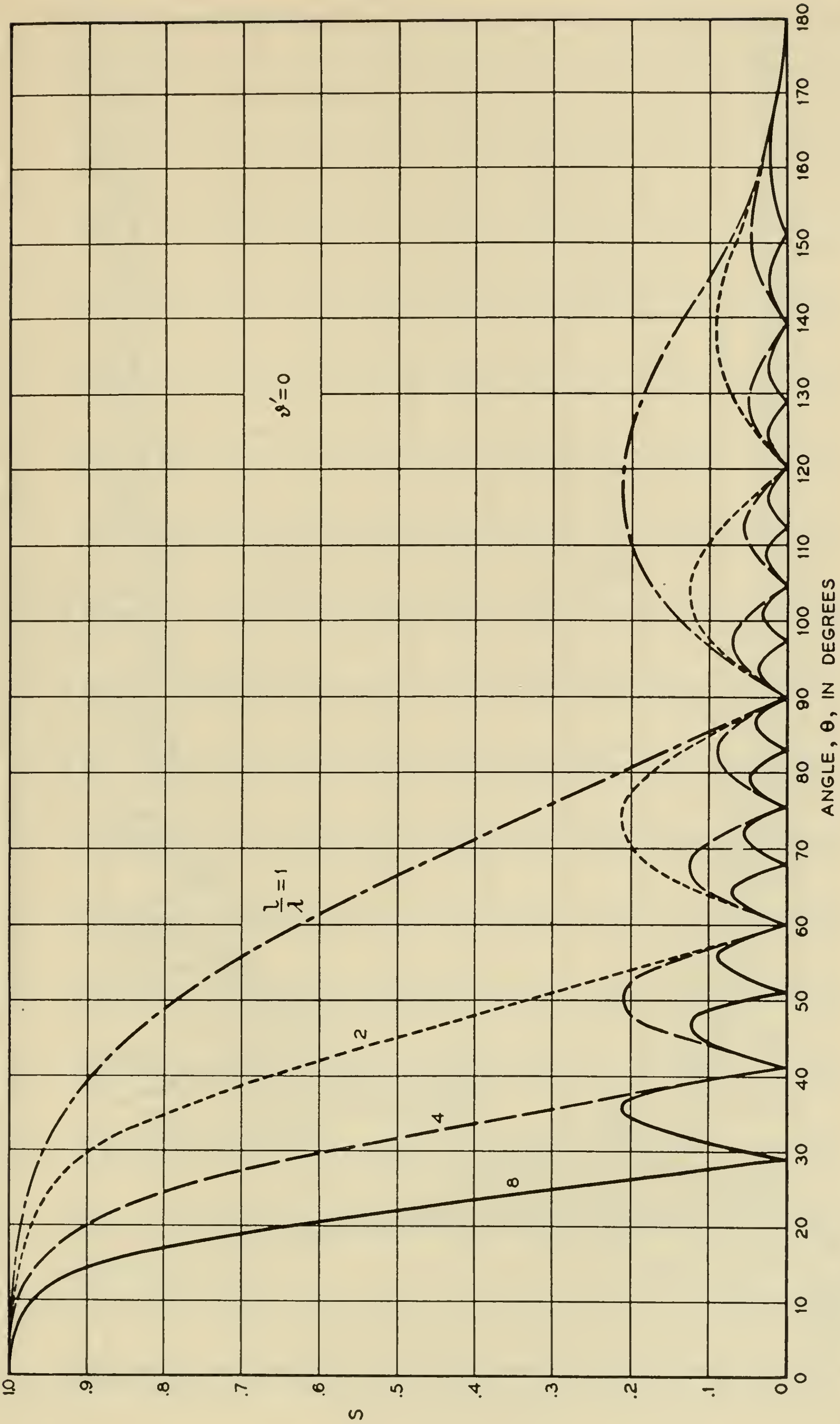


FIG. 15.10 Space factor of fishbone antenna for different antenna lengths when $\vartheta' = 0$.

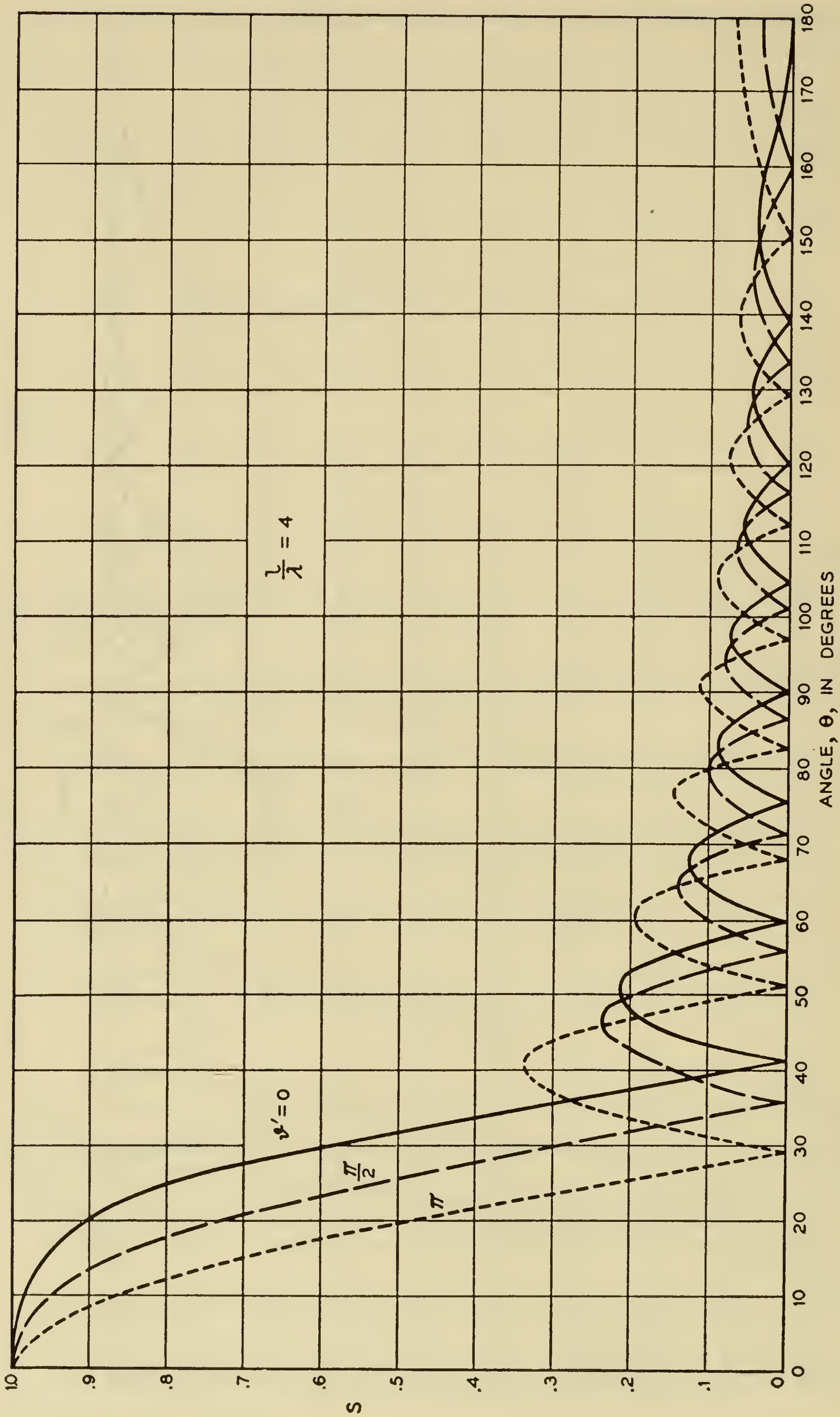


FIG. 15.11 Space factor of fishbone antenna for different values of ϑ' when $l = 4\lambda$.

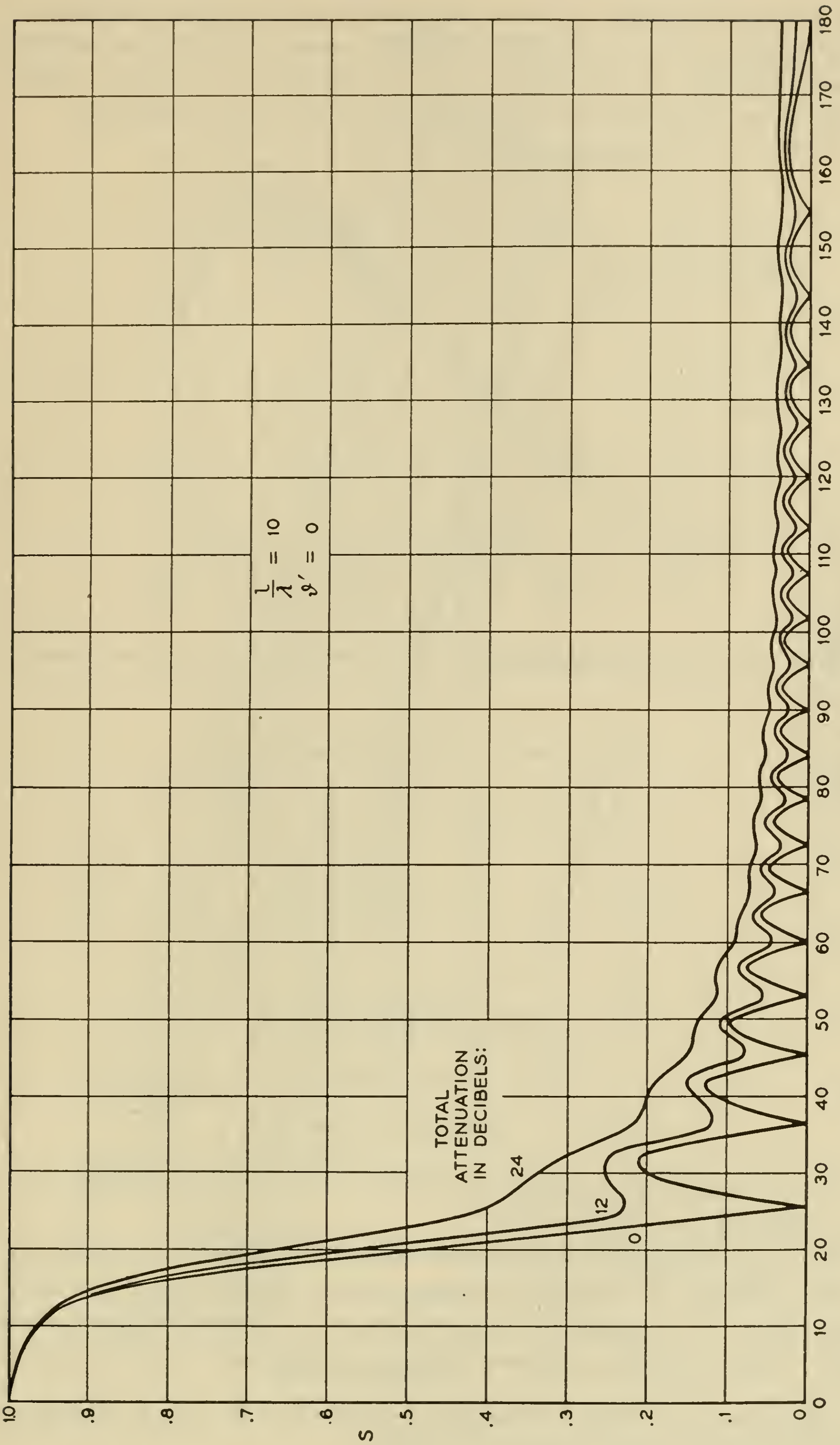


Fig. 15.12 Illustrating the effect of current attenuation in the dipoles of the fishbone antenna.

tenna over a long range of wavelengths because the dipole impedances change rapidly with wavelength. For this reason the fishbone antenna is used primarily for short-wave, 15- to 60-meter, reception where, because of static interference, the efficiency is not very important.

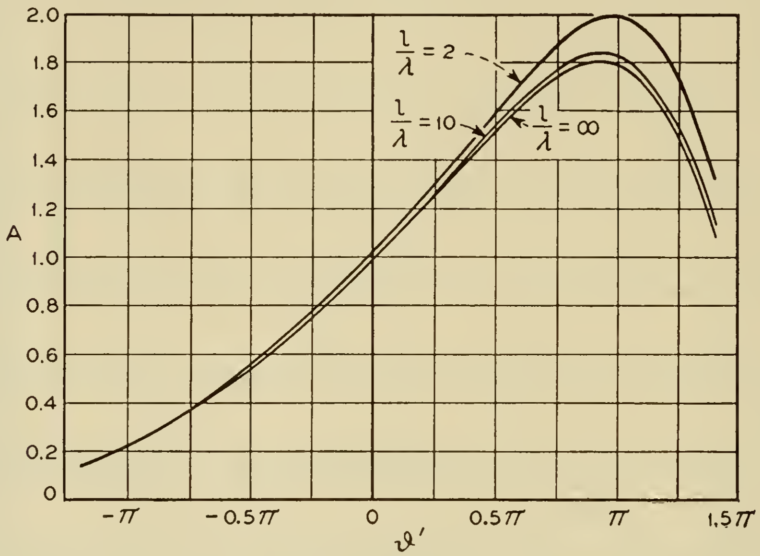


FIG. 15.13 The factor A in equation 3 for the directivity of the fishbone antenna.

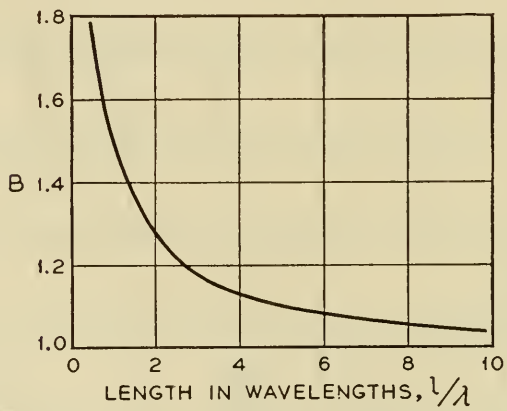


FIG. 15.14 The factor B in equation 3.

Several interesting variants of fishbone antennas are described by Carter, Hansell, and Lindenblad.*

Fishbone antennas may be used as elements of an array (Fig. 15.15).

The fishbone is a balanced antenna, and, when located in a horizontal plane, it is useful for the reception of horizontally polarized waves. Half a fishbone antenna close to the ground is called a *comb*

* Development of directive transmitting antennas by RCA Communications, Inc., *IRE Proc.*, **19**, October 1931, pp. 1773–1842.

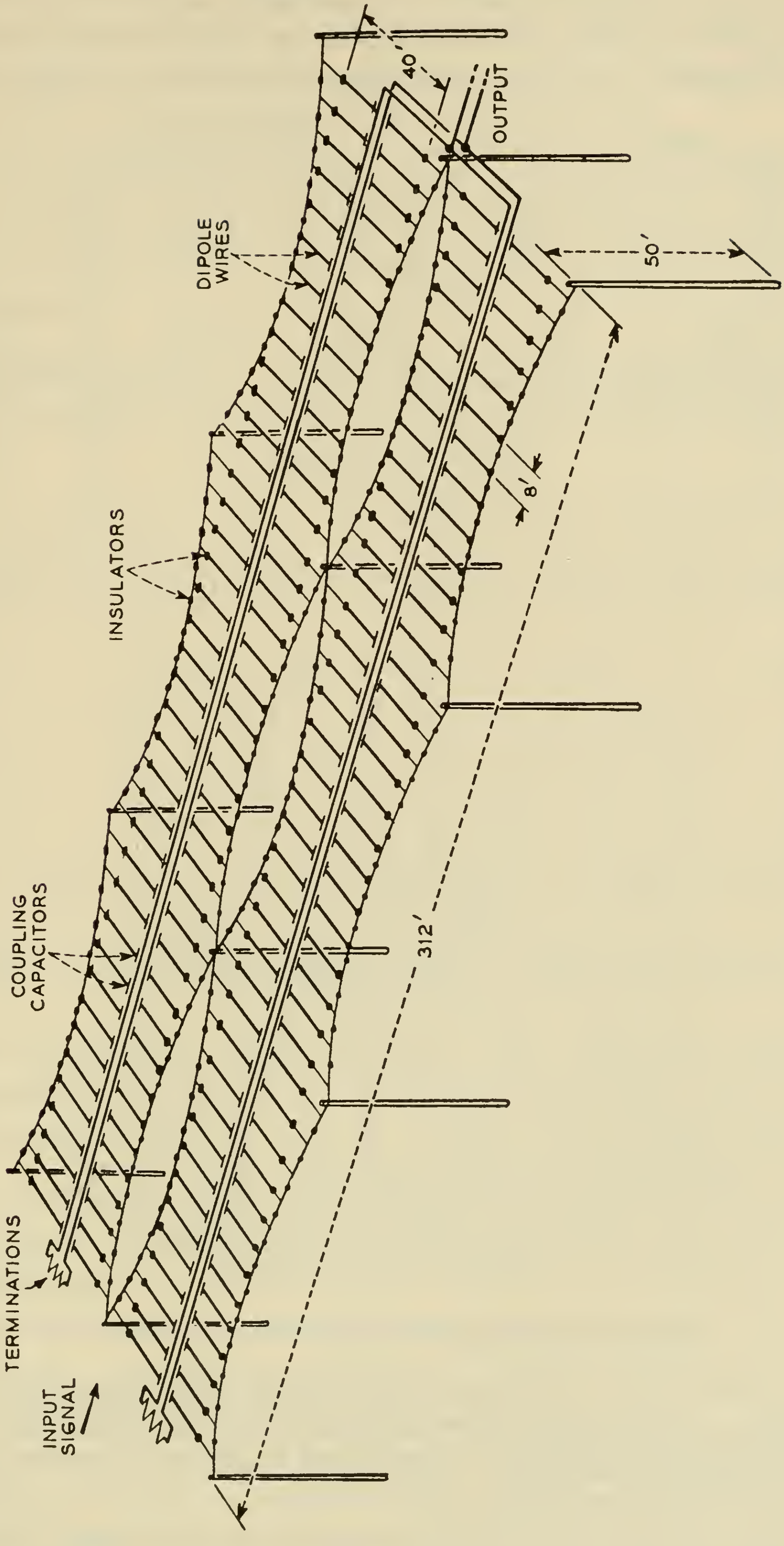


FIG. 15.15 An array of fishbone antennas.

*antenna** and is useful for reception of vertically polarized short waves. The design shown in Fig. 15.16 was arrived at experimentally by A. C. Beck at Holmdel, N. J., by measurements of patterns (Fig. 15.17),

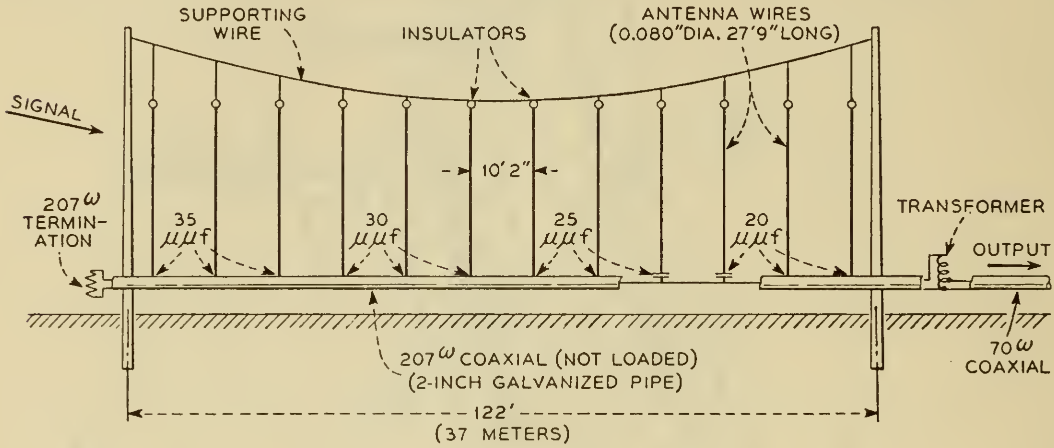


FIG. 15.16 Comb antenna for reception of vertically polarized short waves.

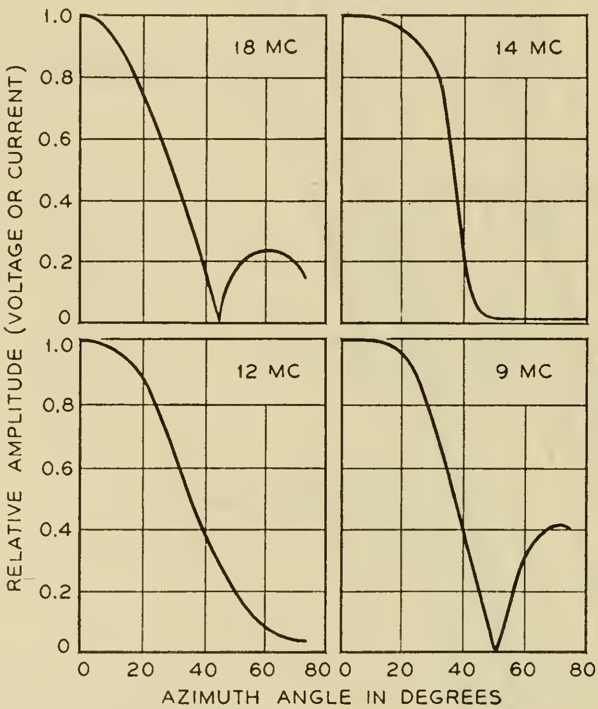


FIG. 15.17 Measured radiation patterns for the comb antenna of Fig. 15.16.

relative levels of the currents in the antenna wires (Fig. 15.18), and directivities. The tapered values of the coupling capacities (Fig. 15.16) tend to equalize the antenna currents and to reduce resonance effects over a large frequency band.

* Ralph Grimm, The comb antenna, *IRE Proc.*, 36, March 1948, pp. 359–362.

The azimuth patterns (Fig. 15.17) were obtained by energizing the antenna and measuring the electric intensity around a circle with half a mile radius. The relative levels of the currents in the antenna wires (Fig. 15.18) also correspond to the transmitting case.

With a field produced by a transmitting antenna located about three quarters of a mile in front of the comb antenna, the relative output of the comb antenna with respect to a nearby half-wave antenna was found to vary between 5 and 7 db over a 9- to 18-Mc frequency range.

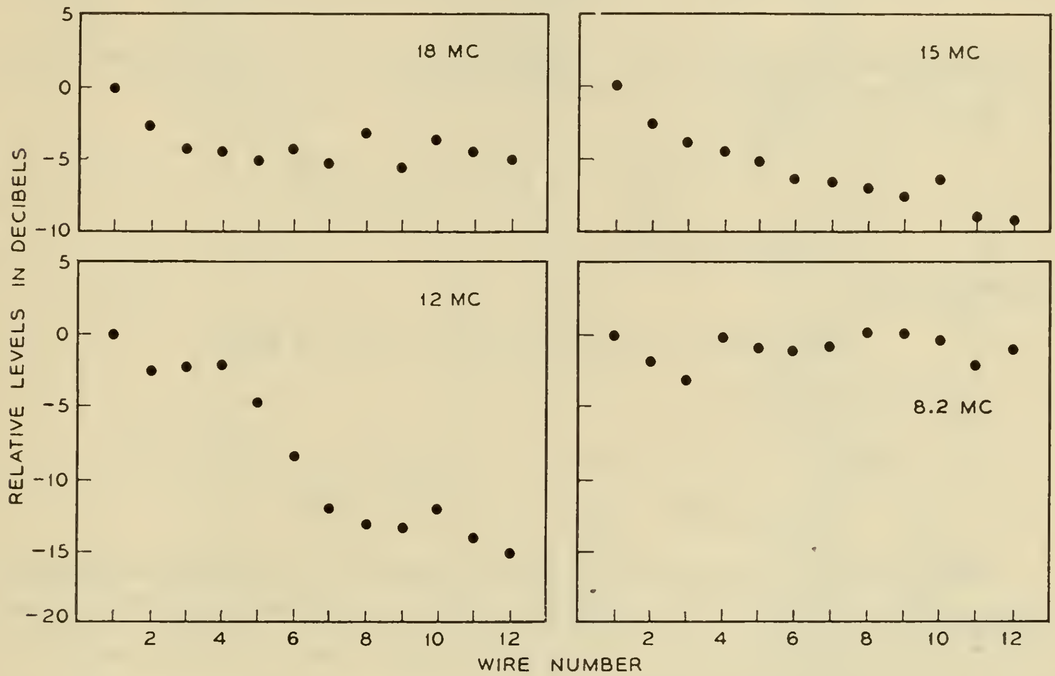


Fig. 15.18 Diagrams showing relative levels of the currents in the wires of the fishbone antenna.

When transatlantic short-wave signals were used for relative gain measurements, between 1 and 2 db lower gain values were obtained. The conservative figure is 4.5 db. This is also the signal gain in free space of a double comb or fishbone antenna of the design shown in Fig. 15.9 with respect to a full-wave antenna (the half-wave antenna and its image). Hence, the signal gain with respect to an isotropic source is approximately

$$G_s = 4.5 + G \text{ (full-wave antenna)} \simeq 8.5 \text{ db}$$

over a frequency range from 9 to 18 Mc.

The heat loss may be estimated by evaluating the difference between the directivity and the signal gain. To obtain the directivity from equation 3, we have to know A. The phase difference ϑ' which determines this factor may be evaluated from the angle θ' corresponding

to the first null in the directive pattern; thus, from equations 1 and 2,

$$\vartheta' = 2\pi\left[1 - \frac{l}{\lambda}(1 - \cos\theta')\right]. \tag{4}$$

For example, at 18 Mc, $\lambda = 16.6$ meters, and, from the measured pattern in Fig. 15.17, we find that $\theta' = 45°$; from Fig. 15.16, l is equal to 37 meters; hence,

$$\vartheta' = 2\pi\left[1 - \frac{37}{16.6}(1 - \cos 45°)\right] = 0.7\pi.$$

The corresponding value of A is 1.8 from Fig. 15.13. From Figure 15.14, we obtain $B = 1.25$ for $l/\lambda = 37/16.6 = 2.24$. Therefore,

$$g = \frac{4 \times 37}{16.6} \times 1.8 \times 1.25 = 20,$$

$$G = 13 \text{ db}.$$

Consequently,

$$\text{Heat loss} = G - G_s \simeq 13 - 8.5 \simeq 4.5 \text{ db}.$$

At 9 Mc. we have

$$\theta' = 50°, \qquad \vartheta' = 1.2\pi, \qquad A \simeq 2, \qquad B = 1.47,$$

$$G \simeq 11 \text{ db}, \quad \text{and} \quad G - G_s = 2.5 \text{ db}.$$

These values for heat loss were not checked by direct measurements of the power delivered to the experimental comb and the power dissipated in its terminating resistance. For this reason the above figures should not be given too much weight.

The comb antenna is one solution the antenna art has produced in the search for a directional antenna that has a constant input impedance, high directivity, and satisfactory efficiency in a broad band of frequencies.

An array of 8 comb antennas has been built for experimental purposes. Figure 15.19 shows a top and side view of the array. The output of each comb is brought into the receiver building through a 72-ohm coaxial transmission line. The lines are cut to lengths such that, for signals arriving along the axis of the array, the currents are in phase when they enter the receiver building. Thus,

$$l_2 = l_1 - d\frac{v}{c}, \qquad l_3 = l_1 - 2d\frac{v}{c}, \qquad l_4 = l_1 - 3d\frac{v}{c},$$

where v is the phase velocity of the lines.

The direction of maximum response of the array is adjusted by switching short lengths of coaxial lines in and out of the lines from the

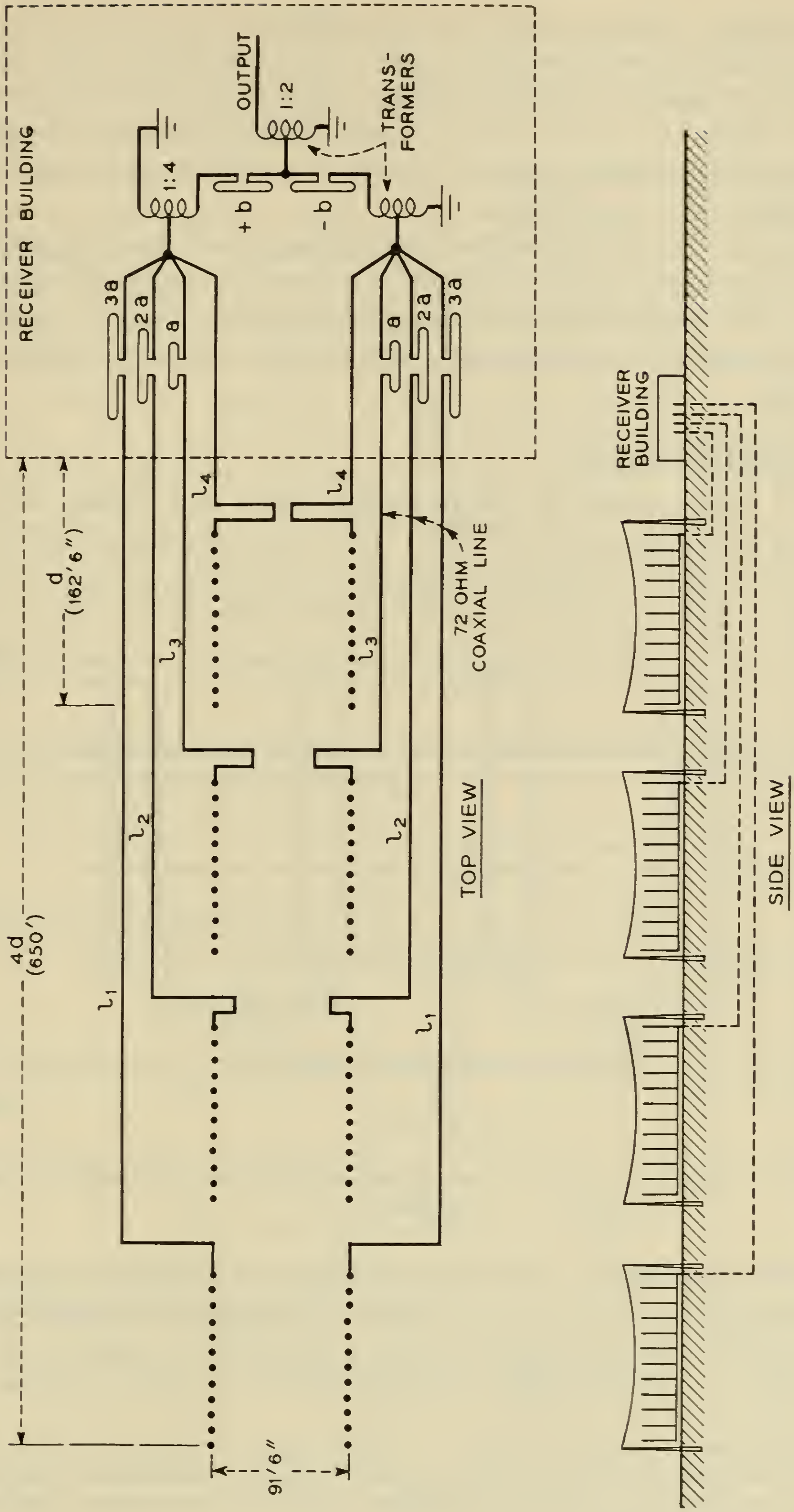


FIG. 15.19 An experimental array of eight comb antennas.

various combs. Vertical steering is accomplished in steps of 0°, 10°, and 20° by switching simultaneously, in and out, the lengths marked a, $2a$, and $3a$, for each row of end-on combs. Horizontal steering is accomplished in steps of 0°, 5°, 10°, and 15° on either side of the axis of the array by means of the lengths marked $+b$ and $-b$. This type of steering is independent of the wavelength.

The array was located on good conducting ground, and it performed very well for signals with variations in horizontal angles of arrival as great as ±20° and for signals with very low angle of arrival. It illustrates the status of the antenna art, but it has not yet found application in practice.

15.7 Pine-tree arrays

Pine-tree arrays* consist of a front curtain and a rear curtain of horizontal one-wave dipoles (Fig. 15.20). The vertical spacing of the dipoles

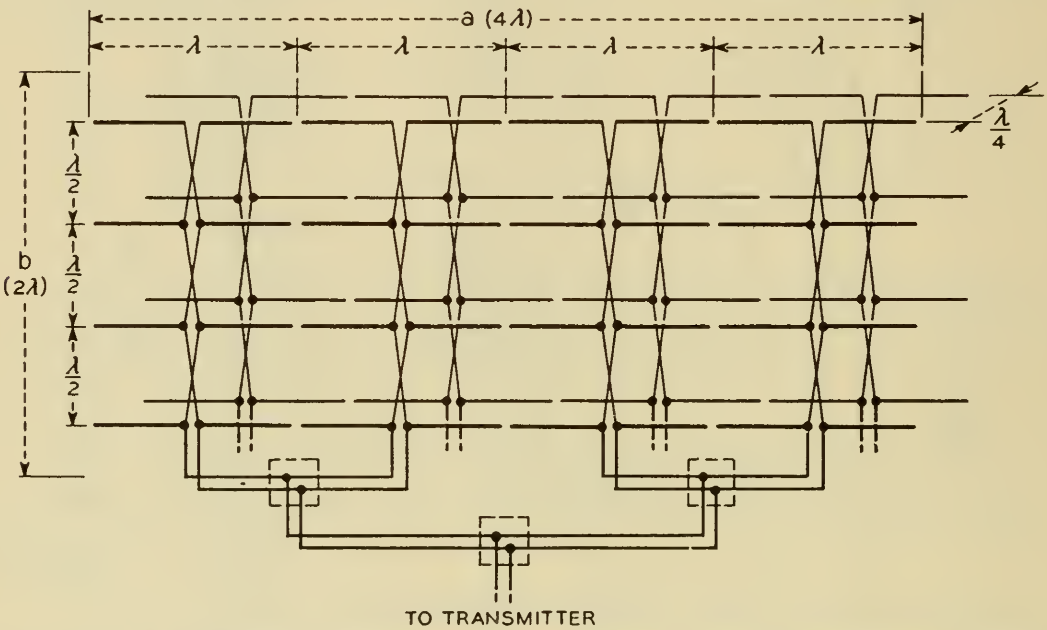


Fig. 15.20 A pine-tree array of horizontal one-wave dipoles, Station DGY, Nauen, Germany.

is one-half wavelength; the horizontal distance between the centers of the adjacent elements is one wavelength. The pine-tree antenna at the Station DGY, Nauen, Germany† was constructed to operate on the wavelength $\lambda = 16.92$ meters. The dipoles in the front curtain are

* From *Tannenbaum* in German.

† M. Baümler, K. Krüger, H. Plendl, and W. Pfitzer, Radiation measurements of a short wave directive antenna at the Nauen high power radio station, *IRE Proc.*, **19**, May 1931, pp. 812–828.

excited as indicated in Fig. 15.20 to obtain the same magnitude and phase of all dipole currents. The rear curtain is used as a reflector. The British Post Office has constructed pine-tree antennas at Rugby of width $a = 8\lambda$ and height $b = 25\lambda$.

The effective directivity area of a large pine-tree antenna in free space is approximately equal to its actual area. To obtain the directional pattern, we should multiply the space factor by the radiation pattern of a single one-wave dipole and by the space factor of the center of the array and its negative image in the ground plane.

An approximate value of the radiation resistance R_a of each half-wave element with respect to the current antinode may be obtained very simply. Let N be the number of half-wave elements (twice the number of one-wave dipoles) in the front curtain; then the radiated power is

$$P = \tfrac{1}{2}NR_aI^2, \tag{5}$$

where I is the maximum amplitude. Equation 5–9 gives the power flow W per unit area in terms of the magnitude of the electric intensity; hence,

$$|E| = \sqrt{240\pi W}. \tag{6}$$

From the transmission formula 6–28, we find that, at a distance r along the normal to the pine-tree antenna,

$$W = P\frac{A}{\lambda^2 r^2}, \tag{7}$$

where A is the effective area of the antenna; therefore,

$$|E| = \frac{\sqrt{240\pi PA}}{\lambda r}. \tag{8}$$

On the other hand, by adding the fields of $2N$ half-wave elements, we obtain

$$|E| = \frac{I\dfrac{\lambda}{2}\dfrac{2}{\pi}2N}{2\lambda r}120\pi = \frac{120NI}{r}. \tag{9}$$

Equating 8 and 9 and noting that

$$N = \frac{ab}{(\lambda/2)^2} = \frac{4A}{\lambda^2}, \tag{10}$$

we find, with the aid of equation 5,

$$R_a = \frac{480}{\pi} = 153. \tag{11}$$

Thus, the radiation resistance (with reference to the current antinode) of each half-wave element in a large pine-tree antenna is aproximately twice as large as its radiation resistance in free space.*

Since the half-wave elements are excited at the current node, the input resistance of each one-wave dipole is (see Chapters 11 and 13)

$$R_i = \frac{K_a(K_a - 146)}{2R_a} = \frac{K_a(K_a - 146)}{306}. \tag{12}$$

Comparing this with the input resistance,

$$R_i' = \frac{K_a(K_a - 146)}{200}, \tag{13}$$

of such a dipole in free space, we find

$$R_i \simeq \tfrac{2}{3}R_i'. \tag{14}$$

If, for example, the input impedance of one of the full-wave dipoles in free space is 3000 ohms, its impedance in a pine-tree antenna will be 2000 ohms. Hence, the impedance presented to transmission lines which connect four vertically stacked dipoles will be only 500 ohms.

Pine-tree antennas are used for transmission and reception of waves in the range from 1 to 70 meters. These antennas have high directivities and are fairly simple to construct. On the other hand, they are frequency selective, for wavelength enters in several places: (1) in the one-wave dipole, (2) in the quarter-wave spacing between the front and rear curtains, and (3) in the transmission lines connecting the vertically stacked dipoles. Besides, it is not easy to provide melting equipment for ice or sleet.

15.8 Wave antennas

In the preceding sections, we have considered a few practical embodiments of the principles of directive radiation expounded in Chapters 5 and 6. In each example, the radiating elements were relatively nondirective antennas. We shall now discuss a radically different type of directive antenna. It is one antenna that *depends for its operation on the finite conductivity of the earth.* Whereas the performance of all other antennas is impaired by the finite conductivity of the earth, the "wave antenna" operates better over a poor ground. Over a perfectly conducting earth, the wave antenna would fail completely.

The *wave antenna*† consists of a horizontal wire, terminated to the

* For another simple and instructive method see Harold A. Wheeler, The radiation resistance of an antenna in an infinite array or waveguide, *IRE Proc.*, **36**, April 1948, pp. 478–487.

† H. H. Beverage, C. W. Rice, and E. W. Kellogg, The wave antenna, *AIEE Trans.*, **42**, 1923, pp. 215–266.

ground at each end in the characteristic impedance of the transmission line formed by the wire and the ground (Fig. 15.21). The performance of the wave antenna is easier to understand when it is used for reception; using the reciprocity theorem, we can then obtain its properties as a transmitting antenna. Consider a plane wave at grazing incidence. Over an imperfect earth there is always a horizontal component of the electric intensity, due to absorption of power by the earth. This component is impressed on the wire and produces, at each element of the wire, two waves traveling in opposite directions. If the direction of the in-

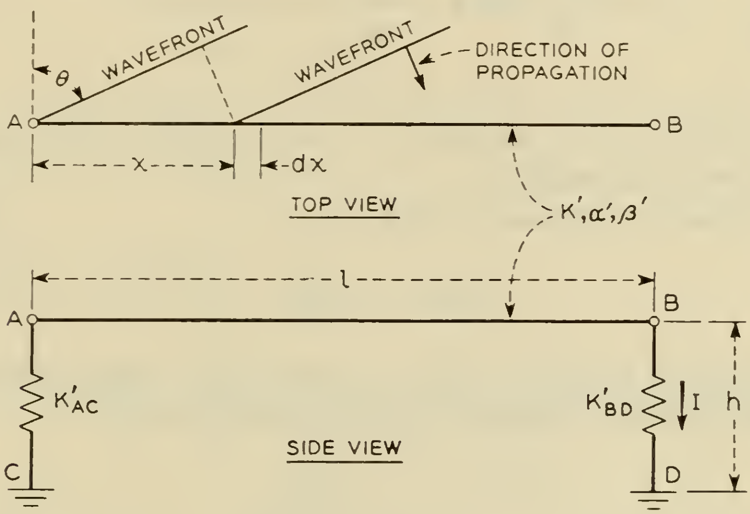


FIG. 15.21 A wave antenna consisting of a horizontal wire grounded at each end.

coming plane wave is parallel to the wire, the induced elementary waves will tend to add at one of the terminals of the wire. They add in phase if the velocity along the line equals that in free space. If the two velocities are not exactly equal, there is some destructive interference. For waves arriving from other directions the interference is greater because the velocity of the incoming plane wave *in the direction* of the wire differs considerably from the wave velocity in the line. Thus, we have a directive antenna.

The voltage induced in an element dx of the wire is

$$E_H \cos\theta \, e^{-j\beta x \cos\theta} \, dx, \tag{15}$$

where θ is shown in Fig. 15.21. The current in the element is obtained if we divide this voltage by $2K$, where K is the characteristic impedance of the line. To obtain the current dI at the far end termination B,D, we must multiply by the propagation factor; thus,

$$dI = \frac{E_H \cos\theta \, e^{-j\beta x \cos\theta} \, dx}{2K} e^{-(\alpha' + j\beta')(l - x)}, \tag{16}$$

where $\alpha' + j\beta'$ is the propagation constant of the line. Hence, the total current is

$$I = \int_{x=0}^{x=l} dI = \frac{E_H \cos\theta}{2K} \frac{e^{[\alpha' + j(\beta' - \beta\cos\theta)]l} - 1}{\alpha' + j(\beta' - \beta\cos\theta)} e^{-(\alpha' + j\beta')l}. \tag{17}$$

If we neglect the attenuation along the wire and assume that the wave velocity along it is the same as in free space, we have

$$|I| = \frac{E_H l}{2K} \cos\theta \frac{\sin\left[\dfrac{\beta l}{2}(1 - \cos\theta)\right]}{\dfrac{\beta l}{2}(1 - \cos\theta)}. \tag{18}$$

The third factor in this expression is seen to be the space factor for a continuous array of length l.

The current is maximum when $\theta = 0$,

$$|I|_{\theta=0} = \frac{E_H l}{2K}. \tag{19}$$

Substituting from equation 7–36, we may express this current in terms of the vertical electric intensity of the incoming wave,

$$|I|_{\theta=0} = \frac{E_V l}{2K\sqrt{60g\lambda}}. \tag{20}$$

This shows that the response in the wave antenna is larger when the conductivity of the earth is smaller.

The current in response to the voltage induced in the vertical wires AC or BD is not affected by the wavefront angle θ; it is

$$I_V = \frac{E_V h}{2K}. \tag{21}$$

For large l this current is much smaller than that in equation 20. If, for example, $l = \lambda$, $g = 1.5 \times 10^{-3}$, $h = 10$ meters, $\lambda = 5000$ meters, then,

$$\frac{|I|_{\theta=0}}{I_V} = 24, \tag{22}$$

so that the difference in levels is 27 db.

In practice, two wires are used as shown in Fig. 15.22. These wires act in parallel, effectively as a single wire of the simple wave antenna in Fig. 15.21; they also act as a balanced two-wire transmission line and bring the signal currents back from the far end. The termination K_{AC} is now near the receiver where it may easily be adjusted for

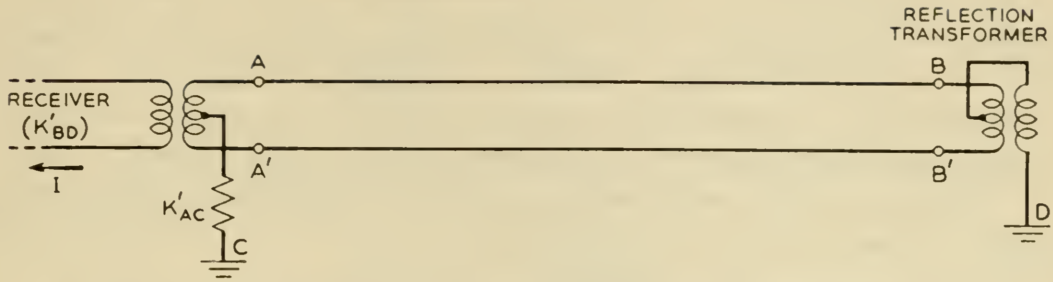


FIG. 15.22 A practical form of wave antenna using two wires in parallel.

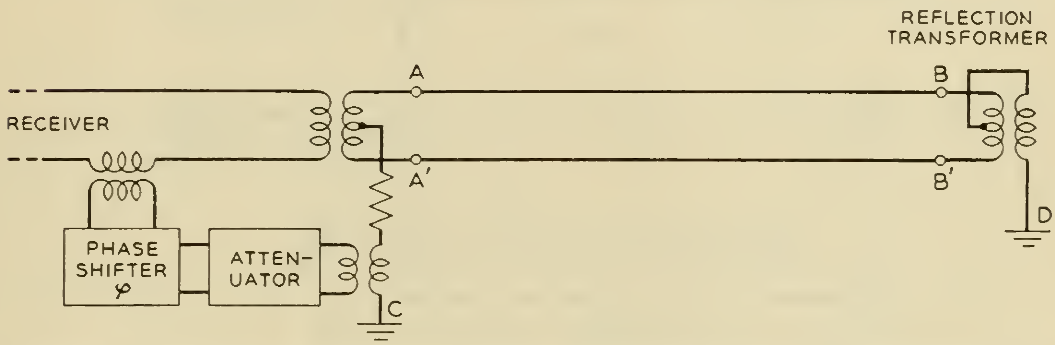
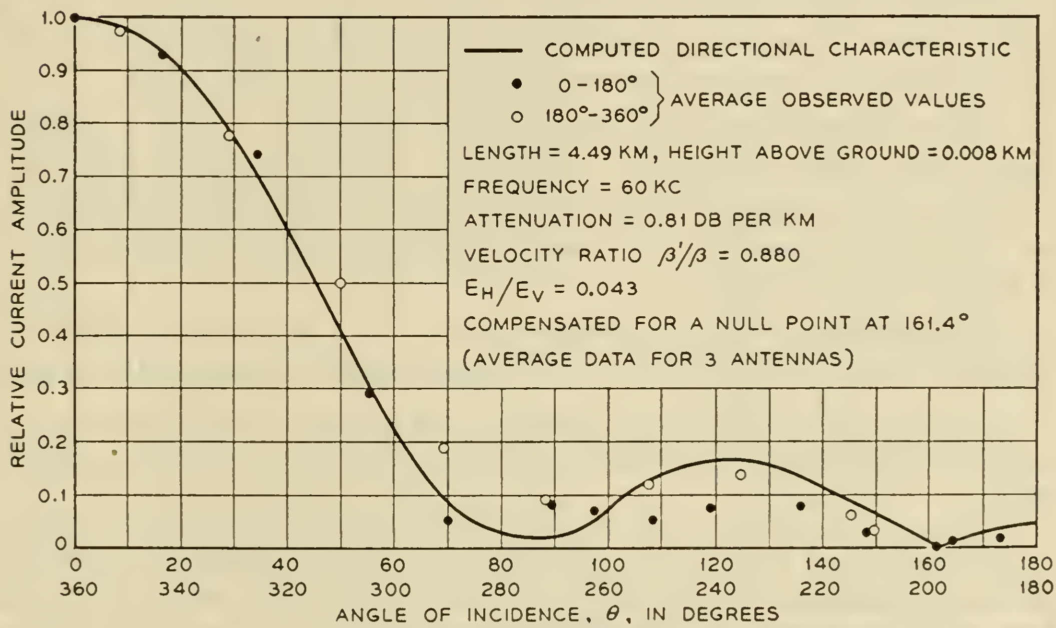


FIG. 15.23 Illustrating a compensating circuit used with the wave antenna to modify the radiation pattern.



Bailey, Dean, and Wintringham, Courtesy IRE

FIG. 15.24 Directional characteristics of the compensated wave antenna of Fig. 15.23.

minimum interference. Further improvement can be obtained by combining, in proper phase and magnitude, the residual current through K_{AC} with the main signal current I, as shown in Fig. 15.23. This "compensation" arrangement makes it possible to produce a null point in the directional pattern in any desired direction (Fig. 15.24*).

An array of four wave antennas of the type shown in Figs. 15.23 and 15.24 has been built and tested at Houlton, Me., for reception of

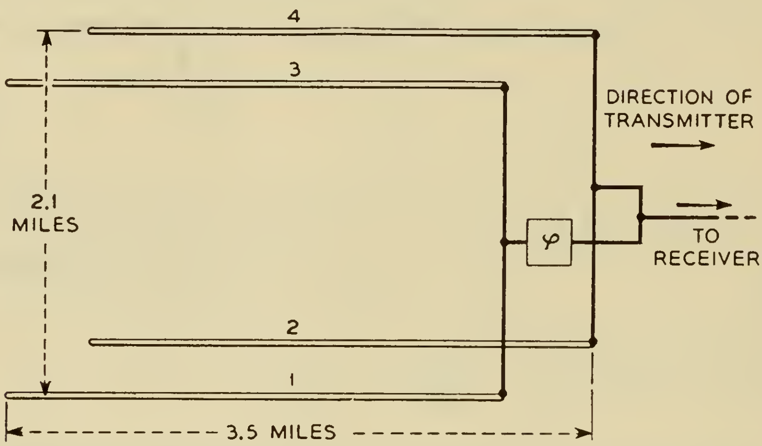


Fig. 15.25 An array of four wave antennas used for long-wave reception.

transatlantic 60-kc telephone signals (Fig. 15.25). Antennas 1 and 3 form a broadside array; 2 and 4 form a second broadside array; the two broadside arrays are then combined into an end-fire array.

Wave antennas are used for long-wave reception. They are aperiodic (broad band) and easy to construct. In practice, they are made as a high-grade telephone line, on 30-ft poles. They require ground with low conductivity and for this reason they have not been as successful in England as in U.S.A.†

The efficiency of a wave antenna is low. Let us compare the power available from the wave antenna with that available from a 100 per cent

* Austin Bailey, S. W. Dean, and W. T. Wintringham, Receiving system for long-wave transatlantic radio telephony, *IRE Proc.*, **16**, December 1928, pp. 1645–1705.

† Arrays of loops have been developed in England for long-wave reception. Two loops, spaced $\frac{1}{4}$ to $\frac{1}{2}$ wavelength in the direction of the transmitter and with their outputs phased for a null point in the opposite direction, have approximately the same directivity as a one-wavelength-long wave antenna. See the preceding reference, Figs. 36 and 77, and H. T. Friis, A new directional receiving system, *IRE Proc.*, **13**, December 1925, pp. 685–707. An array of two such loop units is the only "matchbox" or "supergain" type of antenna that has so far been used in practice. The low efficiency and narrow band limitations of such antennas are not too important factors in long-wave reception.

efficient short vertical antenna over a perfect ground. Thus,

$$P_{\text{vert}} = \frac{1}{2}\frac{E_V{}^2\lambda^2}{640\pi^2}, \qquad P_{\text{wave ant}} = \tfrac{1}{2}K|I|_{\theta=0}{}^2. \tag{23}$$

Substituting from the approximate equation 20, based on the assumption $\alpha' = 0$, $\beta' = \beta$, we have

$$P_{\text{wave ant}} = \frac{E_V{}^2 l^2}{480 g\lambda K}. \tag{24}$$

If $\lambda = 5000$, $l = 3\lambda$, $K = 400$, $g = 1.5 \times 10^{-3}$, then,

$$P_{\text{vert}} = 2000 E_V{}^2, \qquad P_{\text{wave ant}} = 156 E_V{}^2.$$

The figures above indicate that an array of 13 wave antennas, each three wavelengths long, should give as much power output as a perfect short antenna. The approximation $\alpha' = 0$, $\beta' = \beta$ used in obtaining equation 20 from equation 17 makes 24 an optimistic equation. Although β' can be made equal to β by loading the wave antenna with series capacitors, the attenuation constant α' cannot be made equal to zero. Nevertheless, theoretical and experimental studies by Austin Bailey have indicated that at 60 kc an array of 12 loaded wave antennas, each 28 wavelengths long, covering an area about 5 × 9 miles at Bradley, Me., and properly oriented, would produce the same field in England as the R.C.A. Rocky Point vertical antenna. At this frequency the efficiency of the Rocky Point antenna is approximately 50 per cent. The study also showed that the array of wave antennas would transmit signals over a wide frequency band with the same efficiency.

15.9 Shunt-excited antennas

Shunt excitation of antennas (Fig. 15.26) enables us to eliminate base insulators, tower lightning chokes, and other lightning-protective devices.* The dimensions in this illustration are those of Station WWJ which was made available to Morrison and Smith for experimental work through the courtesy of the *Detroit Daily News*. A schematic diagram of connec-

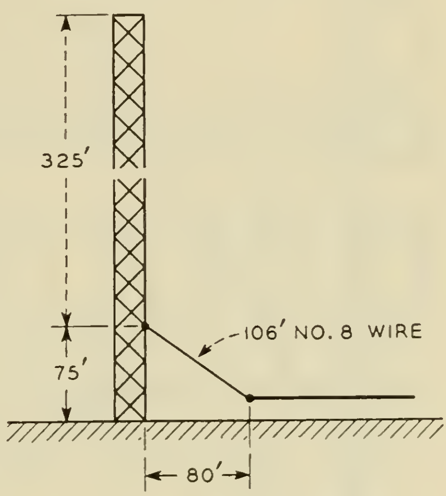
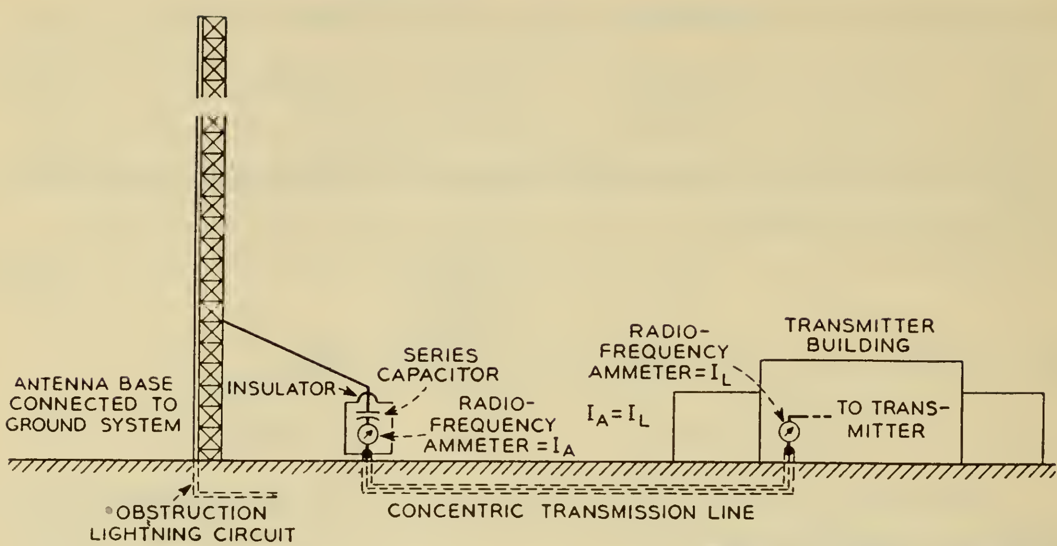


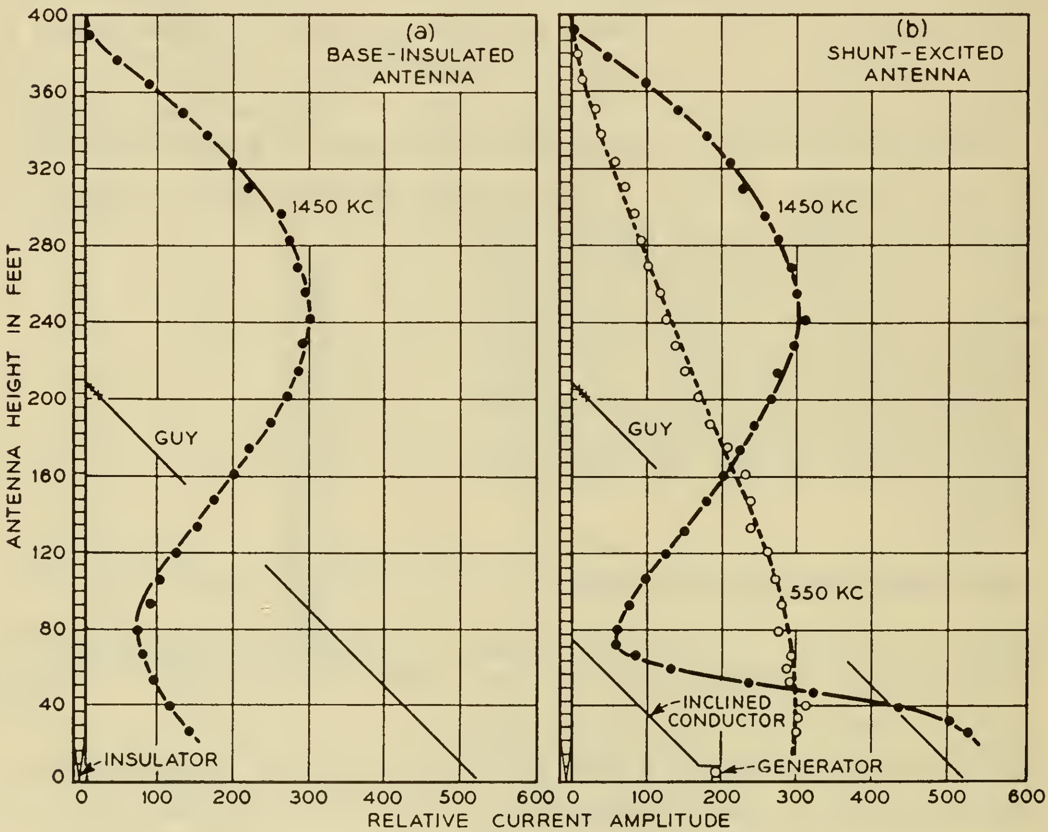
FIG. 15.26 A shunt-excited antenna.

* J. F. Morrison and P. H. Smith, The shunt-excited antenna, *IRE Proc.*, **25**, June 1937, pp. 673–696.



Morrison and Smith, Courtesy IRE

FIG. 15.27 Coupling arrangement for the shunt-excited antenna at Station WWJ.



Morrison and Smith, Courtesy IRE

FIG. 15.28 Current distribution in the antenna of Fig. 15.27: (*a*) for base series excitation; (*b*) for shunt excitation.

tions is shown in Fig. 15.27. Figures 15.28*a* and *b* present measurements of the current distribution in the tower: (*a*) for base-series excitation; (*b*) for shunt excitation.

15.10 Adcock antennas

Consider two equal parallel antennas, and assume that they are perpendicular to the line joining their centers. If the currents in these antennas are equal and opposite, there is no radiation in the plane perpendicular to the line joining their centers; as long as the distance between the antennas does not exceed $\lambda/2$, the maximum radiation is along this line. Hence, the shape of the radiation pattern in the plane perpendicular to the antennas is that of the figure 8. Several methods of feeding two antennas to obtain equal and opposite currents were suggested by

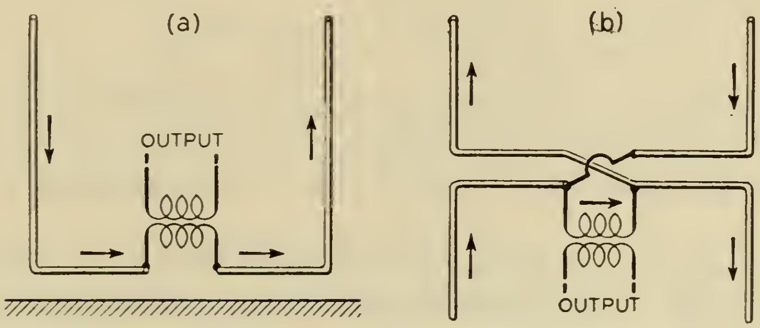


FIG. 15.29 Adcock antennas.

Adcock.* Two of the most obvious arrangements are shown in Fig. 15.29; the arrangement *a* is for vertical antennas above ground and *b* is for antennas in free space.

By the reciprocity theorem we conclude that the shape of the receiving pattern is also that of the figure 8. This may also be concluded from direct inspection of the above arrangement; only the difference between the antenna currents passes through the primary windings of the transformers and induces a response in the secondary windings. The principal application of Adcock antennas is in direction finders.

15.11 V antennas

If a thin antenna is fed in the center and if the length of each antenna arm is less than $\lambda/2$, the currents in various parts of the antenna are

* R. H. Barfield, Some principles underlying the design of spaced-aerial direction-finders, *IEE Jour.*, 76, April 1935, pp. 423–447. *Terman's Handbook*, pp. 880–883.

substantially in phase,* and the radiation is maximum in the equatorial plane. For a longer antenna we have a phase reversal in the current. The field in the equatorial plane will thus begin to decrease, and we might expect a decrease in the directive gain in this plane. However, the radiated power also decreases because the mutual radiation of the antenna arms decreases. To understand this, let us replace each half-wave by a current element of the same moment in the center of the half-wave. Equation 5–77 for the mutual radiation resistance of two colinear current elements indicates clearly that this resistance decreases as the

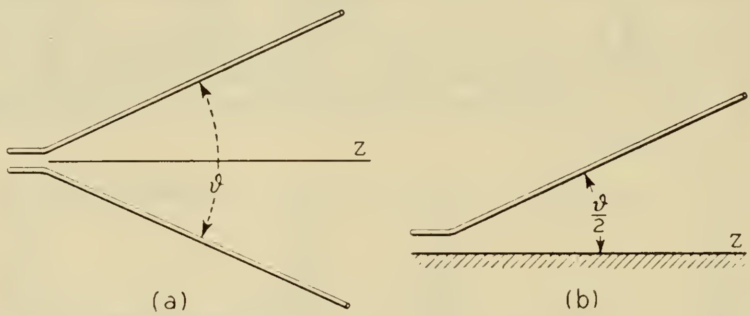


Fig. 15.30 (*a*) A V antenna and (*b*) an inclined wire.

distance between the elements becomes larger than $\lambda/2$. This may also be seen from Fig. 13.17, which gives the radiation resistance of a linear antenna with respect to the current antinode. It was shown by Ballantine† that the gain in the equatorial plane reaches a maximum when $l/\lambda = 0.64$.

The gain in any given direction in the equatorial plane may be increased still further by inclining the wires and forming a *V antenna*‡ (Fig. 15.30*a*). If the equatorial plane is a perfect conductor, we have the case of an inclined wire over perfect ground (Fig. 15.30*b*).

To obtain an approximate formula for the angle ϑ between the wires for maximum radiation in the z direction, we shall assume at first that the current waves in the wires are progressive waves originating at the apex. If β is the phase constant along each wire, the phase constant in the z direction is $\beta \cos(\vartheta/2)$; hence, the difference in the phases of the waves coming from the element at the apex and from the elements at the ends of the wires is $\beta l - \beta l \cos(\vartheta/2)$. These elementary waves reinforce

* Exactly in phase for an infinitely thin antenna.

† Stuart Ballantine, On the optimum transmitting wavelength for a vertical antenna on a perfect earth, *IRE Proc.*, **12**, December 1924, pp. 833–839.

‡ P. S. Carter, C. W. Hansell, and N. E. Lindenblad, Development of directive transmitting antennas by RCA Communications, Inc., *IRE Proc.*, **19**, October 1931, pp. 1773–1842.

each other as long as the phase difference does not exceed π. Therefore, the optimum length l is given by

$$\beta l(1 - \cos \tfrac{1}{2}\vartheta) = \pi. \tag{25}$$

This is an approximate expression for two reasons. The waves reflected from the ends affect the optimum angle. Their principal radiation is in the negative z direction, of course; but there is some radiation in the positive z direction. A more important source of error is the neglect of the variation of the radiated power with l; we have taken into consideration only the variation in the field strength in the z direction. For a straight antenna, $\vartheta = \pi$, and equation 25 gives $\beta l = \pi$, $l = \lambda/2$; actually, $l = \lambda$. For larger values of l/λ the error should be smaller. From equation 25 we find

$$\vartheta = 4 \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{\lambda}{l}}\right). \tag{26}$$

To obtain the radiation intensity, we use the method of Section 12.1. At first we consider only a single wire and assume that the current distribution is sinusoidal,

$$I(s) = I_0 \sin \beta(l - s). \tag{27}$$

The radiation vector is in the direction of the wire,

$$N_s = \frac{I_0(e^{j\beta l \cos \psi} - \cos \beta l - j \cos \psi \sin \beta l)}{\beta \sin^2 \psi}. \tag{28}$$

The radiation vector of the second arm is the negative of this in magnitude. Adding the two vectors, we obtain the radiation vector of the V antenna.

To obtain the gain in the z direction, we note that, for both antenna arms, $\psi = \vartheta$; hence,

$$N_\theta = -N_s \sin \theta = -\frac{2I_0(e^{j\beta l \cos \theta} - \cos \beta l - j \cos \theta \sin \beta l)}{\beta \sin \theta}. \tag{29}$$

For the radiation intensity in the z direction, we have

$$\Phi = \frac{\eta}{8\lambda^2} N_\theta N_\theta^* = \frac{15 I_0^2}{\pi \sin^2 \frac{1}{2}\vartheta} f(\beta l, \vartheta),$$

$$f(\beta l, \vartheta) = 1 + \cos^2 \beta l + \sin^2 \beta l \cos^2 \tfrac{1}{2}\vartheta - 2 \cos \beta l \cos(\beta l \cos \tfrac{1}{2}\vartheta) - 2 \cos \tfrac{1}{2}\vartheta \sin \beta l \sin(\beta l \cos \tfrac{1}{2}\vartheta). \tag{30}$$

The radiated power is

$$P = \tfrac{1}{2} R_a I_0^2, \tag{31}$$

where R_a is the radiation resistance with reference to the current antinode. This resistance may be calculated by integrating the field along the antenna. The directive gain in the z direction is then

$$g = \frac{4\pi\Phi}{P} = \frac{120}{R_a \sin^2 \frac{1}{2}\vartheta} f(\beta l, \vartheta). \tag{32}$$

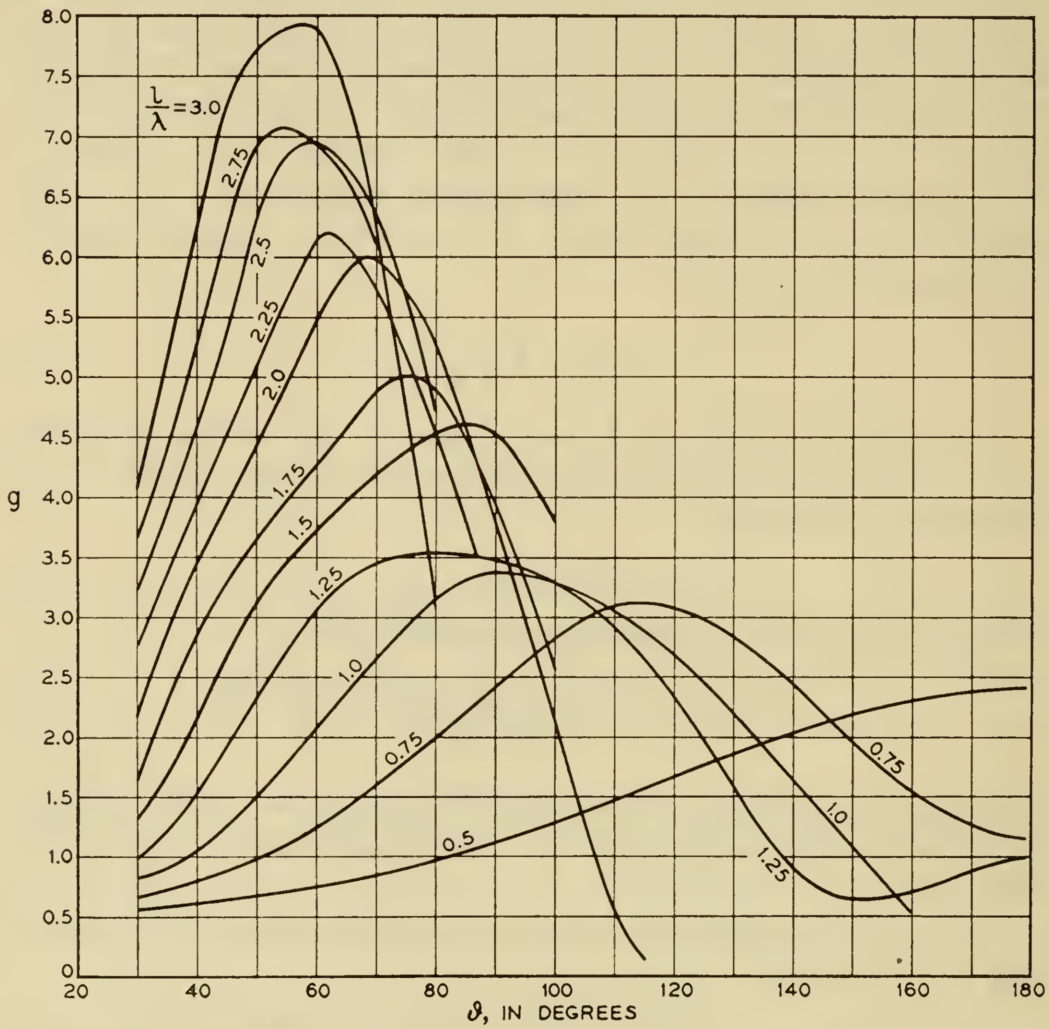


FIG. 15.31 Directivity of V antennas.

Figure 15.31 shows g as a function of ϑ for various lengths (in wavelengths) of the antenna arms. The following table gives the optimum angles and the maximum values of g for various lengths:

$l/\lambda =$	0.5	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$\vartheta =$	180°	114°.5	90°.0	78°.5	85°.0	75°.0	68°.5	62°.0	59°.0	55°.0	60°.0
$g_{\max} =$	2.41	3.12	3.38	3.53	4.61	5.02	6.00	6.20	6.95	7.07	7.94

(33)

If $\vartheta = 180°$, $g = g_{\max} = 3.297$ when $l/\lambda = 0.635$.

15.12 Horizontal omnidirectional antennas

An *omnidirectional antenna* is an antenna that radiates equally in all directions parallel to the surface of the earth. Vertical antennas are omnidirectional antennas. Loops with their planes parallel to the surface of the earth are omnidirectional antennas, provided the current is distributed uniformly around them. Now in a small loop the current is substantially uniform; but in a large loop it is nonuniform unless

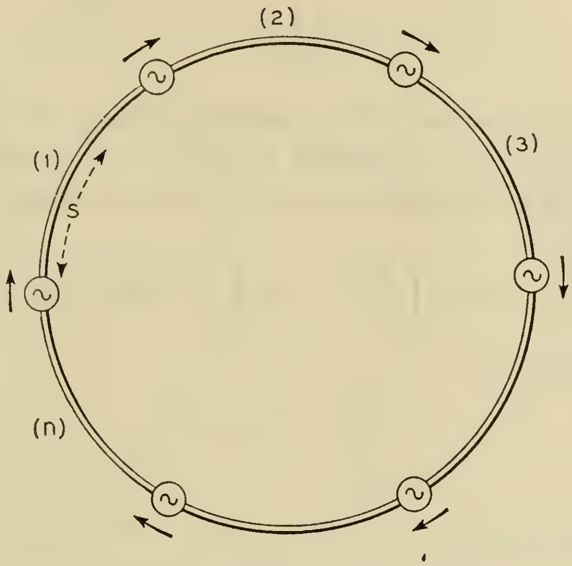


Fig. 15.32 Uniform distribution of voltage round a loop.

we distribute the impressed voltage around the loop (Fig. 15.32). Suppose that we have n generators distributed uniformly around the loop. The principal current in the first section of the loop is then

$$I_1(s) = A \cos \beta(s - s_0), \tag{34}$$

where A is the maximum amplitude and $s = s_0$ is the position of the current antinode. The maximum current must be halfway between the successive generators; hence,

$$s_0 = \frac{\pi a}{n}, \tag{35}$$

where a is the radius of the loop. The current at each generator is

$$I_i = A \cos \beta s_0 = A \cos \frac{\pi \beta a}{n}; \tag{36}$$

therefore,

$$\frac{I_{\max} - I_i}{I_{\max}} = 1 - \cos \frac{\pi \beta a}{n}. \tag{37}$$

From this equation we can determine the degree of nonuniformity in the current distribution for a loop of given radius.

If the reader does not find equation 35 obvious, he can obtain it by routine calculations as follows. By symmetry the current in each section of the loop is the same function of the distance from the generator at the beginning of that section. Hence, the current in the second section is obtained from equation 34 by substituting $s - (2\pi a/n)$ for s,

$$I_2(s) = A \cos \beta \left(s - s_0 - \frac{2\pi a}{n} \right). \tag{38}$$

The current must be continuous in passing through the second generator. The current entering this generator from the first section is $I_1(2\pi a/n)$, and that leaving it into the section is $I_2(2\pi a/n)$; hence,

$$A \cos \beta \left(\frac{2\pi a}{n} - s_0 \right) = A \cos \beta s_0. \tag{39}$$

An obvious solution is

$$\frac{2\pi a}{n} - s_0 = s_0. \tag{40}$$

There are other solutions, but they give the same form of current distribution.

The main problem in design is to find a convenient method of impressing the voltage uniformly around the loop. One solution is the *Alford loop** shown in Fig. 1.49*d* where the loop is formed by the outer conductors of coaxial pairs and the voltage is applied across the gaps at the corners. The shape of the loop may be either circular or polygonal.

Another solution is shown in Fig. 15.33*a* where the voltage is also applied at points equispaced on the loop. From this a cloverleaf antenna (Fig. 15.33*b*) was evolved.† The loop is broken into several sections; the beginnings of the sections are connected to one conductor of a coaxial feed and the ends to the other conductor. Alford loops and cloverleaf loops may be stacked into vertical broadside arrays.

The radiation pattern of a loop may be obtained by the method of Section 12.1. Let the loop be in the xy plane, and let the center be at the origin. The components of the moment of a typical element

* Andrew Alford and Armig G. Kandoian, Ultrahigh-frequency loop antennas, *AIEE Trans.*, 59, 1940, pp. 843–848.

† P. H. Smith, "Cloverleaf" antenna for F. M. broadcasting, *IRE Proc.*, 35, December 1947, pp. 1556–1563.

of the loop at $\varphi = \varphi'$ are:

$$\begin{aligned} dp_x &= -(I\,ds)\sin\varphi' = -Ia\sin\varphi'\,d\varphi', \\ dp_y &= (I\,ds)\cos\varphi' = Ia\cos\varphi'\,d\varphi'. \end{aligned} \tag{41}$$

The radiation vector of this element with reference to the center of the loop is

$$\begin{aligned} dN_x &= -Iae^{j\beta a\sin\theta\cos(\varphi-\varphi')}\sin\varphi'\,d\varphi', \\ dN_y &= Iae^{j\beta a\sin\theta\cos(\varphi-\varphi')}\cos\varphi'\,d\varphi'. \end{aligned} \tag{42}$$

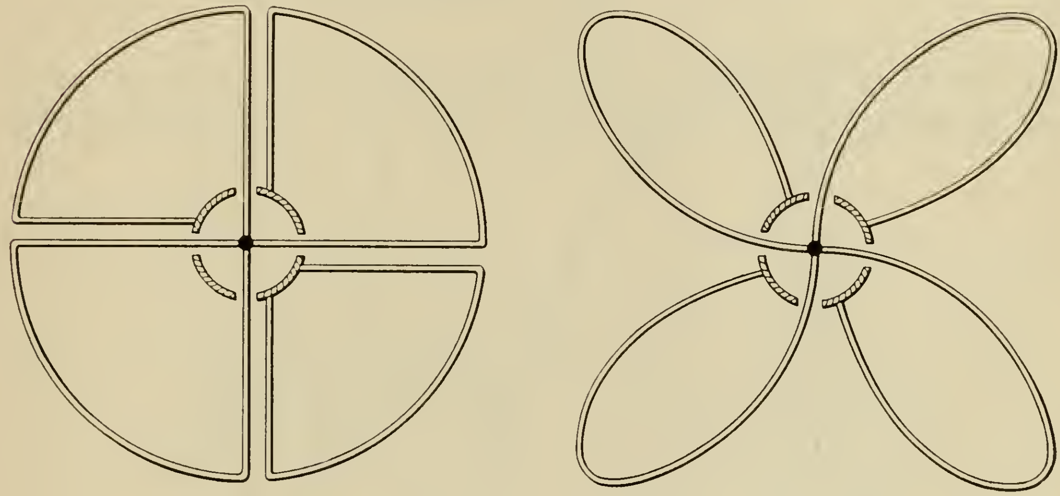


Fig. 15.33 (*a*) A method of applying voltage at points equispaced round the loop and (*b*) a cloverleaf antenna.

Assuming that I is independent of φ' and integrating round the loop, we find

$$N_x = -2j\pi aI\,J_1(\beta a\sin\theta)\sin\varphi, \quad N_y = 2j\pi aI\,J_1(\beta a\sin\theta)\cos\varphi. \tag{43}$$

Hence, by equation 12–9,

$$N_\varphi = 2\pi jaI\,J_1(\beta a\sin\theta), \qquad N_\theta = 0, \tag{44}$$

and, by equation 12–10,

$$\Phi = \frac{60\pi^3}{\lambda^2}\,a^2I^2\,J_1{}^2(\beta a\sin\theta). \tag{45}$$

In the plane of the loop, $\theta = \pi/2$, and

$$\Phi(\tfrac{1}{2}\pi) = \frac{60\pi^3}{\lambda^2}\,a^2I^2\,J_1{}^2(\beta a). \tag{46}$$

This is maximum when

$$\begin{aligned} \frac{d}{d(\beta a)}\,[\beta a\,J_1(\beta a)] &= \beta a\,J_0(\beta a) = 0, \\ \beta a = 2.40, &\qquad 2\pi a = 2.40\lambda. \end{aligned} \tag{47}$$

For a larger loop, the maximum radiation is thrown upward at the angle

$$\theta = \sin^{-1} \frac{2.40\lambda}{2\pi a}\,. \tag{48}$$

The directive gain in the plane of the loop is

$$g = \frac{4\pi\, \Phi(\frac{1}{2}\pi)}{P} = \frac{J_1^2(\beta a)}{\int_0^{\pi/2} J_1^2(\beta a \sin\theta) \sin\theta\, d\theta}\,. \tag{49}$$

FIG. 15.34 Directivity of a loop in its plane.

Expanding the square of the Bessel function in power series and integrating, we have

$$D = \int_0^{\pi/2} J_1^2(\beta a \sin\theta) \sin\theta\, d\theta = \sum_{n=0}^{\infty} \frac{(-)^n(\beta a)^{2n+2}}{(2n+3)(n+2)!\,n!}\,. \tag{50}$$

Hence,

$$\frac{d(\beta a D)}{d(\beta a)} = \sum_{n=0}^{\infty} \frac{(-)^n(\beta a)^{2n+2}}{(n+2)!\,n!} = J_2(2\beta a); \tag{51}$$

and, therefore,

$$D = \frac{1}{\beta a}\int_0^{\beta a} J_2(2u)\,du = \frac{1}{2\beta a}\int_0^{2\beta a} J_2(v)\,dv$$

$$= -\frac{1}{\beta a}J_1(2\beta a) + \frac{1}{2\beta a}\int_0^{2\beta a} J_0(v)\,dv. \tag{52}$$

Thus, we have expressed the denominator in equation 49 in terms of tabulated functions. For large values of $2\beta a$ we may conveniently use

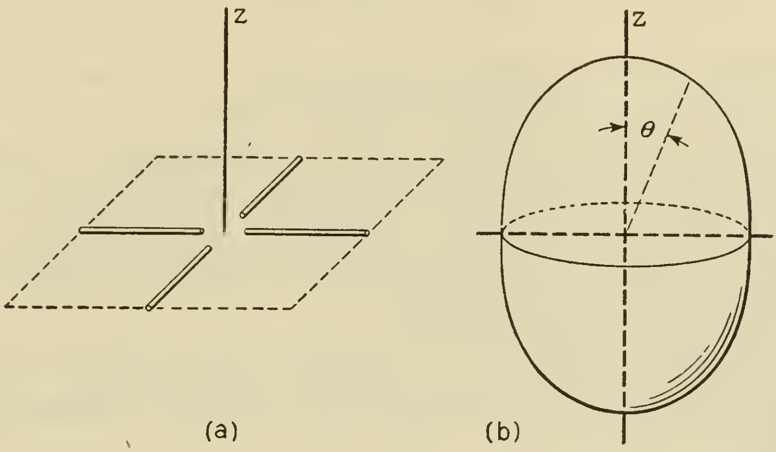


FIG. 15.35 (*a*) A turnstile antenna and (*b*) its radiation pattern.

the asymptotic expansions of these functions; normally, however, only the relatively small values of $2\beta a$ are of practical interest since for the larger values the radiation is thrown upward. Figure 15.34 shows the directivity of a circular loop in its plane as a function of its circumference in wavelengths.

Another type of omnidirectional antenna for horizontally polarized waves is the *turnstile antenna* (Fig. 15.35*a*). It consists of two perpendicular horizontal dipoles operated in quadrature.* Figure 15.35*b* shows the radiation pattern of two perpendicular current elements operated in quadrature. The radiation vectors are

$$N_x = Is, \qquad N_y = jIs; \tag{53}$$

* George H. Brown, The "turnstile" antenna, *Electronics*, **9**, April 1936, pp. 14–17. N. E. Lindenblad, Television transmitting antenna for Empire State building, *RCA Rev.*, **3**, April 1939, pp. 387–408; Antennas and transmission lines at the Empire State television station, *Communications*, **21**, April 1941, pp. 10–14.

hence,

$$N_\theta = Is \cos\theta\, e^{j\varphi}, \qquad N_\varphi = jIse^{j\varphi},$$
$$\Phi = \frac{15\pi}{\lambda^2}(Is)^2(1 + \cos^2\theta). \tag{54}$$

For the radiated power, we find

$$P = 80\pi^2\left(\frac{Is}{\lambda}\right)^2, \tag{55}$$

either by integrating Φ or from the fact that two elements operating in quadrature radiate independently.

The maximum radiation is in the vertical direction where

$$\Phi(0) = \frac{30\pi}{\lambda^2}(Is)^2. \tag{56}$$

The directivity of the turnstile arrangement of elements is seen to be 1.5, that is, the same as the directivity of a single current element.

In order to obtain a 90° phase shift, we may add $\lambda/4$ to the length of the line feeding one of the elements of the turnstile antenna. When the two feeders are connected either in series or in parallel, the variation of the input impedance with frequency is reduced compared with the input impedance of a single element.

The radiation pattern of two dipoles in the turnstile arrangement is given by the product of Φ in equations 54 and the radiation intensity of the dipole.

15.13 Space diversity systems

In a diversity system, the outputs of several spaced antennas are combined after detection so that high-frequency phase angles of the individual outputs are not involved.* Such systems are used to reduce fading due to atmospheric conditions, for it frequently happens that signals induced in antennas 5 to 10 wavelengths apart fade independently.

15.14 Approximate analysis of antenna systems

From the point of view of radiation the important characteristics of an antenna are: (1) the radiation pattern, (2) the radiated power for a given maximum amplitude of current, (3) the input impedance. If we are exploring the possibilities of a given antenna for a certain purpose, we may wish to determine these characteristics rapidly. In such exploratory calculations, errors of 10 to 20 or even 30 per cent are not

* H. H. Beverage and H. O. Peterson, Diversity receiving system of RCA Communications, Inc., for radio telegraphy, *IRE Proc.*, **19**, April 1931, pp. 531–561.

important; the main consideration is simplicity. No general rules can be given; the simplicity of a method depends on the readily available information. Continuous arrays may be approximated by discrete arrays and vice versa, depending on the circumstances. Each current filament of length equal to or less than $\lambda/2$ may be replaced by a current element of the same moment at the center of the filament; and, reciprocally, each current element may be replaced by a current filament of length not exceeding $\lambda/2$. For instance, the moment of a half-wave sinusoidal filament is

$$p = I_0 \int_{-\lambda/4}^{\lambda/4} \cos \beta z \, dz = \frac{2I_0}{\beta} = \frac{\lambda}{\pi} I_0. \tag{57}$$

Hence, the effective length of the element is λ/π. The power radiated by an element of moment p is

$$P = 40\pi^2 \left(\frac{p}{\lambda}\right)^2; \tag{58}$$

hence, the approximate power radiated by the half-wave antenna is $40I_0^2$. The exact power is $36.56I_0^2$. The error is 10 per cent.

If we replace a center-fed full-wave antenna by two colinear current elements, a half wavelength apart, the radiation resistance is $80 + 80 + 48 = 208$ ohms. The exact value is 199 ohms. The error is less than 5 per cent. This close agreement, however, is accidental; the errors in the resistances of the separate arms of the antenna and in their mutual resistance happen to be compensating. If we take the same antenna and feed it at a distance from one end equal to $\lambda/4$, the currents in the two halves will be opposite, and the radiation resistance will be $80 + 80 - 48 = 112$ ohms. The correct value is 93 ohms and the error is 20 per cent.

Next let us consider the Alford loop. If the length of each side is $\lambda/2$, we can calculate the radiation resistance from the formulas for the mutual resistance between parallel antennas, and for the self-resistance of a 90° V antenna. Exact expressions may be obtained for other lengths, but we would have to repeat rather laborious calculations. So we turn to the approximate method. Suppose that each side is $\lambda/4$. The moment is

$$p = I_0 \int_{-\lambda/8}^{\lambda/8} \cos \beta z \, dz = \frac{\lambda}{\pi\sqrt{2}} I_0, \tag{59}$$

and the moment per wavelength is

$$\mu = \frac{p}{\lambda} = \frac{I_0}{\pi\sqrt{2}} \cdot \tag{60}$$

From the equations of Section 5.21, we find

$$R_{\mathrm{rad}} = (4K_{11} + 8K_{12} - 4K_{13})\left(\frac{u}{\lambda}\right)^2, \tag{61}$$

where

$$K_{11} = 80\pi^2, \qquad K_{12} = T\left(\frac{\sqrt{2}}{8}\right) = 44, \qquad K_{13} = S(\tfrac{1}{4}) - T(\tfrac{1}{4}) = 448. \tag{62}$$

Hence,

$$R_{\mathrm{rad}} = 87. \tag{63}$$

This is the radiation resistance with reference to the maximum amplitude of the current (at the mid-point of each side). The current at each corner is $(1/\sqrt{2})I_0$. With reference to this current the radiation resistance is $2 \times 87 = 174$ ohms. A quarter of this resistance, about 42 ohms, is seen at each corner. The length of the coaxial pair from each corner to the center of the square is $\lambda/4$; hence, the impedance $42 + jX$ seen at the corner becomes $K^2/(42 + jX)$ at the center of the loop, assuming that K is the characteristic impedance of the lead-in coaxial pair. The four corners are seen in parallel by the coaxial feed line; the impedance across it will thus be $4K^2/(42 + jX)$.

15.15 Antenna models

The principal reasons for experiments with antennas are: (1) to obtain theoretically unavailable information, (2) to confirm a theory, (3) to check a method of measurement, (4) to appraise an approximate theory when a theoretical appraisal is difficult, (5) to ascertain tolerances permitted in the construction of a system to meet given requirements, assuming again that a theoretical examination is too difficult, (6) to make the final check on the performance of a system built for use. Dimensional considerations are useful in facilitating these experiments and in reducing their number. We have seen, for instance, that the radiation patterns of antennas in free space depend only on the ratios of the physical dimensions to the wavelength; this reduces the number of essential parameters by one. It also enables us to obtain these patterns from experiments on models.* Models are also used in impedance measurements.

The simple rule of preserving the physical dimensions of a system in wavelengths in the construction of a model applies only to perfectly conducting antennas in free space. In the general case, we have to

* George Sinclair, E. C. Jordan, and E. W. Vaughan, Measurement of aircraft-antenna patterns using models, *IRE Proc.*, 35, December 1947, pp. 1451–1462.

scale the conductivity, permeability, and dielectric constant. Proper scaling factors may be obtained from Maxwell's equations,

$$\text{curl } E = -j\omega\mu H, \qquad \text{curl } H = (g + j\omega\varepsilon)E. \tag{64}$$

Let primes denote various quantities in a model; then,

$$\text{curl}' E' = -j\omega'\mu' H', \qquad \text{curl}' H' = (g' + j\omega\varepsilon')E', \tag{65}$$

where the prime after curl denotes differentiation with respect to the scaled coordinates. Suppose that

$$\begin{aligned} &E' = k_E E, \qquad H' = k_H H, \qquad d' = k_d d, \qquad \omega' = k_\omega \omega, \\ &\mu' = k_\mu \mu, \qquad \varepsilon' = k_\varepsilon \varepsilon, \qquad g' = k_g g. \end{aligned} \tag{66}$$

Substituting in equati ons 65, we find

$$\frac{k_E}{k_d} \text{curl } E = -jk_\omega k_\mu k_H \omega\mu H, \quad \frac{k_H}{k_d} \text{curl } H = k_E(k_g g + jk_\omega k_\varepsilon \omega\varepsilon)E,$$

or

$$\begin{aligned} \text{curl } E &= -j\,\frac{k_\omega k_\mu k_H k_d}{k_E}\,\omega\mu H, \\ \text{curl } H &= \left(\frac{k_g k_E k_d}{k_H}\,g + j\,\frac{k_\omega k_\varepsilon k_E k_d}{k_H}\,\omega\varepsilon\right)E. \end{aligned} \tag{67}$$

These equations should be identical with equations 64; hence,

$$\frac{k_\omega k_\mu k_H k_d}{k_E} = 1, \qquad \frac{k_g k_E k_d}{k_H} = 1, \qquad \frac{k_\omega k_\varepsilon k_E k_d}{k_H} = 1; \tag{68}$$

that is,

$$k_\mu = \frac{k_E}{k_H k_\omega k_d}, \qquad k_g = \frac{k_H}{k_E k_d}, \qquad k_\varepsilon = \frac{k_H}{k_E k_\omega k_d}\cdot \tag{69}$$

In practice, experiments on models are performed in free space; hence, we have the following restrictions on the scaling factors:

$$k_\mu = 1, \qquad k_\varepsilon = 1. \tag{70}$$

Consequently,

$$k_E = k_\omega k_d k_H, \qquad k_H = k_\omega k_d k_E;$$

that is,

$$k_\omega k_d = 1, \qquad k_E = k_H. \tag{71}$$

The first of these equations states that, in the construction of a model, we should keep all distances in wavelengths constant. The second equation tells us that the impedances associated with the model equal

the corresponding impedances of the full-scale system. From equations 69 and 71 we obtain

$$k_g = \frac{1}{k_d}. \tag{72}$$

This equation imposes a severe limitation on model experiments. Most antennas (except broadcast antennas) are made with copper conductors. Since the conductivity of copper is exceeded only by that of silver and only by a small percentage, k_g must be smaller than unity. This requires that k_d be greater than unity. That is, in constructing an exact model, we can only increase the dimensions of the actual system. In practice, on the other hand, we most frequently desire to decrease these dimensions. Hence, we are forced to ignore equation 72. The results obtained with model experiments are therefore approximate, and good only to the extent to which the conductivity of the antennas has a negligible effect on the measured quantities. For a more detailed discussion of antenna models the reader is referred to a paper on "Theory of models of electromagnetic systems" by George Sinclair in the *Proceedings of the Institute of Radio Engineers*, November 1948, pp. 1364–1370.

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16

HORNS

16.1 Horns

Horns are expanding waveguides of finite length (Fig. 16.1). When the E vector is vertical, the electric currents are mostly longitudinal and are confined largely to the upper and lower walls of the horn.

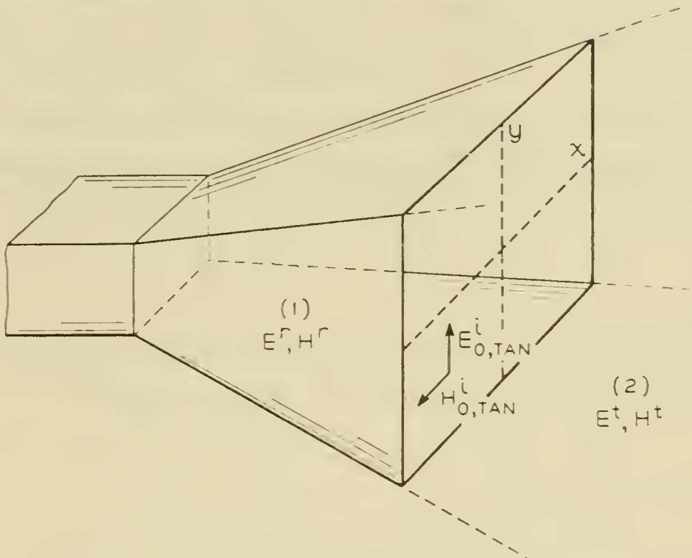

FIG. 16.1 An electric horn.

These currents flow in opposite directions; hence, if we subdivide these two walls into narrow longitudinal strips, we can pair the strips into "V antennas." Thus, we may consider the horn as a system of fanned-out V antennas.

The radiation pattern of a given horn may be obtained from the current distribution in the walls of the horn and of the waveguide feeding it. It is much simpler, however, to obtain it from the field distribution in the aperture of the horn. For this purpose we have to prove a certain induction theorem and derive some auxiliary formulas.

16.2 Induction theorem*

The theorem we are about to prove is a generalization of a more elementary theorem concerning the induction of currents in wires by an impressed electric field. According to this theorem, the impressed field is equivalent to a continuous distribution of series generators of zero internal impedance whose internal emf's are equal to $E^i_{\text{tan}}\,\Delta x$ (Fig. 16.2), where Δx is the length of a typical element of one of the wires.

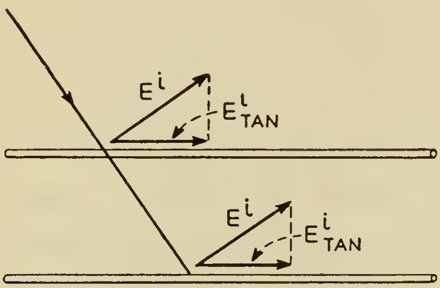

FIG. 16.2 A pair of parallel wires in an impressed electric field.

To obtain the more general theorem, let us assume that we know the field in a horn of infinite length. This field E^i, H^i will be called the primary or the incident field. When we cut the horn to a finite length, the field will be changed; let this field be E, H. Imagine now a surface S over the aperture of the finite horn which separates region 1 "inside" the horn from region 2 "outside" it. We may choose this surface to be plane; we may take it coincident with the wavefront of E^i, H^i; or we may take it to be any other convenient boundary. The actual field in region 2 will be called the *transmitted* field E^t, H^t; thus, *in region* 2,

$$E = E^t, \qquad H = H^t. \tag{1}$$

In region 1 the difference between the actual field and the primary field will be called the *reflected field* E^r, H^r; thus, *in region* 1,

$$E - E^i = E^r, \qquad H - H^i = H^r,$$

or

$$E = E^i + E^r, \qquad H = H^i + H^r. \tag{2}$$

The surface S is just an imaginary surface over the aperture of the horn; there are no sources on it; and the field E, H must be continuous across it. Thus, *at the surface* S, the values of the tangential components obtained from equation 1 must equal those obtained from equation 2,

$$E^t_{0,\text{tan}} = E^i_{0,\text{tan}} + E^r_{0,\text{tan}}, \qquad H^t_{0,\text{tan}} = H^i_{0,\text{tan}} + H^r_{0,\text{tan}}. \tag{3}$$

In view of Maxwell's equations the continuity of the normal components is a necessary consequence of equations 3 and need not be stated

* It is one of several field equivalence theorems analogous to the circuit equivalence theorems discussed in Section 9.9.

explicitly. From equations 3 we have

$$E^i_{0,\tan} - E^r_{0,\tan} = E^t_{0,\tan}, \qquad H^i_{0,\tan} - H^r_{0,\tan} = H^t_{0,\tan}. \tag{4}$$

Consider now the *scattered field* E^s, H^s consisting of the reflected field E^r, H^r in region 1 and the transmitted field E^t, H^t in region 2. We can prove that this field can be generated by a proper distribution of sources over S as well as by the primary field E^i, H^i. The scattered field satisfies Maxwell's equations and the boundary equations at the surface of the horn; but it is discontinuous across S. The discontinuity (equation 4) in the tangential component of E in this field requires a magnetic current sheet; the magnetic current must be perpendicular to $E^i_{0,\tan}$, and its linear density (the current per unit length perpendicular to the lines of flow) must equal $E^i_{0,\tan}$. Similarly, the discontinuity in the tangential component of H requires an electric current sheet of density $H^i_{0,\tan}$. *These currents may be considered as the virtual sources of the scattered field,* defined as *the sum of the reflected and transmitted fields.*

The directions of the currents are obtained from Maxwell's laws. The difference in the tangential transmitted and reflected components of E must be equal to the *negative* of the magnetic current density M, while the corresponding difference in the H components should equal the electric current density* C. Thus, if n is the unit normal to S in the direction of propagation of the incident wave, then,

$$M = E^i_{0,\tan} \times n = E_0{}^i \times n, \qquad C = n \times H^i_{0,\tan} = n \times H_0{}^i. \tag{5}$$

In particular,

$$M_x = E_{0,y}, \qquad C_y = H_{0,x};$$
$$M_y = -E_{0,x}, \quad C_x = -H_{0,y}. \tag{6}$$

Thus we have arrived at the following:

Induction Theorem. *The reflected and transmitted fields may be generated by an appropriate distribution of electric and magnetic currents distributed over the "surface of reflection." The linear densities of these currents are given by the tangential components of the incident field.*

In obtaining the field of these currents the medium must be left unchanged; for example, the horn in Fig. 16.1 must be left in its place.

The importance of the induction theorem warrants another proof. Again we shall begin by considering an infinite horn and assume that the field in it is known. The calculation of this field is much simpler

* These densities are *linear* densities of current sheets and are equal to the currents per unit length normal to the lines of flow.

than it would be for a finite horn. Next let us imagine a surface S dividing the infinite horn transversely. Let the source of power be to the left of S. Let us now see if we can make the surface S a perfect absorber. If it is a perfect absorber, then the field to the right of it will be identically equal to zero and the field to the left unchanged. Hence, the tangential components of E and H will be discontinuous across S. In accordance with Maxwell's equations, these discontinuities require electric and magnetic current sheets over S; Maxwell's equations also give the linear densities of these as equal to* E^i_{tan}, H^i_{tan}. The action of the perfect absorber may now be interpreted as a cancelation of the original field to the right of S by the field of the current sheets over S. If we now reverse the directions of the current sheets, their field will be exactly equal to the original field on the right of S. Thus, we may replace the actual generator in the interior of the horn to the left of S by a system of virtual generators distributed over S.

Thus far we have been assuming that the horn is infinite. However, once we have determined the distribution of virtual sources necessary to make S a perfect absorber, the field of these virtual sources cancels the original field to the right of S, and there are no currents in that part of the walls of the horn that extends to the right of S. Consequently, we may remove this part of the horn without disturbing the conditions to the right of S. Hence, the field of our electric and magnetic current sheets still cancels the field produced by the actual generator in the interior of the horn. The latter field must thus be equal in magnitude and opposite in sign to the field of the virtual sources over S.

16.3 Field equivalence theorems

There is an obvious corollary to the induction theorem. We now assume that we know the field produced by the actual generator in the interior of a given *finite* horn, and enclose the horn by a closed surface S. The arguments of the preceding section may be applied to this surface. Thus, we can obtain a distribution of electric and magnetic currents on S which will make it a perfect absorber. The field of these currents will cancel the original field in the exterior of S without affecting the field in the interior. Hence, the interior fields of the electric and magnetic currents on S must cancel each other. Consequently, in calculating the exterior field of these currents we are permitted to change the conditions in the interior of S; for example, we may remove the horn and the generator inside it.

The horn is incidental to our main arguments. The surface S may be a closed surface surrounding an electric current element, or an

* The directions of these currents are also prescribed by Maxwell's equations.

antenna, or any given electric current distribution. In such cases we can verify the above field equivalence theorem (or the equivalence principle, as we have frequently called it) by direct calculation.

This theorem asserts, in effect, that any field in a source-free region may be determined from the components of E and H tangential to the boundary of the region. It should be noted, however, that these tangential components cannot be chosen arbitrarily, since they belong to a field that satisfies Maxwell's equations. It can be shown that either the tangential component of E or the tangential component of H alone is sufficient to define a unique field in the region. When taken separately, these tangential components may be chosen arbitrarily.

The field equivalence theorem is an extension to vector waves of Kirchhoff's theorem* for scalar waves. Neither of these theorems should be confused with the induction theorem, of which they are special cases. In Kirchhoff's theorem and in the field equivalence theorem the surface S *must* be a closed surface; but in the induction theorem S may be an open surface. In the former case, the field of the virtual sources is calculated *as if these sources were in free space,* with the boundaries surrounding the primary source removed; but, in the latter case, these boundaries must be retained when the field is calculated. These are the conditions for obtaining *exact* results. However, these theorems are used principally in approximate calculations; the nature of the approximations is often such as to remove, in effect, the difference between the induction theorem and the field equivalence theorem.

For further details about these theorems and their applications, the reader is referred elsewhere.† Some of these theorems can also be proved by setting up explicit expressions for the fields of the actual sources in terms of retarded potentials and then transforming the integrals until they yield the desired results.‡ This type of proof is fairly long and laborious, and less general; but some may have more confidence in it than in the strictly logical proofs given here.

* A. G. Webster, *Partial Differential Equations of Mathematical Physics*, G. E. Stechert, New York, 1927, p. 216.

† S. A. Schelkunoff, Some equivalence theorems of electromagnetics and their application to radiation problems, *Bell Sys. Tech. Jour.*, **15**, January 1936, pp. 92–112; On diffraction and radiation of electromagnetic waves, *Phys. Rev.*, **56**, August 15, 1939, pp. 308–316; A general radiation formula, *IRE Proc.*, **27**, October 1939, pp. 660–666.

‡ Julius Adams Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, pp. 464–470.

16.4 Elementary sources in free space

According to the induction theorem, the problem of obtaining the fields generated by horns may be solved by integrating the fields of *known** electric and magnetic current elements situated on some surface covering the mouth of the horn. The fields of these elementary sources must be calculated, not in free space, but in the presence of the horn itself. This is a difficult problem. However, if we take the free-space fields of the elementary sources and integrate them over a large area, we shall find that the field is strong in the directions away from the horn and weak in the directions leading into the horn itself. The reaction of the horn will then be relatively small, and our results will constitute an approximate solution of our problem. In this method we start with the known field over the aperture of the horn and are forced to make approximations when evaluating the field of an equivalent distribution of sources.

We can also take a *closed* surface S which covers the aperture, as in the preceding case, and, in addition, encloses the horn itself. If we knew the tangential E and H on this surface, we could, by using the equivalence principle, evaluate the field outside S *exactly*; for, in this case, the horn may be removed as explained in the preceding section, and the field of the virtual sources on S may be calculated as if these sources were in free space. But, in the present case, *we do not know* the field on S, and, therefore, we do not know the exact distribution of the virtual sources. In this method we are forced to make approximations in this distribution. When the dimensions of the aperture are large compared with the wavelength, a reasonable approximation would be to assume that the field over the aperture is substantially equal to the incident field and that the field over the rest of S is negligible. If we make this approximation, we obtain the same result as by the first method. The approximations in the two methods seem to be different, but their final effect is the same.

The free-space field of an electric current element is given by equations 4–82. In the arrangement shown in Fig. 16.1, the moment of a typical electric current element in the plane of the aperture is $C_y\,dx\,dy = H_{0,x}\,dx\,dy$. Since the present current element is parallel to the y axis and not to the z axis, it is simpler to start by evaluating the dynamic component of E, and then use the fact that the total electric intensity at a distant point equals the projection of the dynamic component on the tangent plane to the sphere centered at the element and passing through the point under consideration. For the dynamic com-

* Known from a prior solution of the problem for an infinite horn.

ponent, we have

$$F_y = -\frac{j\omega\mu C_y\,dx\,dy}{4\pi r}\,e^{-j\beta r} = -j\,\frac{60\pi H_{0,x}\,dx\,dy}{\lambda r}\,e^{-j\beta r}. \tag{7}$$

Hence,

$$\begin{aligned} E_\theta &= F_y \cos\theta \sin\varphi = -j\,\frac{60\pi H_{0,x}\,dx\,dy}{\lambda r}\cos\theta\sin\varphi\,e^{-j\beta r}, \\ E_\varphi &= F_y \cos\varphi = -j\,\frac{60\pi H_{0,x}\,dx\,dy}{\lambda r}\cos\varphi\,e^{-j\beta r}. \end{aligned} \tag{8}$$

The field of a magnetic current element may be obtained by analogy with that of an electric current element. Thus, for the dynamic component of the magnetic intensity of an element of moment $M_x\,dx\,dy = E_{0,y}\,dx\,dy$, we have

$$F_x^{(m)} = -\frac{j\omega\varepsilon M_x\,dx\,dy}{4\pi r}\,e^{-j\beta r} = -j\,\frac{E_{0,y}\,dx\,dy}{240\pi\lambda r}\,e^{-j\beta r}. \tag{9}$$

From this we obtain the total magnetic intensity (at large distances from the element),

$$\begin{aligned} H_\theta &= F_x^{(m)}\cos\theta\cos\varphi = -j\,\frac{E_{0,y}\,dx\,dy}{240\pi\lambda r}\cos\theta\cos\varphi\,e^{-j\beta r}, \\ H_\varphi &= -F_x^{(m)}\sin\varphi = j\,\frac{E_{0,y}\,dx\,dy}{240\pi\lambda r}\sin\varphi\,e^{-j\beta r}. \end{aligned} \tag{10}$$

Hence, the corresponding electric intensity is

$$\begin{aligned} E_\theta &= 120\pi H_\varphi = j\,\frac{E_{0,y}\,dx\,dy}{2\lambda r}\sin\varphi\,e^{-j\beta r}, \\ E_\varphi &= -120\pi H_\theta = j\,\frac{E_{0,y}\,dx\,dy}{2\lambda r}\cos\theta\cos\varphi\,e^{-j\beta r}. \end{aligned} \tag{11}$$

The total field of the double source representing an element of the wavefront is the sum of equations 8 and 11.

16.5 Huygens source in free space

For uniform plane waves in free space,

$$E_{0,x} = 120\pi H_{0,y}, \qquad E_{0,y} = -120\pi H_{0,x}. \tag{12}$$

These are also approximate equations for horns with large apertures. In honor of Huygens, who was the first to introduce the idea that a

wavefront may be considered as a system of secondary sources, the double source *corresponding to the particular ratio of electric and magnetic intensities given by* equations 12 has been named the "Huygens source." Substituting from equations 12 in equations 8 and 11 and adding, we obtain the distant field of the Huygens source *for the case in which E is parallel to the y axis,*

$$E_\theta = \frac{jE_{0,y}\,dx\,dy}{2\lambda r}(1 + \cos\theta)\sin\varphi\, e^{-j\beta r},$$
$$E_\varphi = \frac{jE_{0,y}\,dx\,dy}{2\lambda r}(1 + \cos\theta)\cos\varphi\, e^{-j\beta r}. \tag{13}$$

Along the axis of the Huygens source, $\theta = 0$, and

$$E_\theta = \frac{jE_{0,y}\,dx\,dy}{\lambda r}\sin\varphi\, e^{-j\beta r},$$
$$E_\varphi = \frac{jE_{0,y}\,dx\,dy}{\lambda r}\cos\varphi\, e^{-j\beta r}. \tag{14}$$

In Cartesian coordinates,

$$E_y = \frac{jE_{0,y}\,dx\,dy}{\lambda r}e^{-j\beta r}, \qquad E_x = 0. \tag{15}$$

In this direction the field is parallel to the original field. The phase retardation is less than that corresponding to the distance r by 90°. The reason for this difference lies in the higher phase velocity of spherical waves in the vicinity of their source.

16.6 Radiation patterns

From the expression for the field of a Huygens source, we can obtain the radiation pattern of any given distribution of Huygens sources. Let us calculate, for example, the radiation pattern for an open end of a large waveguide carrying a dominant wave. This will also be the radiation pattern of a horn (Fig. 16.1) with a lens in its aperture which transforms the spherical wavefront into a plane wavefront. The field of the dominant wave varies as follows,

$$E_{0,y} = E_0 \cos\frac{\pi x}{a}, \tag{16}$$

where x is measured from the center.

In this case the elementary Huygens sources are similarly arranged, and the radiation intensity is the product of the space factor and the

radiation intensity of a Huygens source of unit moment ($E_{0,y}\,dx\,dy = 1$). The latter is

$$\Phi_0 = \frac{(1 + \cos\theta)^2}{960\pi\lambda^2}. \qquad (17)$$

For the former we have

$$S^2 = E_0^2 \left| \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \cos\frac{\pi x}{a}\, e^{j\beta(x \sin\theta \cos\varphi + y \sin\theta \sin\varphi)}\, dx\, dy \right|^2. \qquad (18)$$

Therefore,

$$\Phi = S^2\Phi_0 =$$

$$\frac{E_0^2 a^2 \sin^2\left(\frac{\pi b}{\lambda}\sin\theta\sin\varphi\right)\cos^2\left(\frac{\pi a}{\lambda}\sin\theta\cos\varphi\right)}{240\pi \sin^2\theta \sin^2\varphi(\pi^2 - \beta^2 a^2 \sin^2\theta\cos^2\varphi)^2}(1 + \cos\theta)^2. \qquad (19)$$

This formula is for *large* apertures. For smaller apertures we may operate close to the cutoff, in which case the ratio of E to H at the aperture is quite different from the free-space ratio, and we must use* equations 6 instead of 12. Very close to the cutoff the magnetic intensity at the aperture is very small, and the virtual sources are largely magnetic current elements; then the radiation in the backward direction is almost as great as in the forward direction.

Likewise, if the guide or the horn is narrow in the direction of the E lines, there is a large impedance mismatch at the aperture, such that E is relatively large and H is small. In this case also, we have mostly magnetic currents in the secondary source system, and the field of each element will be given by equations 11 rather than by the sum of equations 8 and 11. The radiation pattern of the guide in the plane perpendicular to the long dimension is practically a circle.

16.7 Directivity

In the case of an open waveguide of large cross section, most of the power delivered to the open end is radiated. This power is

$$P = \frac{E_0^2}{240\pi} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \cos^2\frac{\pi x}{a}\, dx\, dy = \frac{E_0^2 ab}{480\pi}. \qquad (20)$$

The average radiation intensity is $P/4\pi$. To obtain the maximum intensity, we let $\theta = 0$ in equation 19; thus,

$$\Phi_{\max} = \frac{E_0^2 a^2 b^2}{60\pi^3\lambda^2}. \qquad (21)$$

* *Electromagnetic Waves*, p. 359.

Hence,

$$g = \frac{\Phi_{\max}}{\Phi_{\mathrm{av}}} = \frac{32ab}{\pi\lambda^2}\cdot \tag{22}$$

For the effective area, we find

$$A_{\mathrm{eff}} = \frac{g\lambda^2}{4\pi} = \frac{8}{\pi^2}\,ab; \tag{23}$$

that is, the effective area of a horn with a lens which straightens the wavefront of the guided wave is about four fifths of the area of the aperture.

Without the lens in the aperture, the front of the emerging wave is curved, the Huygens sources are at different distances from a distant point in the direction of the horn, the corresponding elementary waves arrive at that point in different phases, and the total radiation intensity is reduced. Therefore, without the correcting lens, the directivity and the effective area are smaller.

If the curvature of the wavefront is small, we may assume that the electric intensity is parallel to the xy plane and is given by equation 16. The radiation intensity will be

$$\Phi = \frac{E_0{}^2(1+\cos\theta)^2}{960\pi\lambda^2}\times$$

$$\left|\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\cos\frac{\pi x}{a}\,e^{j\beta(x\sin\theta\cos\varphi+y\sin\theta\sin\varphi+z\cos\theta)}\,dx\,dy\right|^2, \tag{24}$$

where (x, y, z) is a point on the wavefront. The value of Φ in the z direction is obtained when $\theta = 0$,

$$\Phi_0 = \frac{E_0{}^2}{240\pi\lambda^2}\left|\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\cos\frac{\pi x}{a}\,e^{j\beta z}\,dx\,dy\right|^2\cdot \tag{25}$$

Let the equation of the wavefront be

$$z = f(x, y) = a_0 + a_1x + b_1y + c_{11}x^2 + c_{12}xy + c_{22}y^2 + \cdots. \tag{26}$$

Choosing the origin $(0, 0, 0)$ on the wavefront, we obtain $a_0 = 0$. Since the xy plane has already been chosen tangential to the wavefront, we must have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \quad \text{when} \quad x = y = 0. \tag{27}$$

Therefore, $a_1 = b_1 = 0$, and

$$z = c_{11}x^2 + c_{12}xy + c_{22}y^2 + \cdots. \tag{28}$$

The c's may be expressed in terms of the principal radii of curvature of the wavefront. In the arrangement shown in Fig. 16.1, the principal radii of curvature are R_m and R_e in the magnetic or xz plane, and electric or yz plane, respectively. The first radius R_m is the length of the dihedral horn that would be formed by extending those faces of the given horn that are normal to the magnetic plane until they meet (Fig. 16.3); similarly, R_e is the length of the dihedral horn that would be formed by extending the top and bottom faces which are perpendicular to E (Fig. 16.4). To obtain the radius of curvature of the wavefront in the plane

$$y = kx \tag{29}$$

at the origin, we have to evaluate

$$R = \pm \frac{\left[1 + \left(\frac{dz}{ds}\right)^2\right]^{3/2}}{\frac{d^2z}{ds^2}} \tag{30}$$

in this plane. Substituting from equation 29 in equation 28, we have

$$z = (c_{11} + c_{12}k + c_{22}k^2)x^2 + \cdots; \tag{31}$$

then,

$$\begin{aligned} ds &= [dx^2 + dy^2 + dz^2]^{1/2} = \left[1 + k^2 + \left(\frac{dz}{dx}\right)^2\right]^{1/2} dx, \\ \frac{dz}{dx} &= 2(c_{11} + c_{12}k + c_{22}k^2)x + \cdots, \\ \frac{dz}{ds} &= \frac{dz}{dx}\frac{dx}{ds} = \\ &\quad 2(c_{11} + c_{12}k + c_{22}k^2)\, x \left[1 + k^2 + \left(\frac{dz}{dx}\right)^2\right]^{-1/2} + \cdots, \\ \frac{d^2z}{ds^2} &= \frac{d}{dx}\left(\frac{dz}{ds}\right)\frac{dx}{ds} = \\ &\quad 2(c_{11} + c_{12}k + c_{22}k^2) \left[1 + k^2 + \left(\frac{dz}{dx}\right)^2\right]^{-1} + \cdots. \end{aligned} \tag{32}$$

In all these equations only the principal terms are displayed. At the origin,

$$R = \pm \frac{1 + k^2}{2(c_{11} + c_{12}k + c_{22}k^2)}. \tag{33}$$

In the xz plane, $k = 0$ and in the yz plane, $k = \infty$; hence,

$$R_m = \pm \frac{1}{2c_{11}}, \qquad R_e = \pm \frac{1}{2c_{22}};$$
$$c_{11} = \pm \frac{1}{2R_m}, \qquad c_{22} = \pm \frac{1}{2R_e}. \tag{34}$$

The signs depend on the relationship between the positive direction of the z axis and the directions of curvature. In our case, c_{11} and c_{22} are negative.

Since $k = \tan \varphi$, where φ is the angle between the plane $y = kx$ and the xz plane, equation 33 may be written as

$$R = \pm \frac{1}{2(c_{11} \cos^2 \varphi + c_{12} \sin \varphi \cos \varphi + c_{22} \sin^2 \varphi)}. \tag{35}$$

To obtain the angles for which R is either maximum or minimum, we equate to zero the derivative of $1/R$ with respect to φ; thus,

$$\tan 2\varphi = \frac{c_{12}}{c_{11} - c_{22}}. \tag{36}$$

In the arrangement shown in Fig. 16.1, the principal planes of curvature are $\varphi = 0$ and $\varphi = \pi/2$; hence, $c_{12} = 0$.

Thus we finally have

$$z = -\frac{x^2}{2R_m} - \frac{y^2}{2R_e} + \cdots \tag{37}$$

for the equation of the wavefront.

Neglecting the higher powers of x and y, we substitute from equation 37 in equation 25,

$$\Phi_0 = \frac{E_0^2}{240\pi\lambda^2} \left| \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \cos \frac{\pi x}{a} \exp\left(-j \frac{\pi x^2}{\lambda R_m}\right) dx \right|^2 \times \left| \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \exp\left(-j \frac{\pi y^2}{\lambda R_e}\right) dy \right|^2. \tag{38}$$

Both integrals may be expressed in terms of Fresnel integrals,*

$$\int_0^x \exp(\tfrac{1}{2}j\pi t^2)\, dt = C(x) + j\, S(x),$$

$$C(x) = \int_0^x \cos(\tfrac{1}{2}\pi t^2)\, dt, \qquad S(x) = \int_0^x \sin(\tfrac{1}{2}\pi t^2)\, dt. \tag{39}$$

* *Applied Mathematics*, Chapter 19.

Thus, for the second integral, we find

$$\left|\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \exp\left(-j\,\frac{\pi y^2}{\lambda R_e}\right) dy\right|^2 = 2\lambda R_e \left|\int_0^{\frac{1}{2}b} \exp\left(-j\,\frac{\pi y^2}{\lambda R_e}\right) d\left(\frac{y\sqrt{2}}{\sqrt{\lambda R_e}}\right)\right|^2$$
$$= 2\lambda R_e\left[C^2\left(\frac{b}{\sqrt{2\lambda R_e}}\right) + S^2\left(\frac{b}{\sqrt{2\lambda R_e}}\right)\right]. \qquad (40)$$

In the first integral in equation 38 the cosine function should be expressed in terms of exponential functions; then, by simple linear substitutions, the integrands may be reduced to the standard form (equations 39). In this manner we find

$$\Phi_0 = \frac{R_m R_e}{240\pi} E_0^2 \{[C(u) - C(v)]^2 + [S(u) - S(v)]^2\}[C^2(w) + S^2(w)],$$
$$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_m}}{a} + \frac{a}{\sqrt{\lambda R_m}}\right), \qquad (41)$$
$$v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_m}}{a} - \frac{a}{\sqrt{\lambda R_m}}\right), \qquad w = \frac{b}{\sqrt{2\lambda R_e}}.$$

The radiated power is given by equation 20; thus, the gain in the z direction is

$$g = \frac{8\pi R_m R_e}{ab} \{[C(u) - C(v)]^2 + [S(u) - S(v)]^2\}[C^2(w) + S^2(w)]. \qquad (42)$$

If the horn is flared only in the magnetic plane, $R_e = \infty$. For small values of w, $C(w) \simeq w$ and $S(w) \propto w^3$; hence, equation 42 becomes

$$g_m = \frac{4\pi b R_m}{\lambda a} \{[C(u) - C(v)]^2 + [S(u) - S(v)]^2\}. \qquad (43)$$

Figure 16.3 shows $g_m\lambda/b$. It should be noted that, in deriving these formulas, we assumed that the aperture is large; hence, b must not be small.

If the horn is flared only in the electric plane, $R_m = \infty$, and the simplest method of determining g_e is to re-evaluate Φ_0 from equation 38. Then we find

$$g_e = \frac{64 a R_e}{\pi\lambda b} [C^2(w) + S^2(w)]. \qquad (44)$$

Figure 16.4 shows $g_e\lambda/a$.

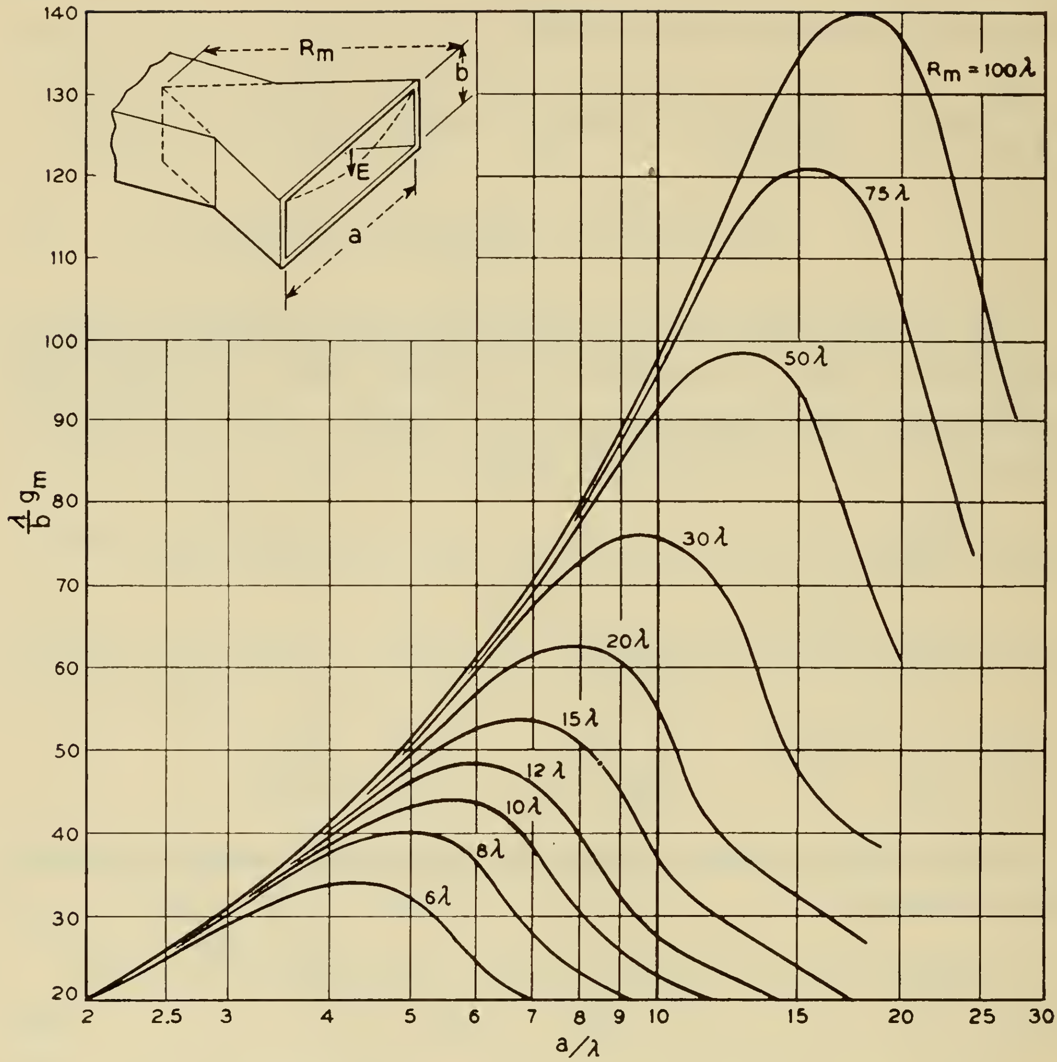

Fig. 16.3 Directivity of a large horn flared in the magnetic plane.

Multiplying equations 43 and 44 and comparing with equation 42, we find

$$g = \frac{\pi}{32}\left(\frac{g_m\lambda}{b}\right)\left(\frac{g_e\lambda}{a}\right). \tag{45}$$

Hence, to obtain the directive gain of a horn flared in both planes we need only divide the product of two readings from Figs. 16.3 and 16.4 by $32/\pi = 10.2$. When the directivities are expressed in decibels, we have to subtract 10 db from the sum of the readings.

16.8 Dihedral horns

In the principal mode of wave propagation in a dihedral horn (Fig. 16.5), the electric lines are circles coaxial with its apex line, and the field is independent of the distance along it. In this mode the only nonvanish-

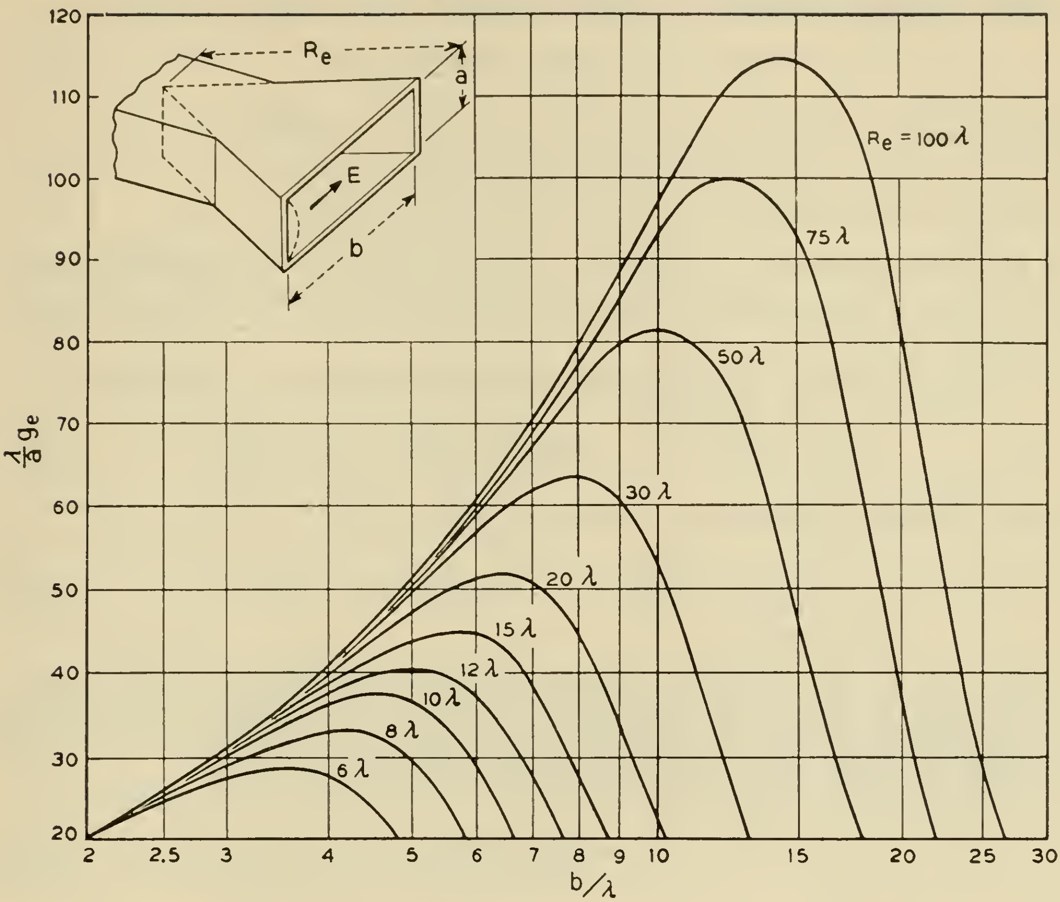

FIG. 16.4 Directivity of a large horn flared in the electric plane.

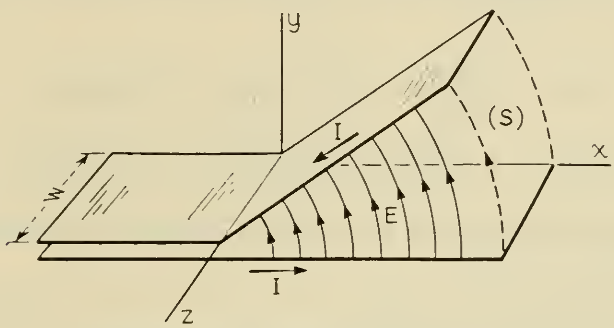

FIG. 16.5 A dihedral horn.

ing components of the field are H_z and E_φ, and Maxwell's equations become

$$\frac{dH_z}{d\rho} = -j\omega\varepsilon E_\varphi, \qquad \frac{d}{d\rho}(\rho E_\varphi) = -j\omega\mu\rho H_z. \tag{46}$$

Substituting for E_φ from the first equation into the second, we have

$$\rho\frac{d^2H_z}{d\rho^2} + \frac{dH_z}{d\rho} + \beta^2\rho H_z = 0. \tag{47}$$

This is Bessel's equation of order zero, and its general solution is

$$H_z = A\, J_0(\beta\rho) + B\, N_0(\beta\rho). \tag{48}$$

Substituting in the first equation in the set 46, we find

$$E_\varphi = -j\eta[A\, J_1(\beta\rho) + B\, N_1(\beta\rho)]. \tag{49}$$

If the horn extends to infinity, the waves must be progressive and $B = -jA$. If the horn is large, the reflection from its aperture is small and $B \simeq -jA$.

The transverse voltage between the boundaries of a large horn is

$$V(\rho) = \rho\psi\, E_\varphi = -j\eta\psi\rho A[J_1(\beta\rho) - j\, N_1(\beta\rho)], \tag{50}$$

where ψ is the horn angle. For small values of $\beta\rho$,

$$J_1(\beta\rho) = \tfrac{1}{2}\beta\rho, \qquad N_1(\beta\rho) = -\frac{2}{\pi\beta\rho}\cdot \tag{51}$$

Hence, when the distance between the parallel strips is very small, the coefficient A may be expressed in terms of the voltage $V(0)$ at the junction between the strip transmission line and the horn,

$$A = \frac{\pi\beta}{2\eta\psi}\, V(0). \tag{52}$$

If the width of the horn is w, the current is $I = wH_z$. Using the foregoing equations, we find

$$\begin{aligned} I(\rho) &= \frac{\pi\beta w}{2\eta\psi}\, V(0)[J_0(\beta\rho) - j\, N_0(\beta\rho)], \\ V(\rho) &= -\tfrac{1}{2}j\pi\beta\rho\, V(0)[J_1(\beta\rho) - j\, N_1(\beta\rho)]. \end{aligned} \tag{53}$$

If the distance b between the parallel strips is very small, the input admittance of the horn is

$$\begin{aligned} Y_i = \frac{I(b)}{V(0)} &= \frac{\pi\beta w}{2\eta\psi}\,[J_0(\beta b) - j\, N_0(\beta b)] \\ &\simeq \frac{\pi\beta w}{2\eta\psi}\,\left[1 - j\frac{2}{\pi}\,(\log \beta b - 0.116)\right]. \end{aligned} \tag{54}$$

Since the characteristic admittance of the parallel-strip transmission line varies inversely as b, whereas Y_i varies only as $\log \beta b$, the mismatch at the junction is large. For very small values of b, the transmission

line is almost "open" electrically. This fact has an important effect on the application of the equivalence principle to narrow horns.

16.9 Narrow horns

Seen from the aperture, the external broad surfaces of a narrow horn (Fig. 16.6) form a dihedral horn ($\psi = 2\pi$). The internal broad surfaces form a strip transmission line feeding the dihedral horn. In the preceding section we found that this line is almost open electrically. Hence, the electric intensity at the aperture is strong, and the magnetic intensity is relatively weak. Thus, the secondary sources at the aperture will be almost entirely magnetic currents.

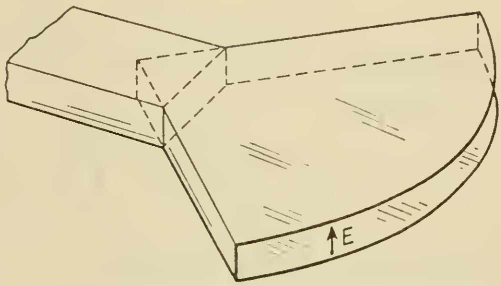

FIG. 16.6 A narrow horn flared in the magnetic plane.

First let us calculate the radiation pattern for the case in which the flare angle is zero as in a wave-guide open at the end. The electric intensity at the aperture is given by equation 16. The moment of a typical magnetic current element is $bE_0\,dx$, and we can use equations 9, 10, and 11 if we let $dy = b$. Hence, the radiation intensity is

$$\Phi = \frac{E_0{}^2b^2}{960\pi\lambda^2}\,(1 - \sin^2\theta\cos^2\varphi)\left|\int_{-\frac{1}{2}a}^{\frac{1}{2}a} \cos\frac{\pi x}{a} e^{j\beta x \sin\theta\cos\varphi}\,dx\right|^2 \cdot \qquad (55)$$

Integrating,

$$\Phi = \frac{\pi E_0{}^2a^2b^2\cos^2\left(\dfrac{\pi a}{\lambda}\sin\theta\cos\varphi\right)}{240\lambda^2(\pi^2 - \beta^2a^2\sin^2\theta\cos^2\varphi)^2}\,(1 - \sin^2\theta\cos^2\varphi). \qquad (56)$$

In the present case we may not assume that all power delivered to the aperture by a progressive wave in the guide is radiated. In fact, we have shown that most of it is reflected back into the guide. To obtain the radiated power we shall have to integrate equation 56. The integration may be greatly facilitated by changing the coordinates. We note that

$$\cos\theta' = \sin\theta\cos\varphi \qquad (57)$$

is the cosine of the angle between the x axis and a typical direction. Let us, therefore, rotate our coordinates and choose the x axis as the

z axis in the new system; then,

$$P = \iint \Phi \, d\Omega$$

$$= \frac{\pi E_0{}^2 a^2 b^2}{240\lambda^2} \int_0^\pi \int_0^{2\pi} \frac{\cos^2\left(\dfrac{\pi a}{\lambda} \cos\theta'\right)}{(\pi^2 - \beta^2 a^2 \cos^2\theta')^2} \sin^3\theta' \, d\theta' \, d\varphi'. \tag{58}$$

Integrating with respect to φ' and changing the form of the numerator, we obtain

$$P = \frac{\pi^2 a^2 b^2 E_0{}^2}{240\lambda^2} \int_0^\pi \frac{1 + \cos(\beta a \cos\theta')}{(\pi^2 - \beta^2 a^2 \cos^2\theta')^2} \sin^3\theta' \, d\theta'. \tag{59}$$

Introducing a new variable

$$t = \beta a \cos\theta', \tag{60}$$

we find

$$P = \frac{\pi a b^2 E_0{}^2}{240\lambda} \int_0^{\beta a} \frac{1 + \cos t}{(\pi^2 - t^2)^2} \left(1 - \frac{t^2}{\beta^2 a^2}\right) dt. \tag{61}$$

Integrating,

$$P = \frac{ab^2 E_0{}^2}{960\pi\lambda} \left\{ \left(1 - \frac{\lambda^2}{4a^2}\right) [\mathrm{Si}(\pi + \beta a) - \mathrm{Si}(\pi - \beta a)] + \frac{1}{\pi}\left(1 + \frac{\lambda^2}{4a^2}\right) [\mathrm{Cin}(\pi + \beta a) - \mathrm{Cin}(\pi - \beta a)] - \frac{2(1 + \cos\beta a)}{\beta a} \right\}. \tag{62}$$

To obtain the maximum radiation intensity, we let $\theta = 0$ in equation 56; thus,

$$\Phi_{\max} = \frac{a^2 b^2 E_0{}^2}{240\pi^3\lambda^2}. \tag{63}$$

Hence, for a narrow horn ($b \ll \lambda$) with zero flare angle,

$$g = \frac{4\pi\Phi_{\max}}{P} = \frac{16a}{\pi\lambda} \left\{ \left(1 - \frac{\lambda^2}{4a^2}\right) [\mathrm{Si}(\pi + \beta a) - \mathrm{Si}(\pi - \beta a)] + \frac{1}{\pi}\left(1 + \frac{\lambda^2}{4a^2}\right) [\mathrm{Cin}(\pi + \beta a) - \mathrm{Cin}(\pi - \beta a)] - \frac{2(1 + \cos\beta a)}{\beta a} \right\}^{-1}. \tag{64}$$

As $a/\lambda \to \infty$, we find

$$g \to \frac{16a}{\pi^2\lambda}. \tag{65}$$

To obtain the directivity for any comparatively small flare angle ψ we shall assume that the radiated power is not affected by the angle of

flare. The maximum radiation intensity, on the other hand, will be reduced by the destructive interference of elementary waves arriving from different points in the aperture of the horn. Thus we have the following approximate relationship:

$$g(\psi) = g(0)\,\frac{\Phi_{\max}(\psi)}{\Phi_{\max}(0)}\,. \tag{66}$$

If R is the radius of curvature of the horn, $R = a\psi$, and

$$\begin{aligned}\frac{\Phi_{\max}(\psi)}{\Phi_{\max}(0)} &= \left|\frac{2}{\psi}\int_0^{\psi/2} \exp\left(-j\,\frac{\pi a}{\psi\lambda}\,\theta^2\right) d\theta\right|^2 \\ &= \frac{2\lambda}{\psi a}\left[C^2\left(\sqrt{\frac{\psi a}{2\lambda}}\right) + S^2\left(\sqrt{\frac{\psi a}{2\lambda}}\right)\right].\end{aligned} \tag{67}$$

16.10 Dielectric waveguide antennas

A *dielectric waveguide antenna* is a finite section of a dielectric cylinder. For such a cylinder the cutoff frequency of the dominant mode (TE_{11}

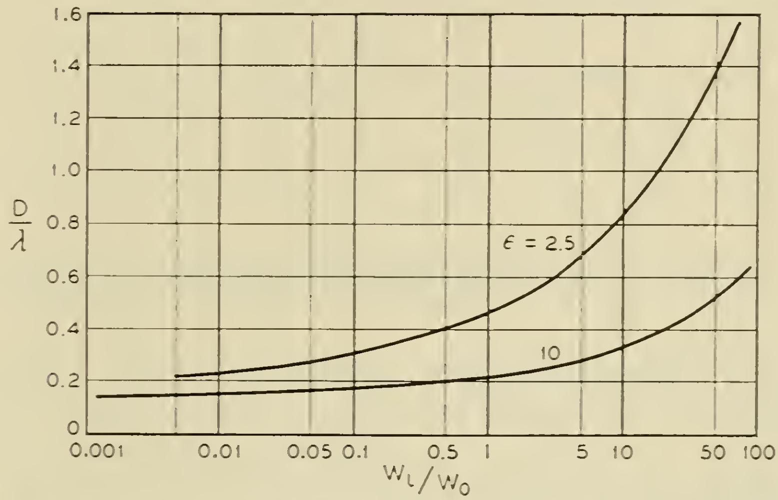

FIG. 16.7 The diameter in wavelengths as a function of the power ratio W_i/W_0, where W_i is the power inside the dielectric waveguide and W_0 is the power outside.

mode in a circular cylinder and TE_{10} in a rectangular cylinder) is zero.* However, at low frequencies most of the guided energy is outside the guide and is only loosely coupled to the guide. In a practical sense, this energy can hardly be called "guided," for it will escape its ties at any but the very slightest discontinuity (such as a bend). Figure 16.7 shows how the ratio of the power inside a circular dielectric wire to that

* *Electromagnetic Waves*, p. 428.

outside varies with the ratio of the diameter to the wavelength.* When most of the power is traveling outside the wire, the phase velocity of the wave is substantially equal to the free-space velocity. As this power is drawn inside the wire by increasing the diameter, the phase velocity approaches the value appropriate to the dielectric. This is illustrated in Fig. 16.8.

It is evident from these curves that there is a kind of critical frequency in the sense that for substantially lower frequencies most of the "guided" energy is outside the guide and for substantially higher frequencies most of it is inside the guide. Except when the relative dielectric constant is near unity, the transition region is narrow. This

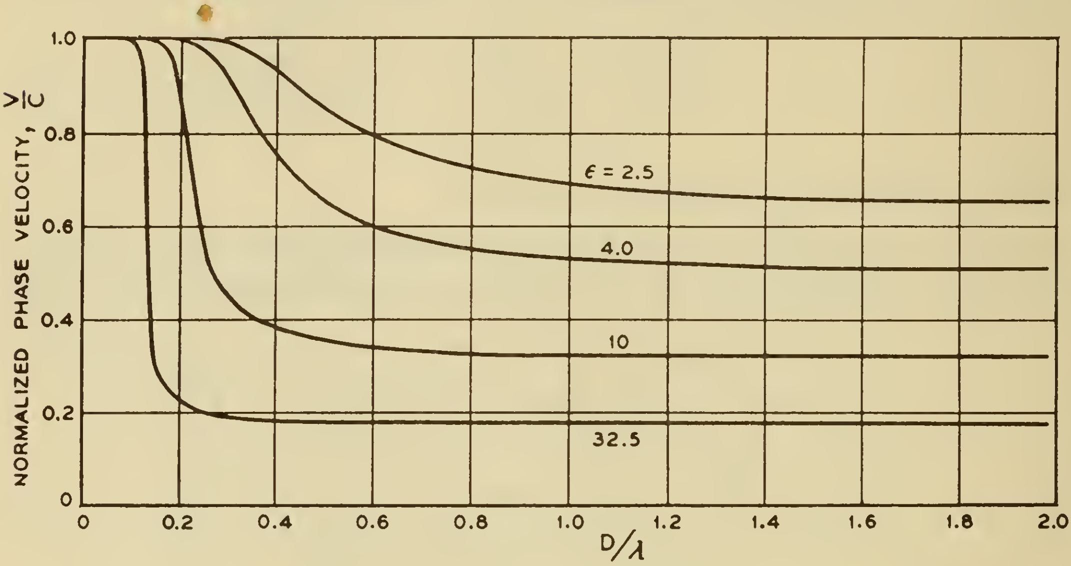

FIG. 16.8 The ratio of the phase velocity along the dielectric waveguide to the velocity of light as a function of the diameter in wavelengths.

property can be used to design dielectric waveguide antennas. The guide is tapered. At one end its diameter is so large that it acts truly as a waveguide. At this end we may insert a dipole antenna or any other coupling device to the source of power (or to the load). At the other end the diameter is so small that most of the guided energy is forced out of the guide, and the area of the wavefront is large. When the guide is terminated, this wavefront, acting as an array of Huygens sources, produces a sharply directive pattern. The principle of operation is very similar to that in an ordinary horn. The dielectric waveguide antenna is a "horn turned inside out" in the sense that the material boundary of the horn is in the interior of the guided wave.

Dielectric waveguide antennas may be conveniently used in broadside rectangular arrays where ordinary horns could not be employed

* For Figs. 16.7 and 16.8, the authors are indebted to Miss Marion C. Gray.

effectively on account of their material boundaries; for, in order to obtain a maximum gain from the space factor, the distance between the centers of horn apertures may have to be smaller than permitted by the rigid walls of the horns.

Another way of interpreting a dielectric waveguide antenna is to consider it as an end-fire array of doublets. Polarization currents radiate just as effectively as conduction currents. From this point of view, the taper in the guide is introduced in order to obtain the phasing proper to an end-fire array. The polarization current is the difference between the actual displacement current in the guide and that which would flow in free space in response to the same electric intensity. Polarization currents may be calculated for an infinitely long waveguide of uniform cross section. We may use them in approximate calculations of the performance of dielectric waveguide antennas.

For further details the reader is referred to a paper by G. E. Mueller and W. A. Tyrrell, Polyrod antennas, *Bell Sys. Tech. Jour.*, **26**, October 1947, pp. 837–851.

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17

SLOT ANTENNAS

Classification of antennas into mutually exclusive types is impossible, for we can look at the same antenna from different points of view. A very thin dipole antenna, for example, is a horn, even though this may not be immediately apparent. Thus, Fig. 4.8 shows an omnidirectional biconical horn; however, if the internal angle 2ψ of this horn is small, we have a thin conical dipole. The shape of a horn need not be conical; hence, a thin cylindrical wire is still "a horn." If the internal angle of the biconical horn is nearly equal to 180°, we have a biconical capacitor antenna. By a slight deformation in shape we shall obtain a capacitor antenna formed by two parallel disks. The narrow aperture of this antenna is also a slot in a metal pillbox. Hence, a capacitor antenna is both a horn and a "slot antenna." An open end of a narrow rectangular waveguide is a slot antenna, for it is a slot in a large metal surface. The outer surface of the waveguide is also the wall of a horn whose solid angle is nearly 4π. We can recognize linear antennas, horns, slots in metal surfaces or *slot antennas*, reflectors, lenses, etc., when we see them; but we are unable to subdivide them into mutually exclusive categories.

From the analytical point of view, this overlapping has its advantages, since it suggests different methods of treatment of similar antennas. Before proceeding with the subject of this chapter, we shall consider magnetic currents. We have used this concept in our analysis of horns; but we can use it to even greater advantage in the case of slot antennas.

17.1 Electric and magnetic currents; electromotive and magnetomotive forces

A stationary electric charge is surrounded by a field of force, and we are accustomed to regard the charge as the "cause" of the field. If the charge is moving, the field is different. It is natural to think that the

motion of the charge, that is, the electric current, is the cause of a field which is superimposed on the field of the stationary charge. But in order to move the charge, we have to exert a force (in addition to the force that may be needed to overcome the inertia of the body containing the charge); hence, we may consider this force as the primary cause of the field.

In dynamics the force acting on a particle is not always the best quantity to deal with. In rotational dynamics, for instance, torque is preferable to force. Convenient concepts in general dynamics are "generalized forces" and "generalized displacements," so chosen that their products give work. Thus, in rectilinear dynamics, the product of the force and the linear displacement gives work; in rotational dynamics, the product of the torque and the angular displacement gives work. The *generalized force* is defined as the work per unit generalized displacement. In studying electric phenomena in an electric circuit we are not concerned with the length of the circuit. It is the charge displaced round the circuit and its time rate of change, the electric current, that claim our attention. Hence, we choose the charge as the generalized displacement and define the corresponding generalized force as the work per unit charge. This generalized force is called the electromotive force or voltage, as an abbreviation. In a continuously distributed field, electric intensity is introduced to represent voltage per unit length. In addition to electric current, defined as moving charge, Maxwell introduced electric displacement current defined as time rate of change of electric displacement (or electric flux).

In most respects magnetic fields are similar to electric fields. There are, however, no magnetic charges and no magnetic currents in the sense of moving charges. There exist only magnetic displacement currents representing time rates of change of magnetic displacement (or magnetic flux) analogous to Maxwellian electric displacement currents. According to Maxwell's equations, the physical dimensions of magnetic current are identical with those of electromotive force. The magnetomotive force is a generalized force so defined that the product of the force and the corresponding magnetic displacement gives work. The physical dimensions of magnetomotive force are those of electric current. Thus, there exists a dual relationship between electric and magnetic quantities which enables us to replace a set of electric quantities by an equivalent magnetic set and vice versa. This duality is a direct consequence of Maxwell's equations. It enables us to obtain solutions of certain problems directly from already known solutions of other problems. Solutions for capacitor antennas and annular slots in conducting planes may be obtained at once from the solutions for

electric current loops. In general, slot antennas are duals of ordinary linear antennas. In some cases we shall find it more convenient to treat slots as such; in other cases we shall replace them by magnetic linear antennas and use the already available information about their electric counterparts.

In electric circuit theory we use two kinds of ideal generators: (1) series generators of zero internal impedance, across which the current is continuous, and which introduce into the circuit a fixed (that is, independent of the circuit) voltage discontinuity; and (2) shunt

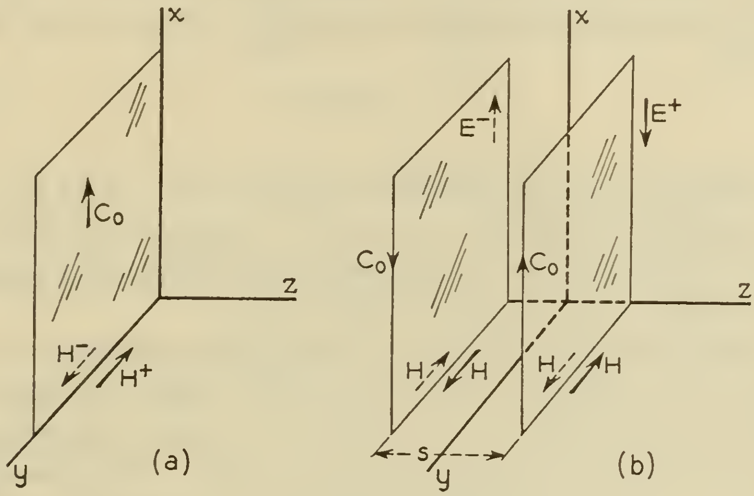


FIG. 17.1 (*a*) An electric-current sheet, and (*b*) a double electric current sheet or simple magnetic current sheet.

generators of infinite impedance, across which the voltage is continuous, and which introduce into the circuit a fixed current discontinuity. There is no loss in generality when we use such ideal generators, since a finite internal impedance in the first case and internal admittance in the second may always be considered as elements of the circuit. In distributed fields, discontinuities in E and H are introduced, respectively, by magnetic and electric current sheets. Thus, if we imagine an infinite electric current sheet in the xy plane (Fig. 17.1*a*), and if the linear density of the current is

$$C_x = C_0, \tag{1}$$

then, by the Ampère-Maxwell law, we have

$$H_y^+ - H_y^- = -C_0. \tag{2}$$

If the medium outside the current sheet is homogeneous,

$$H_y^- = -H_y^+, \qquad H_y^+ = -\tfrac{1}{2}C_0. \tag{3}$$

In other cases, the magnetic intensities on the two sides of the sheet are not equal and opposite, but the difference is still C_0.

Consider now two parallel electric current sheets (Fig. 17.1*b*), and let the distance s between them be infinitesimal. In passing across this double current sheet the magnetic intensity is seen to be continuous; but the electric intensity is discontinuous. By the Faraday-Maxwell law,

$$E_x^+ - E_x^- = -C_y^m, \tag{4}$$

where C_y^m is the linear density of the magnetic current, which may be expressed in terms of the magnetic intensity H_y^{in} between the electric current sheets,

$$C_y^m = j\omega\mu s H_y^{\text{in}}. \tag{5}$$

Since s is infinitesimal, H_y^{in} must be infinite if C_y^m is to be finite. In any case, only the product of j, ω, μ, s, H_y^{in} determines the discontinuity in the electric intensity and not the separate factors; this product gives the linear density of the *magnetic current sheet.*

The return path for the current flowing in an electric current sheet is formed by the media on both sides of the sheet. These media are electrically in parallel since E is continuous across the sheet and the current in the sheet is divided between the media. The electric current sheet acts as a shunt generator of infinite impedance. Similarly, a magnetic current sheet acts as a series generator of zero internal impedance, in that it impresses an electric intensity in such a way that the media on the two sides of the sheet are electrically in series.

17.2 Magnetic current elements

A *magnetic current element* is an infinitely thin solenoid of length s. The magnetic moment of the element is Vs, where V is the magnetic current through the solenoid.* The field of this element may be obtained by the method used in Section 4.13 for the electric current element. Instead of electric charges q, $-q$ at the ends of the electric current element, we have the magnetic displacements Φ, $-\Phi$ at the ends of the magnetic element. Instead of equations 4–67 for the electric field in the immediate vicinity of an electric current element, we now have analogous equations for the magnetic field in the vicinity of the

* It should be noted that, when the cross section of the solenoid is finite, only half of the magnetic flux passing through the central cross section emerges or enters through its ends; the rest leaks out. Most of the leakage is near the ends, in a region comparable to a few diameters; hence, the ends of an *infinitely thin* solenoid are true point sources.

magnetic current element,

$$H_r = \frac{\Phi s \cos\theta}{2\pi\mu r^3}, \qquad H_\theta = \frac{\Phi s \sin\theta}{4\pi\mu r^3}, \qquad H_\varphi = 0. \tag{6}$$

The complete expressions at any distance should be of the same form as the corresponding expressions for E_r and E_θ in equations 4–81 except that Is should be replaced by a constant of integration A which is then determined by comparing with equations 6. Thus,

$$H_r = \frac{\eta A}{2\pi r^2}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r} \cos\theta. \tag{7}$$

As $\sigma r \to 0$,

$$H_r \to \frac{\eta A}{2\pi\sigma r^3} \cos\theta. \tag{8}$$

Comparing with equations 6, we find

$$\frac{\eta A}{\sigma} = \frac{\Phi s}{\mu}, \qquad A = \frac{\sigma \Phi s}{\mu\eta}. \tag{9}$$

Since $j\omega\Phi = V$,

$$A = \frac{\sigma V s}{j\omega\mu\eta} = \frac{V s}{\eta^2}. \tag{10}$$

Thus, by substituting $Is = A = Vs/\eta^2$ in the expressions 4–81 for E_θ and E_r, we shall obtain H_θ and H_r for the present case; E_φ may then be obtained from equation 4–11 or from H_φ after making a comparison between equations 4–8 and 4–11. The latter shows that we should reverse the algebraic sign of H_φ and replace Is by Vs: the factor $1/\eta^2$ in the preceding transformation is canceled by another factor η^2 arising out of the right-hand side terms in equations 4–8 and 4–11. Making the required changes in equations 4–81, we obtain the field of a magnetic current element of moment Vs meter-volts,

$$\begin{aligned}
H_\theta &= \frac{(g + j\omega\varepsilon)Vs}{4\pi r}\left(1 + \frac{1}{\sigma r} + \frac{1}{\sigma^2 r^2}\right) e^{-\sigma r} \sin\theta, \\
E_\varphi &= -\frac{\sigma V s}{4\pi r}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r} \sin\theta, \qquad (11) \\
H_r &= \frac{Vs}{2\pi\eta r^2}\left(1 + \frac{1}{\sigma r}\right) e^{-\sigma r} \cos\theta.
\end{aligned}$$

In nondissipative media,

$$H_\theta = \frac{j\omega\varepsilon Vs}{4\pi r}\left(1 + \frac{1}{j\beta r} - \frac{1}{\beta^2 r^2}\right) e^{-j\beta r} \sin\theta,$$

$$E_\varphi = -\frac{j\beta Vs}{4\pi r}\left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \sin\theta, \tag{12}$$

$$H_r = \frac{Vs}{2\pi\eta r^2}\left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \cos\theta.$$

At great distances,

$$H_\theta = \frac{j\omega\varepsilon Vs}{4\pi r} e^{-j\beta r} \sin\theta, \qquad E_\varphi = -\eta H_\theta. \tag{13}$$

17.3 Radiation from magnetic currents

The method of Section 12.1 for obtaining the distant field and the radiation intensity of a given electric current distribution may be extended to include the magnetic currents. The radiation vector L associated with a magnetic current element of moment $p = Vs$ at the point (r', θ', φ') with respect to $(0, 0, 0)$ is defined by an equation identical with equation 12–5,

$$L = pe^{j\beta r' \cos\psi}. \tag{14}$$

The rules for the evaluation of the radiation vector for any system of elements are the same as those given by equations 12–7, 12–8, and 12–9. But the equations for obtaining the distant field and the radiation intensity involve a different factor. Instead of equations 12–12, we now have, from equations 13,

$$\begin{aligned} H_\theta &= -j\frac{\omega\varepsilon}{4\pi} L_\theta \frac{e^{-j\beta r_0}}{r_0} = -\frac{j}{240\pi\lambda} L_\theta \frac{e^{-j\beta r_0}}{r_0}, \\ H_\varphi &= -j\frac{\omega\varepsilon}{4\pi} L_\varphi \frac{e^{-j\beta r_0}}{r_0} = -\frac{j}{240\pi\lambda} L_\varphi \frac{e^{-j\beta r_0}}{r_0}. \end{aligned} \tag{15}$$

Substituting in equation 5–10, we have the radiation intensity

$$\Phi = \frac{1}{960\pi\lambda^2}\,(L_\theta L_\theta{}^* + L_\varphi L_\varphi{}^*). \tag{16}$$

This differs from equation 12–10 by a factor $1/\eta^2 = 1/(120\pi)^2$.

In the case of mixed electric and magnetic current distributions, we have mutual radiation. To obtain the most general radiation formula it is best to obtain E_θ and E_φ from equations 15 and add them

to equations 12–12 before using equation 5–10. Thus,

$$E_\theta = \left(-j\,\frac{60\pi}{\lambda}\,N_\theta - \frac{j}{2\lambda}\,L_\varphi\right)\frac{e^{-j\beta r_0}}{r_0},$$
$$E_\varphi = \left(-j\,\frac{60\pi}{\lambda}\,N_\varphi + \frac{j}{2\lambda}\,L_\theta\right)\frac{e^{-j\beta r_0}}{r_0}\cdot \tag{17}$$

Hence,

$$\Phi = \Phi_{11} + 2\Phi_{12} + \Phi_{22}, \tag{18}$$

where Φ_{11} is given by equation 12–10, Φ_{22} by equation 16, and

$$\Phi_{12} = \frac{1}{8\lambda^2}\,\mathrm{re}(N_\theta L_\varphi{}^* - N_\varphi L_\theta{}^*). \tag{19}$$

We have seen that the expressions 12–10 and 16 for the radiation intensity differ by a factor $1/\eta^2$; that is, *if we have two similar distributions of electric and magnetic currents and if the strengths of these distributions when expressed in mks units are equal, the ratio of the power radiated by the electric current distribution to that radiated by the magnetic current distribution is* η^2.

17.4 Uniform magnetic current filaments and uniformly energized slots

A long thin solenoid carrying uniform current is a magnetic current filament. If the filament is infinitely long, the field is particularly simple. Let us assume that the filament is along the z axis; then the only nonvanishing component of H is H_z, and it is a function of ρ only. The electric field is also uniform, and its only nonvanishing component is E_φ. Hence, the field equations are the same as for TEM waves in a dihedral horn (equations 16–46). The only difference is that in the horn the field exists only in the interior region, the walls of the horn effectively shielding the exterior. If the parallel planes in the transmission line feeding the horn are close together, and if the horn angle is nearly equal to 2π (Fig. 17.2*a*), the field is identical with that of the magnetic current filament everywhere except in a small region between the planes. If the angle of the horn is equal to π as in Fig. 17.2*b*, the field inside the horn equals that of a magnetic filament in front of a perfectly conducting plane; for the same voltage across the slot E is twice as large as in the case of a 360° horn.

Letting $b = \frac{1}{2}s$ in equation 16–54, we obtain the admittance of the horn in Fig. 17.2*a* as seen from the slot,

$$Y_i \simeq \frac{w}{240\lambda}\left[1 + j\,\frac{2}{\pi}\left(\log\frac{\lambda}{s} - \log\pi + 0.116\right)\right]\cdot \tag{20}$$

The radiated power is thus

$$P = \frac{w}{480\lambda} VV^*, \tag{21}$$

where V is the voltage across the slot.

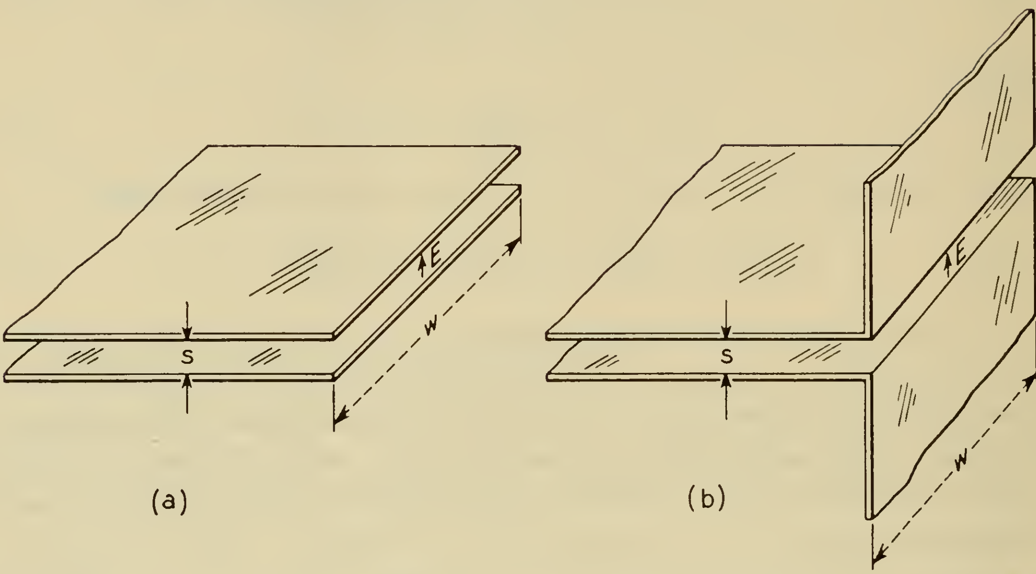


FIG. 17.2 Slot antennas: (*a*) a 360° dihedral horn, (*b*) a 180° dihedral horn.

The characteristic admittance of the strip transmission line feeding the horn is

$$A = \frac{w}{\eta s} = \frac{w}{120\pi s} \cdot \tag{22}$$

Hence, the voltage reflection coefficient at the slot is

$$q = \frac{A - Y_i}{A + Y_i} = \frac{1 - \dfrac{\pi s}{2\lambda} - j\dfrac{s}{\lambda}\left(\log\dfrac{\lambda}{s} - \log\pi + 0.116\right)}{1 + \dfrac{\pi s}{2\lambda} + j\dfrac{s}{\lambda}\left(\log\dfrac{\lambda}{s} - \log\pi + 0.116\right)} \cdot \tag{23}$$

For small values of s/λ, the reflection coefficient is almost unity.

The admittance of the slot in a plane (Fig. 17.2*b*) is twice as great as that given by equation 20.

For a slot of finite length w (Fig. 17.2*a*), the maximum radiation intensity is found from equation 16,

$$\Phi_{\max} = \frac{1}{960\pi\lambda^2} VwV^*w = \frac{w^2}{960\pi\lambda^2} VV^*. \tag{24}$$

Hence, we have the directivity of a long slot,

$$g = \frac{4\pi\Phi_{\max}}{P} = \frac{2w}{\lambda}. \tag{25}$$

As it should, this directivity equals that of a long electric current filament. The radiation patterns are also identical. The only difference is in polarization.

The length a of the feed line (Fig. 17.3) does not affect either the radiation pattern or the directivity. The circular lines of force normal to the external surfaces of the parallel conducting sheets simply become continuous circles round the guide as their radius increases. We shall consider this structure in more detail in connection with slotted waveguides.

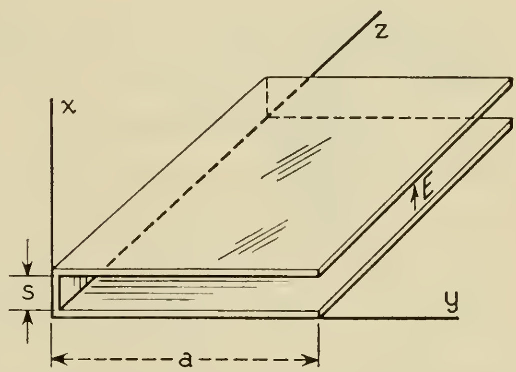


FIG. 17.3 A slotted waveguide.

Equation 25 for g was derived on the assumption that the voltage distribution along the slot is uniform. If the distribution is sinusoidal, the radiated power will be only half of equation 21, assuming that V is the maximum amplitude of the voltage. The radiation vector is only $2/\pi$ times as great as the former radiation vector; hence, the radiation intensity is $4/\pi^2$ times as great as before. The directivity, therefore, is multiplied by $8/\pi^2$. This gives equation 16–65.

17.5 Capacitor antennas, circular slots in conducting sheets

Figure 17.4 shows capacitor antennas of radius a in free space and above a perfectly conducting plane. In free space the antenna is equivalent to a magnetic current loop of radius a. Since we have already considered an electric current loop (Section 15.12), we have little to add. The radiation patterns are the same, but the polarizations of distant fields are opposite (with respect to the plane of the loop). In terms of the voltage V across the edges of the capacitor antenna (and, hence, the magnetic current in the equivalent loop), the radiation intensity is

$$\Phi = \frac{\pi a^2 V^2}{240\lambda^2} J_1{}^2(\beta a \sin\theta). \tag{26}$$

This is obtained from equation 15–45 by replacing I by V and dividing by $\eta^2 = (120\pi)^2$. Equation 15–49 for the directivity remains the same.

The case of a circular plate above a conducting plane is essentially the same as above. We can reduce it to the free-space case by introducing an image plate and removing the conducting plane. Conversely, we can insert a perfectly conducting plane halfway between the plates of the capacitor antenna in free space.

A narrow circular slot in an infinite conducting plane (Fig. 17.5) is essentially equivalent to the capacitor antenna above such a plane

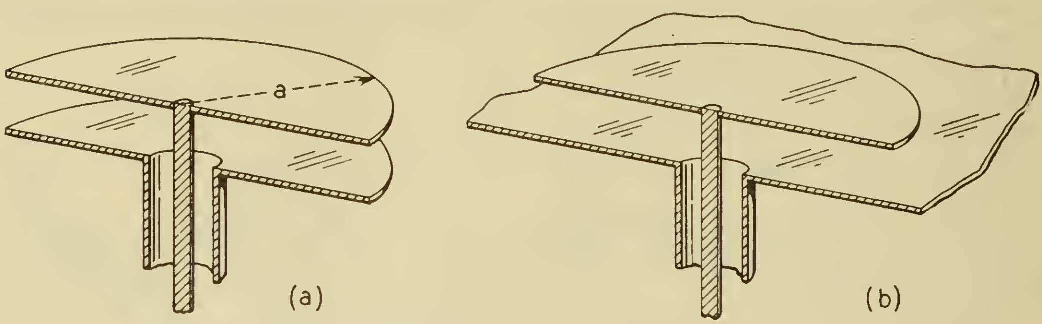


Fig. 17.4 Capacitor antennas.

(Fig. 17.4*b*). For a direct treatment of this type of slot antenna, not using the dual properties of electromagnetic waves, the reader is referred elsewhere.*

The conductance seen by the capacitor antenna (or the circular slot) is obtained from the radiated power. The susceptance can also

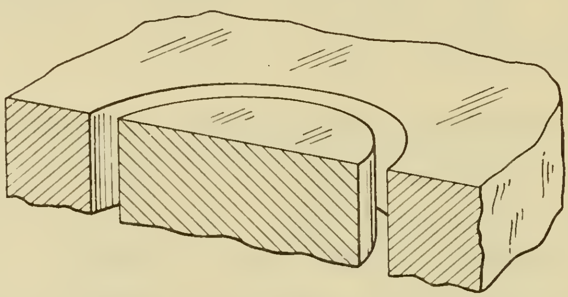


Fig. 17.5 A circular slot antenna fed by a coaxial transmission line.

be obtained very simply when the radius a is small or when it is large. In the first case, we have a capacitance between the external surfaces of the plates; in the latter case, we take the susceptance of a straight slot (Fig. 17.2*a*), of length $w = 2\pi a$. In the intermediate case, the calculations are not quite so simple; however, they are still analogous to the calculations of the reactance of an electric current loop.

The calculation of the admittance looking backward from the

* A. A. Pistolkors, Theory of the circular diffraction antenna, *IRE Proc.*, **36**, January 1948, pp. 56–60.

slot into the feed line is straightforward. In the case of the circular slot antenna (Fig. 17.5), this admittance equals the admittance of the coaxial line in parallel with a small susceptance associated with the edge effect, with a somewhat greater charge density near the edges. In the case of capacitor antennas, we have a disk transmission line terminated into a coaxial line, and the series inductance at the junction of the two lines is easily calculated.

17.6 Slotted waveguides

A section of a simple slotted waveguide is shown in Fig. 17.3. The condition for guided waves is the continuity of E and H across the slot. Hence, the voltage across the slot must be continuous. Likewise, the current flowing toward the edge on the internal surface of the waveguide must equal the current flowing away from the edge on the external surface. Expressing it differently, the average values of E and H must be continuous if E and H are continuous. If the height s of the waveguide is small compared with the width a and wavelength λ, the departure of the field from a uniform distribution is confined to the neighborhood of the slot, and we shall neglect it. When it seems desirable, we can evaluate this departure approximately, and represent it by a small lumped susceptance across the slot. Here, however, we shall not complicate our calculations by such refinements, and we shall assume that inside the guide the field is uniform in the direction of the x axis (parallel to the short side of the cross section). We have to consider three field components: E_x, H_y, and H_z. Maxwell's equations reduce to

$$H_y = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z},$$
$$H_z = \frac{1}{j\omega\mu}\frac{\partial E_x}{\partial y}, \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x. \tag{27}$$

Eliminating H_y and H_z,

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\beta^2 E_x. \tag{28}$$

For guided waves E_x is proportional to $\exp(-\Gamma z)$ where Γ is the propagation constant along the guide. Hence, the appropriate solution of this equation is

$$E_x = E_0 \sin(\sqrt{\Gamma^2 + \beta^2}\, y)e^{-\Gamma z}. \tag{29}$$

The cosine term is omitted because E_x should vanish at the perfectly conducting face $y = 0$. The voltage across the slot (in the positive x direction) is

$$V = E_0 s \sin(\sqrt{\Gamma^2 + \beta^2}\, a)e^{-\Gamma z}. \tag{30}$$

The current flowing toward the edges of the slot is determined by H_z. From equations 27 and 29, we have,

$$H_z = \frac{E_0\sqrt{\Gamma^2 + \beta^2}}{j\omega\mu} \cos\left(\sqrt{\Gamma^2 + \beta^2}\, y\right)e^{-\Gamma z}. \tag{31}$$

In the slot,

$$H_z = \frac{E_0\sqrt{\Gamma^2 + \beta^2}}{j\omega\mu} \cos\left(\sqrt{\Gamma^2 + \beta^2}\, a\right)e^{-\Gamma z}. \tag{32}$$

Except for a distortion near the slot, the field outside the guide is given by E_φ, H_ρ, and H_z. Furthermore, E_φ is independent of φ. Maxwell's equations reduce to

$$H_\rho = \frac{1}{j\omega\mu}\frac{\partial E_\varphi}{\partial z}, \quad H_z = -\frac{1}{j\omega\mu\rho}\frac{\partial}{\partial\rho}(\rho E_\varphi), \quad \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = j\omega\varepsilon E_\varphi. \tag{33}$$

For guided waves the field is proportional to* $\exp(-\Gamma z)$; hence equations 33 become

$$H_\rho = -\frac{\Gamma}{j\omega\mu}E_\varphi, \quad H_z = -\frac{1}{j\omega\mu\rho}\frac{\partial}{\partial\rho}(\rho E_\varphi), \quad -\Gamma H_\rho - \frac{\partial H_z}{\partial \rho} = j\omega\varepsilon E_\varphi. \tag{34}$$

Eliminating H_ρ, we have

$$\frac{\partial H_z}{\partial\rho} = -j\omega\varepsilon\left(1 + \frac{\Gamma^2}{\beta^2}\right)E_\varphi, \qquad \frac{\partial}{\partial\rho}(\rho E_\varphi) = -j\omega\mu\rho H_z. \tag{35}$$

These equations are obtained from equations 16–46 if we replace ε by $\varepsilon[1 + (\Gamma^2/\beta^2)]$. Therefore, in view of equations 16–53, we obtain

$$H_z = \frac{\beta^2 + \Gamma^2}{4\beta\eta} V_1[J_0(\sqrt{\Gamma^2 + \beta^2}\rho) - j\, N_0(\sqrt{\Gamma^2 + \beta^2}\rho)]e^{-\Gamma z}, \tag{36}$$

where V_1 is the voltage in the positive φ direction, that is, from the upper face to the lower.

For small ρ,

$$H_z = \frac{\beta^2 + \Gamma^2}{4\beta\eta} V_1 \left\{1 - \frac{2j}{\pi}\left[\log(\sqrt{\Gamma^2 + \beta^2}\rho) + 0.577 - \log 2\right]\right\} e^{-\Gamma z}. \tag{37}$$

To match the internal and external values of H_z at the edges of the slot, we should substitute $\rho = s/2$ in equation 37 and equate it to 32. A better approximation is obtained if we note that each linear element of

* The continuity along the slot requires that the propagation constants in the direction of the slot should be the same inside and outside the guide.

the slot is a source of external cylindrical waves. To obtain H_z at a typical point x in the slot, we should replace V by $V\,dx'/s$ and ρ by $|x - x'|$ and integrate from $x' = 0$ to $x' = s$. Then, to obtain the average value of H_z, we integrate with respect to x from $x = 0$ to $x = s$. Since

$$\frac{1}{s^2}\int_0^s dx \int_0^s \log|x - x'|\,dx' = \log s - 1.5, \tag{38}$$

we have the average H_z,

$$H_{z,\text{av}} = \frac{\Gamma^2 + \beta^2}{4\beta\eta} V_1 \left\{1 - \frac{2j}{\pi}\left[\log(\sqrt{\Gamma^2 + \beta^2}\, s) - 1.616\right]\right\} e^{-\Gamma z}. \tag{39}$$

Equating to 32, and noting that $V_1 = -V$, we obtain an equation for the propagation constant along the guide,

$$\frac{\sqrt{\Gamma^2 + \beta^2}}{j\omega\mu s}\cot(\sqrt{\Gamma^2 + \beta^2}\, a)$$
$$= -\frac{\Gamma^2 + \beta^2}{4\beta\eta}\left\{1 - \frac{2j}{\pi}\left[\log(\sqrt{\Gamma^2 + \beta^2}\, s) - 1.616\right]\right\} \tag{40}$$

or

$$\cot(\sqrt{\Gamma^2 + \beta^2}\, a)$$
$$= -\tfrac{1}{4}j\sqrt{\Gamma^2 + \beta^2}\, s \left\{1 - \frac{2j}{\pi}\left[\log(\sqrt{\Gamma^2 + \beta^2}\, s) - 1.616\right]\right\}. \tag{41}$$

In the first approximation (for the first mode),

$$\sqrt{\Gamma^2 + \beta^2}\, a = \frac{\pi}{2}, \quad \Gamma = \sqrt{\frac{\pi^2}{4a^2} - \frac{4\pi^2}{\lambda^2}} = \frac{2\pi j}{\lambda}\sqrt{1 - \frac{\lambda^2}{16a^2}}. \tag{42}$$

To this order of approximation the cutoff wavelength is

$$\lambda_c = 4a, \tag{43}$$

and there is no radiation. To obtain the next approximation we write

$$\sqrt{\Gamma^2 + \beta^2}\, a = \frac{\pi}{2} + \delta, \tag{44}$$

so that

$$\Gamma = \sqrt{-\beta^2 + \frac{1}{a^2}\left(\frac{\pi}{2} + \delta\right)^2}. \tag{45}$$

Since δ is small, we shall neglect its square, when equation 45 becomes

$$\Gamma = \sqrt{\frac{\pi^2}{4a^2} - \frac{4\pi^2}{\lambda^2} + \frac{\pi\delta}{a^2}}. \tag{46}$$

To obtain δ we substitute from equation 44 in equation 41,

$$\tan\delta = j\frac{\pi s}{8a}\left(1 + \frac{2}{\pi}\delta\right)\left[1 - \frac{2j}{\pi}\left(\log\frac{\pi s}{2a}\left(1 + \frac{2}{\pi}\delta\right) - 1.616\right)\right]. \tag{47}$$

In the first approximation,

$$\delta = j\frac{\pi s}{8a} - \frac{s}{4a}\left(\log\frac{2a}{\pi s} + 1.616\right). \tag{48}$$

Sufficiently above the cutoff,

$$\Gamma = \frac{2\pi j}{\lambda}\sqrt{1 - \frac{\lambda^2}{16a^2}}\sqrt{1 - \frac{\delta\lambda^2}{4\pi a^2}\left(1 - \frac{\lambda^2}{16a^2}\right)^{-1}}$$

$$\simeq \frac{2\pi j}{\lambda}\sqrt{1 - \frac{\lambda^2}{16a^2}} - \frac{j\delta\lambda}{4a^2}\left(1 - \frac{\lambda^2}{16a^2}\right)^{-\frac{1}{2}}; \tag{49}$$

that is,

$$\Gamma = \frac{\pi s\lambda}{32a^3}\left(1 - \frac{\lambda^2}{16a^2}\right)^{-\frac{1}{2}} +$$

$$\frac{2\pi j}{\lambda}\left[\sqrt{1 - \frac{\lambda^2}{16a^2}} + \frac{s\lambda^2}{32\pi a^3}\left(\log\frac{2a}{\pi s} + 1.616\right)\left(1 - \frac{\lambda^2}{16a^2}\right)^{-\frac{1}{2}}\right]. \tag{50}$$

Thus, the escape of energy from the internal part of the guide into free space causes a slight change in the wave velocity and a certain amount of attenuation.

At the cutoff (in the vicinity of $\lambda = 4a$) the wave velocity is very small, and the voltage is distributed almost uniformly along the slot.

Cylindrical slotted waveguides are shown in Fig. 17.6. When the angle ψ between the radial planes is fairly small, the guide in Fig. 17.6*b* may be analyzed accurately. As in the preceding case, the voltage across the gaps between *AC* and *BD*, and between *MP* and *NQ*, and the average H_z should be continuous. Hence, the sum of the admittances looking in opposite directions from either gap must equal zero. This gives an equation for the propagation constant along the guide. The radial planes *ACPM* and *BDQN* form a wedge transmission line (of angle ψ) terminated on one side by the admittance looking outward across the slot *MPQN* and on the other side by the admittance looking inward across the slot *ABDC*. The latter is the admittance of the wedge transmission line of angle $2\pi - \psi$ which is short-circuited by the cylinder at distance a from the axis. The former admittance may be obtained by expanding E_φ across the gap *MNQP* in a Fourier series,

and matching it to the solution of Maxwell's equations appropriate to the region external to the cylinder. For small ψ, the higher-order waves in the wedge of angle ψ are highly attenuated, and the energy

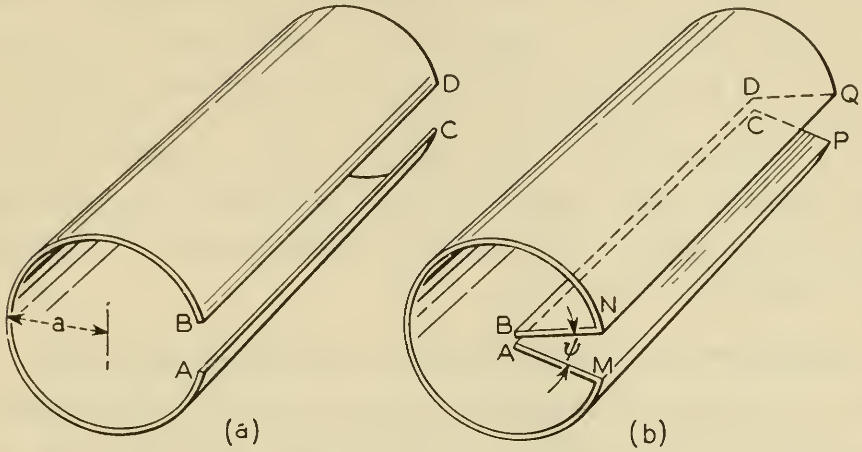


FIG. 17.6 Slotted waveguides.

content of these waves is small; hence, in solving our problem, we may assume that E_φ in the slot is independent of φ.

The case of large ψ, let us say $\psi = \pi$ as in Fig. 17.7, may also be treated rigorously, although not so simply as the case of small ψ. The method of treatment is analogous to the method of treatment of biconical antennas.* We imagine the "boundary cylinder" of radius a equal to the radius of the guide. A part of this cylindrical surface is a perfect conductor where E_φ vanishes. Across the complementary part, E_φ and H_z must be continuous. Outside the boundary cylinder the field may be expressed in cylindrical coordinates. The region inside the boundary cylinder is separated by the radial conducting planes into two subregions, in each of which the field may be expressed in cylindrical coordinates. The fields at the junctions between different regions can then be matched. In an approximate solution for large ψ, we can assume that the impedance seen outward from the slot equals the impedance of an infinite wedge transmission line; adding

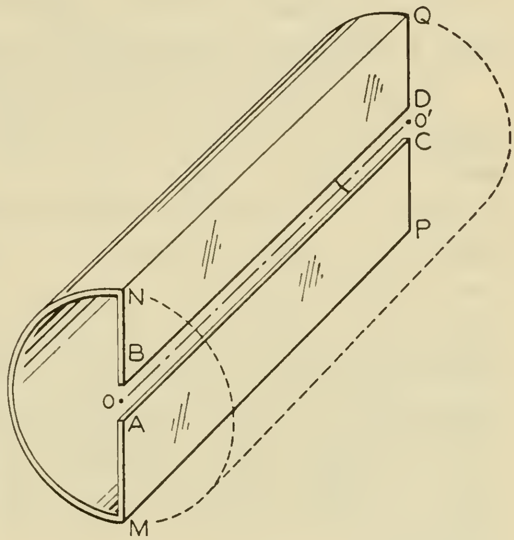


FIG. 17.7 A slotted waveguide.

* *Electromagnetic Waves*, Chapter 11.

this impedance to that looking inward and equating the sum to zero, we obtain the equation for the propagation constant along the slot.

The case shown in Fig. 17.6*a* is similar but simpler. We have two regions in which the field may be expressed in cylindrical coordinates. The two fields may then be matched "on the average" across the narrow slot. In a more refined solution we may assume that the electric field in the slot is distributed as the static field is distributed across a slot in an infinite plane. This assumption is justified because cylindrical wave functions behave as static functions for small values of ρ.

Slotted waveguides may be used to form magnetic V antennas and magnetic rhombic antennas.

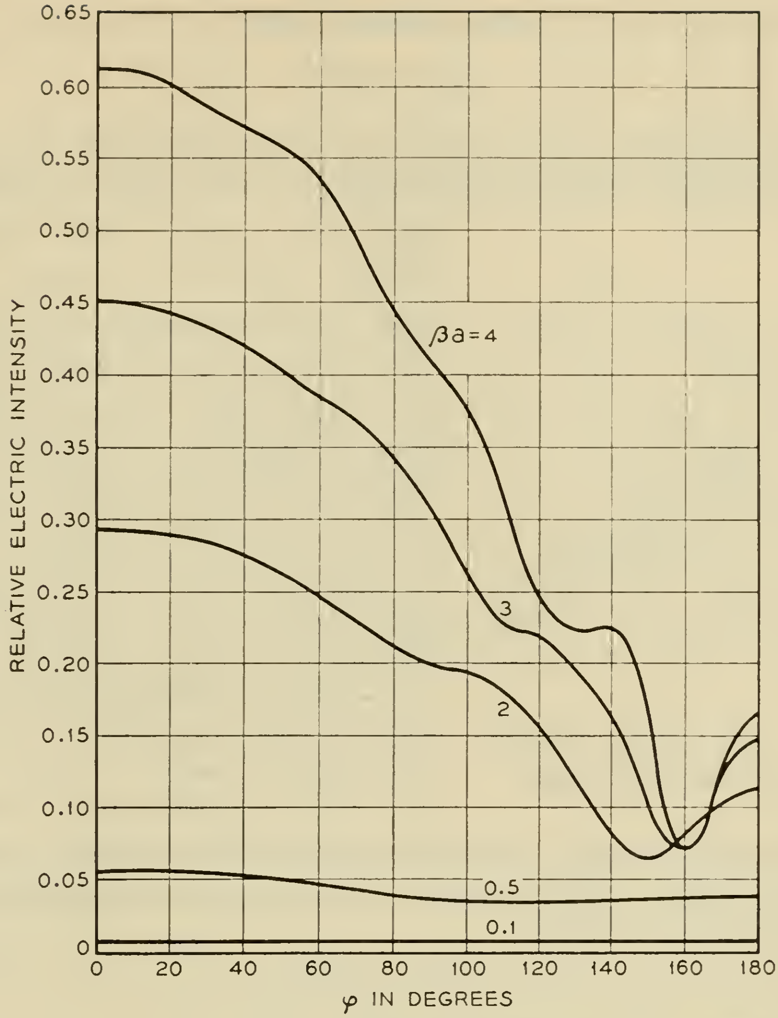
17.7 Radiation patterns of slotted waveguide antennas

A section of a slotted waveguide of length l will act as an antenna. Its radiation pattern is the product of two factors: one depending on z and the other on φ. The factor depending on z is exactly the same as in the case of an electric current filament of the same length, if the current is proportional to the voltage along the slot. The factor depending on φ depends on the shape of the guide. For rectangular guides (Fig. 17.3) this factor is substantially independent of φ; the electric lines starting at the slot are approximately circles cutting the upper and lower faces of the guide, and, as they progress backward, they easily slide off the guide and become continuous expanding circles.

In the case of cylindrical slotted waveguides, the radiation pattern depends on the radius of the guide. When $2\pi a/\lambda \ll 1$, the pattern is nearly circular; when $2\pi a/\lambda \gg 1$, the pattern is approximately a cardioid. A number of representative polar radiation patterns are exhibited in a paper by Sinclair.* Figure 17.8 shows rectangular plots of the radiation patterns of slotted cylinders of various radii for a slot angle of 0.1 radian as calculated by Papas and King.† The increase in the maximum values at $\varphi = 0$ with increasing radius is due to an assumption that the voltage across the slot is proportional to the radius. If the voltage is kept constant, the maximum radiation intensity is constant for large values of a; but the back radiation decreases as a increases.

* George Sinclair, The patterns of slotted-cylinder antennas, *IRE Proc.*, **36**, December 1948, pp. 1487–1492.

† Charles H. Papas and Ronold King, Currents on the surface of an infinite cylinder excited by an axial slot, *Quart. Appl. Math.*, 7, July 1949, pp. 175–182.



Papas and King, Courtesy Quart. Appl. Math.

FIG. 17.8 Radiation patterns of slotted cylinders.

17.8 Magnetic dipole antennas

Slots in conducting planes may be energized by balanced transmission lines (Fig. 17.9). Such slots are counterparts of dipole antennas fed by series generators, and thus may be called *magnetic dipole antennas.* Across the feed point of the electric dipole antenna the current is continuous; the voltage is discontinuous because of an impressed electromotive force. Across the feed point of the magnetic dipole antenna the voltage is continuous; the current is discontinuous because of an impressed current, that is, an impressed magnetomotive force.

To obtain the impedance of a magnetic dipole antenna, we shall first consider the electric currents in the conducting plane. These currents may be considered as flowing on the two faces of the plane. The two faces are electrically in parallel; hence, the shunt admittance of

the plane is twice the admittance of one face,

$$Y_p^{\mathrm{sh}} = 2Y_f. \tag{51}$$

The magnetic lines of force produced by currents in the plane are normal to the slot. Hence, theoretically, we can place a perfect magnetic strip over the slot without disturbing the field, provided that we feed the slot symmetrically from the two faces and supply half the current from each side. The magnetic intensity tangential to the slot is zero before the strip is introduced, and its value is unaltered by the strip. The tangential electric intensity is continuous across the slot and across the strip; hence, there are no magnetic currents in the strip. Since no energy can pass either through a perfect electric screen or through a perfect magnetic screen, the entire space has been divided into two independent halves. Let us now reverse the direction of the feed current to one face. The two faces will then be in series, and the series admittance of the plane is

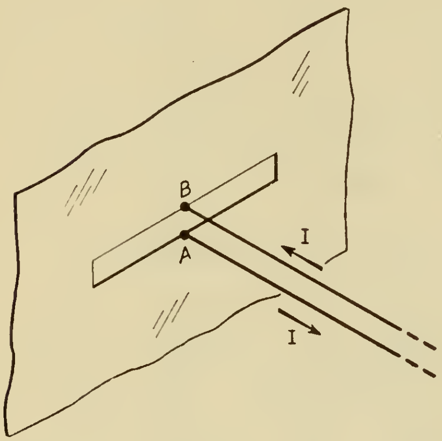


Fig. 17.9 A slot antenna fed by a balanced transmission line.

$$Y_p^{\mathrm{se}} = \tfrac{1}{2}Y_f. \tag{52}$$

Hence,

$$Y_p^{\mathrm{sh}} = 4Y_p^{\mathrm{se}}. \tag{53}$$

In the new arrangement the magnetic intensity tangential to the strip is still zero. The reversal of currents in one face requires also a reversal of the electric intensity tangential to the strip; hence, the tangential electric intensity is now discontinuous, and in the strip there exist magnetic currents. In the conducting plane, however, there are now no electric currents; consequently, we may remove this plane without disturbing the field. Thus, we are left with a true magnetic dipole antenna formed by the plane strip of the same dimensions as the slot. Its admittance is Y_p^{se}.

In Section 17.3 we have seen that there exists a simple relationship between dual electric and magnetic antennas. This relationship was found incidentally while calculating radiation intensities, but it is so important that it is worth while to derive it directly from Maxwell's equations. If the medium is nondissipative, these equations are in-

variant under the following transformation,

$$E \to H, \qquad H \to -E, \qquad \mu \to \varepsilon, \qquad \varepsilon \to \mu. \tag{54}$$

Under this transformation the intrinsic phase constant remains unchanged, and the intrinsic impedance goes into intrinsic admittance,

$$\eta \to \frac{1}{\eta} \cdot \tag{55}$$

The input impedance of an electric antenna is defined as the ratio of the impressed electromotive force to the electric current. In order to keep the physical dimensions of impedance the same, we shall define the impedance of the magnetic antenna as the ratio of the magnetic current to the impressed magnetomotive force. From the transformations of equation 54, we conclude that, in all wave impedances associated with the fields of electric currents and all wave admittances associated with the fields of the corresponding magnetic currents, the factors depending on the geometry of the distributions are the same. Hence, we have the important theorem,

$$\frac{Z_e}{Y_m} = \frac{\eta}{1/\eta} = \eta^2, \qquad Z_e Z_m = \eta^2. \tag{56}$$

The zeros and poles of the corresponding impedances are interchanged. One impedance equals the other transformed by a quarter-wave line of characteristic impedance η.

Thus, if Z is the impedance of an electric strip antenna, the impedance of the corresponding magnetic antenna and, hence, the series impedance of the slotted plane, is

$$Z_p^{\text{se}} = \frac{\eta^2}{Z} \cdot \tag{57}$$

From equation 53 we obtain the shunt admittance and impedance of the plane,

$$Y_p^{\text{sh}} = \frac{4Z}{\eta^2}, \qquad Z_p^{\text{sh}} = \frac{\eta^2}{4Z}, \tag{58}$$

and the admittance and impedance of each face,

$$Y_f = \frac{2Z}{\eta^2}, \qquad Z_f = \frac{\eta^2}{2Z} \cdot \tag{59}$$

The impedance Z may be obtained from the equations of Chapter 13 if we use the equivalent radius,

$$a = \tfrac{1}{4}w, \tag{60}$$

of a cylindrical antenna whose characteristic impedance equals the characteristic impedance of the strip antenna of width w. The above equation is derived from a consideration of principal spherical waves along conducting strips.

In actual operation, slots are made in metal sheets of finite dimensions. Sometimes these dimensions are large, as in the case of slots in aircraft. In such cases, the impedance of the external face is substantially equal to the impedance of one face of the conducting plane. On the other side the slot is usually encased in a structure of relatively small dimensions, a cavity resonator or a waveguide. Hence, there will be a wide variation in impedances seen from slots in the direction of the generator (or the load, when slots are used for reception). These impedances are obtained by calculating the interaction between magnetic strips, and the normal modes of oscillation in resonators or normal modes of propagation in waveguides, as the case may be.*

Slots in coaxial transmission lines and in waveguides may be arranged into antenna arrays for the purpose of increasing the directivity. In the array calculations we have to know primarily the properties of the individual slots; the principle of superposition enables us to evaluate the arrays.

17.9 Input regions

If the current impressed on a slot antenna is concentrated in a small region, we encounter a problem similar to that of the gap in the case of dipole antennas. In the case of idealized slots, freed from their connection to the feeders, we must provide a path for the electrons from one edge of the slot to the other (Fig. 17.10). The current in this region is prescribed — it is the *impressed current.* Thus, in a theoretical analysis, the slot antenna is supposed to be fed by a generator of infinite impedance, just as a linear antenna is supposed to be fed by a generator of zero impedance. In a slot antenna, a generator of finite internal impedance is represented by a generator of infinite impedance in parallel with its internal impedance, just as, in a linear antenna, a generator of finite impedance is represented by a generator of zero impedance in series with its internal impedance. If the slot is uniform (Fig. 17.10a), but very narrow, the input impedance is relatively independent of the width s of the gap, as long as this width is small compared with the length of the slot but not too small compared with the width a. As s approaches zero, the input impedance approaches infinity.

* S. A. Schelkunoff, Representation of impedance functions in terms of resonant frequencies, *IRE Proc.*, **32**, February 1944, pp. 83–90; Impedance concept in waveguides, *Quart. Appl. Math.*, **2**, April 1944, pp. 1–15.

In tapered slots (Figs. 17.10*b*, *c*) there are no input singularities, that is, when the planes are infinitely thin. In planes of finite thickness

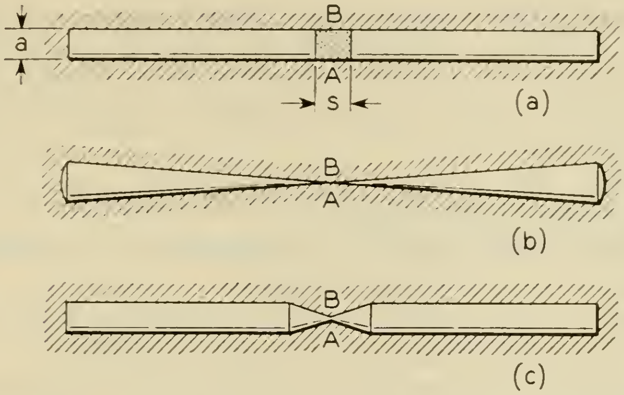


FIG. 17.10 Slot antennas with idealized input regions.

we shall have a narrow wedge (which has a singularity) unless the input tips are tapered also at right angles to the planes to form conical tips.

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PROBLEMS

17.1 Consider a large biconical antenna. Suppose that the length of each cone along the generators is l, the maximum diameter is $2a$, and the height of the antenna is h. Assume that the antenna is large and that $l - a \ll \frac{1}{2}\lambda$. Find the directivity.

Ans.
$$g = \frac{2h}{\lambda}\cdot$$

17.2 Obtain the distant field of an infinitely long uniform electric current filament.

Ans. $H_\varphi = -\frac{1}{2}I(\lambda\rho)^{-\frac{1}{2}}\, e^{-j\beta\rho - j3\pi/4}, \qquad E_z = -\eta H_\varphi,$

where I is the current.

17.3 Obtain the distant field of an infinitely long magnetic current filament.

Ans. $E_\varphi = \frac{1}{2}V(\lambda\rho)^{-\frac{1}{2}}\, e^{-j\beta\rho - j3\pi/4}, \qquad H_z = \eta^{-1}E_\varphi.$

17.4 Obtain the distant field of an infinitely long electric current blade of width dx. Assume that the current per unit length is C.

Ans. $H_z = \frac{1}{2}\pi C\, dx(\lambda\rho)^{-\frac{1}{2}}\, e^{-j\beta\rho - j3\pi/4} \sin\varphi, \qquad E_\varphi = \eta H_z.$

17.5 Obtain the distant field of an infinitely long magnetic current blade of width dx. Let the magnetic current per unit length be M.

Ans. $E_z = -\frac{1}{2}\pi M\, dx\,(\lambda\rho)^{-\frac{1}{2}}\, e^{-j\beta\rho - j3\pi/4} \sin\varphi, \qquad H_\varphi = -\eta^{-1}E_z.$

17.6 Obtain the equation for the propagation constants in a slotted half-cylinder (Fig. 17.7).

Ans. $$\frac{N_1(u)}{J_1(u)} - \frac{4}{\pi}\log u + \frac{6.464}{\pi} + \frac{4}{\pi}\log\frac{a}{s} = j, \qquad u = a\sqrt{\Gamma^2 + \beta^2}.$$

17.7 From the equation in the preceding problem, obtain the propagation constant of the dominant mode on the assumption that $\log(a/s)$ is large.

Ans. $$\Gamma = \left[\left(\frac{u}{a}\right)^2 - \beta^2\right]^{\frac{1}{2}},$$

$$u = 3.83 - \frac{\pi N_1(3.83)}{4J_1'(3.83)}\left(\log\frac{a}{s} + 0.273 - j\frac{\pi}{4}\right)^{-1}.$$

18

REFLECTORS

18.1 Reflectors

An antenna has two functions: one is to couple the source of power to free space and the other to direct this power in some preferred directions. Some antennas are relatively nondirective; others are highly directive. Nondirective antennas may be combined into highly directive arrays. The directivity is obtained by using the principle of interference of waves from the various elements of the array arriving

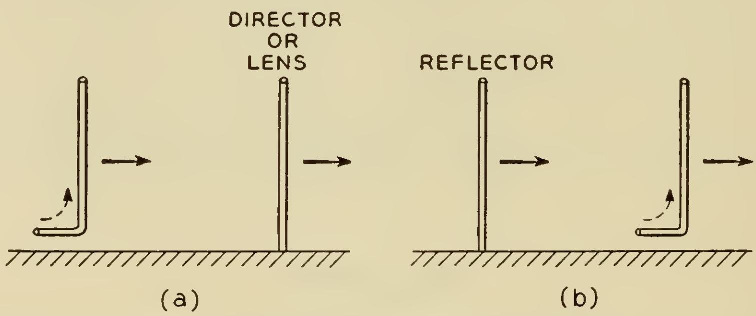


Fig. 18.1 (*a*) A director or a lens is a reradiating antenna placed in front (as far as the main beam of radiation is concerned) of the antenna directly connected to a source of power; (*b*) a reflector (in a narrow sense) is a reradiating antenna placed behind the antenna directly connected to a source of power.

at a given point in space in different phases. The required variation in phase is produced by a suitable spatial arrangement of the individual antennas in the array, and by a suitable phasing of these antennas with respect to some particular element of the array. In an antenna array, each element is coupled directly to the source of power or to the transmission line connected to it.

Suppose, however, we energize only one antenna and ground the other (Fig. 18.1). The field of the first will excite the second. The fields of the two will interfere and produce a nonuniform radiation pat-

tern in the horizontal plane. By changing the length of the "parasitic" or reflecting antenna, or by inserting a tuning circuit between it and the ground, we can control the phase of the current and, hence, the direction of maximum radiation. In either of the two relative positions of the grounded antenna with respect to the one connected to the source of power which are shown in Fig. 18.1, we can obtain the phase difference required to direct more power from left to right than in other directions. Although the grounded antenna, if close to the other, affects the radiated power, its main function is to alter the directive pattern. The grounded antenna acts as a receiving antenna reradiating all the received power (assuming for the present a perfect ground, etc.); it is a "virtual source," acting on borrowed power. In Chapter 8 such antennas were called "reflecting antennas." Frequently, however, the term "reflector" is employed in a narrower sense for antennas that are *behind* the source of primary radiation with respect to the direction of maximum radiation (Fig. 18.1*b*). The "reflectors" in front of the primary source (Fig. 18.1*a*) are called *directors*; they are, in effect, vestigial lenses. This distinction between directors and reflectors is entirely superficial.

Reflectors may be single wires as above, or they may be systems of wires, screens, and conducting sheets of various shapes. If the sheets are large compared with the wavelength, they can be shaped to exert a profound effect on the radiation pattern of the primary source. Reflectors enable us to obtain an immense variety of radiation patterns. This is an important property which is utilized in radar antennas.*

18.2 Radiation patterns

In such simple arrangements as those shown in Fig. 18.1, the radiation patterns are obtained from the current distribution in the wires. In the first approximation the current in the primary antenna is assumed to be unaffected by the induced current in the reflecting antenna. The induced current is calculated as explained in Chapter 8. For greater accuracy we can calculate the current induced in the primary antenna by the field reradiated by the reflecting antenna.

Essentially the same method may be used for all reflectors. Let

$$f(x, y, z) = 0 \quad \text{or} \quad z = F(x, y) \tag{1}$$

be the equation of the reflecting surface. If $\vec{C}(x, y, z)$ is the linear density of electric current induced in the reflector by the primary source, the moment of a typical current element is $\vec{C}\, dS$ where dS is an element of area. Using the method explained in Section 12.1, we can calculate

* H. T. Friis and W. D. Lewis, Radar antennas, *Bell Sys. Tech. Jour.*, **26**, April 1947, pp. 219–317

the Cartesian components of the radiation vector (N_x,N_y,N_z) by integrating the corresponding components of the current moment $(C_x\,dS, C_y\,dS, C_z\,dS)$ over the reflecting surface. From the radiation vector we then obtain the distant field of the reflector. Adding the primary field, we have the total field.

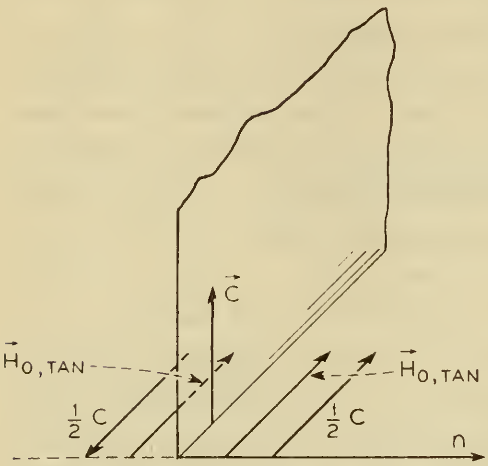


FIG. 18.2 The density of current induced in an infinite plane is twice the tangential component of the incident magnetic field.

There is no simple method for obtaining the exact current in reflectors except when they are infinite planes. The density of current induced in an infinite perfectly conducting plane *is twice the tangential component of the magnetic intensity* $\vec{H}_0(x,y,z)$ *of the primary field.* To obtain this result we note that the primary field cannot penetrate an infinite reflecting plane. Consequently, the currents of density $\vec{C}$ induced in the plane (Fig. 18.2) must cancel the primary field on the other side of the plane. Since the tangential magnetic vectors produced by these currents are equal and opposite on the two sides of the plane, the induced current density must be twice as large as the tangential intensity of the primary field in order that the cancelation may be complete. Therefore,

$$\vec{C} = 2\vec{n} \times \vec{H}_{0,\mathrm{tan}}, \tag{2}$$

where $\vec{n}$ is the unit normal from the plane toward the source of the primary field. We may also write

$$\vec{C} = 2\vec{n} \times \vec{H}_0 \tag{3}$$

since the normal component $\vec{H}_{0,\mathrm{nor}}$ does not affect the value of the vector product in this equation.

As we have already stated, there is no simple method for calculating the current density in finite reflectors, whether curved or plane. If, however, the reflecting surface is large, we may assume that the current induced in each element of the surface equals the current that would have been induced if the element were part of an infinite tangent plane. The errors involved in this approximation are large near the edges of the reflector, since the current density normal to the edge must vanish and the tangential density is increased by the proximity to the edge. However, the effect of these errors on the total field is proportional to

the perimeter of the reflector while the magnitude of the field of a large current sheet depends on the area of the sheet. The edge effect will be important only in those directions in which the main contributions from the area cancel each other by destructive interference. The effect on the major radiation lobe will be small. Although it is possible to make reasonable approximations to the current density near the edge of the reflector and thus obtain a better result for the reflected field, normally it is not necessary.

Differentiating equation 1, we have

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0. \tag{4}$$

This equation shows that the vector whose components are $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is perpendicular to the components (dx, dy, dz) of an infinitesimal displacement in the reflecting surface; hence, the first vector is in the direction of the normal, and the components of the unit normal may be expressed as follows:

$$\begin{gathered} n_x = D^{-1}\frac{\partial f}{\partial x}, \qquad n_y = D^{-1}\frac{\partial f}{\partial y}, \qquad n_z = D^{-1}\frac{\partial f}{\partial z}, \\ D = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right]^{1/2}. \end{gathered} \tag{5}$$

From these expressions and from the given primary field, we obtain the approximate current density by using equation 3.

Some reflectors are similar to horns. The radiation pattern of such a reflector may also be calculated from the field over the aperture by the method explained in Chapter 16. This method is practicable only when the conditions are such that we can make reasonable *ab initio* assumptions with regard to the field over the aperture. Such conditions are actually met in many important cases.*

18.3 Directivity

In calculating the directivity of a large reflector we may assume that the reflector redistributes the power radiated by the primary source without affecting the total amount of radiation. The sequence of calculations is then as follows: (1) the magnetic field of the primary source is expressed in terms of the radiated power P; (2) the current density in the reflector is obtained from equation 3; (3) the maximum radiation intensity is obtained from the currents in the reflector; (4) the direc-

* *Ibid.*

tivity and the effective area are finally calculated from their definitions,

$$g = \frac{4\pi\Phi_{\max}}{P}, \qquad A = \frac{\lambda^2\Phi_{\max}}{P}. \tag{6}$$

Thus, we can calculate g without obtaining the entire radiation pattern.

Instead of expressing the primary field in terms of P, we may express P in terms of the primary field at the reflector. If we assume that all of the primary radiation falls on the reflector, then,

$$P = \tfrac{1}{2}\eta \iint \vec{H}_0 \cdot \vec{H}_0^* \cos\psi \, dS, \tag{7}$$

where ψ is the angle between the normal to dS and the reflected ray. Substituting in equation 6, we have

$$A = \frac{2\lambda^2\Phi_{\max}}{\eta \iint \vec{H}_0 \cdot \vec{H}_0^* \cos\psi \, dS}. \tag{8}$$

As shown in Section 12.1, the radiation intensity may be expressed in terms of the radiation vector $\vec{N}$; thus,

$$\Phi = \frac{\eta}{8\lambda^2}(N_\theta N_\theta^* + N_\varphi N_\varphi^*), \tag{9}$$

$$\vec{N} = \iint \vec{C} e^{j\beta r \cos\vartheta} \, dS = 2 \iint (\vec{n} \times \vec{H}_0) e^{j\beta r \cos\vartheta} \, dS, \tag{10}$$

where r is the radius from the origin to a typical reflecting element and ϑ is the angle made by this radius with a typical direction (θ, φ) in space. Let

$$\vec{H}_0 = H_0(x, y, z)\, e^{jp(x,y,z)}\, \vec{h}(x, y, z), \tag{11}$$

where the first factor is the magnitude of the magnetic intensity at the reflector, the second is the phase factor, and the third is the unit vector in the direction of $\vec{H}_0$. Substituting in equation 10, we have

$$\vec{N} = 2 \iint (\vec{n} \times \vec{h}) H_0(x, y, z)\, e^{jp(x,y,z) + j\beta r \cos\vartheta} \, dS. \tag{12}$$

Expressing the effective area in terms of the radiation vector, we find

$$A = \frac{(N_\theta N_\theta^* + N_\varphi N_\varphi^*)_{\max}}{4 \iint [H_0(x, y, z)]^2 \cos\psi \, dS} = \frac{(N_{\tan} \cdot N_{\tan}^*)_{\max}}{4 \iint [H_0(x, y, z)]^2 \cos\psi \, dS}. \tag{13}$$

Thus, the effective area of a *large reflector* has been expressed in terms of the primary field at the reflecting surface for the case in which *most*

of the primary radiation falls on the reflecting surface. If a certain quantity of power P_1 is "spilled over" the reflector, the effective area as given above should be multiplied by the ratio of the difference $P - P_1$ to the total radiated power P.

Whenever $\vec{n} \times \vec{h}$ and $\cos \psi$ are substantially constant over the surface, the above equations become simpler, since these factors can be taken outside the integral signs. It is to be noted that the magnitude of $\vec{n} \times \vec{h}$ equals the sine of the angle between the magnetic intensity and the normal to the surface, while ψ is the angle between this normal and the reflected ray. Further simplifications can be made for special types of reflectors.

If the reflector resembles a horn and is such that over its aperture, we may assume an essentially plane reflected wave (see Figs. 18.4, 18.5, 18.6), the effective area may be expressed in terms of the electric intensity over the aperture,

$$E(x, y) = E_0\, a(x, y) e^{j\varphi(x,y)}. \tag{14}$$

Thus, we find

$$A = \frac{\left| \iint a(x, y)\, e^{j\varphi(x,y)}\, dS \right|^2}{\iint [a(x, y)]^2\, dS}. \tag{15}$$

The effects of nonuniform amplitude and phase distributions of the reflected field may thus be studied.*

18.4 Reflecting properties of parabolas

To obtain the maximum radiation intensity in a given direction (θ, φ) we must make the phase factor in equation 12 independent of the position of the reflecting element dS,

$$p(x, y, z) + \beta r \cos \vartheta = \text{const.} \tag{16}$$

If the primary source is a point source at F (Fig. 18.3), and if the shortest distance OF is large enough for the field to vary substantially as $1/r$, then,

$$p(x, y, z) = -\beta r + \text{const.} \tag{17}$$

If we wish to obtain the maximum radiation intensity in the direction of the z axis, $\cos \vartheta = \cos \theta$ and $r \cos \vartheta = z$ if the origin is taken at F; therefore equation 16 becomes

$$r - z = \text{const}, \quad \text{or} \quad \sqrt{\rho^2 + z^2} - z = 2l, \tag{18}$$

* *Ibid.*

where $\rho^2 = x^2 + y^2$. The value of the constant is obtained from its value at $\rho = 0$, when $r = l$ and $z = -l$. From equation 18 we find

$$\rho^2 = 4lz + 4l^2. \tag{19}$$

This equation represents a parabola in one plane and a paraboloid of revolution in all planes.

In the z direction,

$$N_\theta N_\theta^* + N_\varphi N_\varphi^* = N_x N_x^* + N_y N_y^*. \tag{20}$$

Only those currents in the reflector that are perpendicular to the z axis are effective in this direction. In many cases of symmetric primary

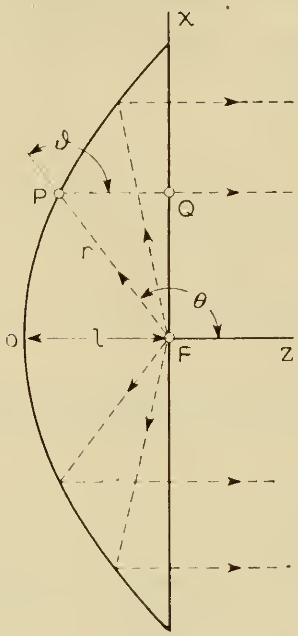


Fig. 18.3 Reflecting properties of parabolas.

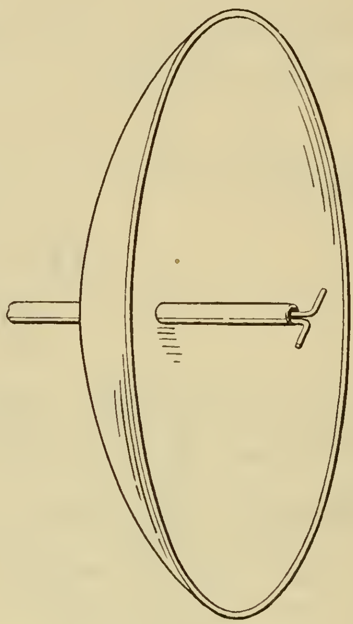


Fig. 18.4 A paraboloidal reflector with a dipole primary antenna.

fields, only N_x or N_y is different from zero. In such cases, equation 13 becomes

$$A = \frac{\left[\iint (\vec{n} \times \vec{h})_x H_0(x, y, z)\, dS\right]^2}{\iint [H_0(x, y, z)]^2 \cos\psi\, dS}, \tag{21}$$

where ψ is now the angle between the z axis and the normal to dS. This angle equals the angle between the xy plane and the plane of dS; hence $\cos\psi\, dS$ is the projection of dS on the xy plane.

In some cases $(\vec{n} \times \vec{h})_x$ is substantially equal to $\cos \psi$, and

$$A = \frac{\left[\iint H_0(x, y, z) \cos \psi \, dS\right]^2}{\iint [H_0(x, y, z)]^2 \cos \psi \, dS}. \tag{22}$$

If H_0 is constant, A equals the area of the aperture of the reflector (the area of the reflector projected on the xy plane).

If the primary field is not a point source, the reflecting surface for maximum field in the z direction is not parabolic. When a parabolic surface is used, the phase factor in equation 12 is not constant and must be included in the above equations for A. This phase factor reduces the effective area.

Figure 18.4 shows a paraboloidal reflector with a dipole antenna at its focus. A small reflector is generally added to the dipole to prevent

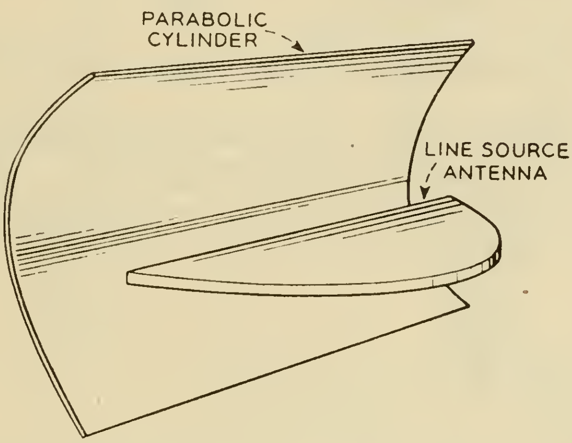


FIG. 18.5 A parabolic cylinder serving as a reflector for waves emitted from a line source.

direct radiation from it. Open-ended waveguides are also used as primary sources. The effective area of a well-designed parabolic reflector has been found experimentally to be approximately two thirds of the projected area of the reflector. Figure 18.5 shows a parabolic cylinder serving as a reflector for a line source. The line source itself is made up of a point source in the focus of a narrow parabolic horn. Reflector antennas have been used extensively in radar; for further theoretical and practical information the reader is referred to the monograph by H. T. Friis and W. D. Lewis on radar antennas already cited.

18.5 Corner reflectors, dihedral and trihedral

A *dihedral corner reflector* consists of two intersecting reflecting planes (Fig. 18.6). A *trihedral corner reflector* consists of three such planes.

An approximate radiation pattern and radiation resistance may be calculated by the theory of images. For example, in the case of a 90°

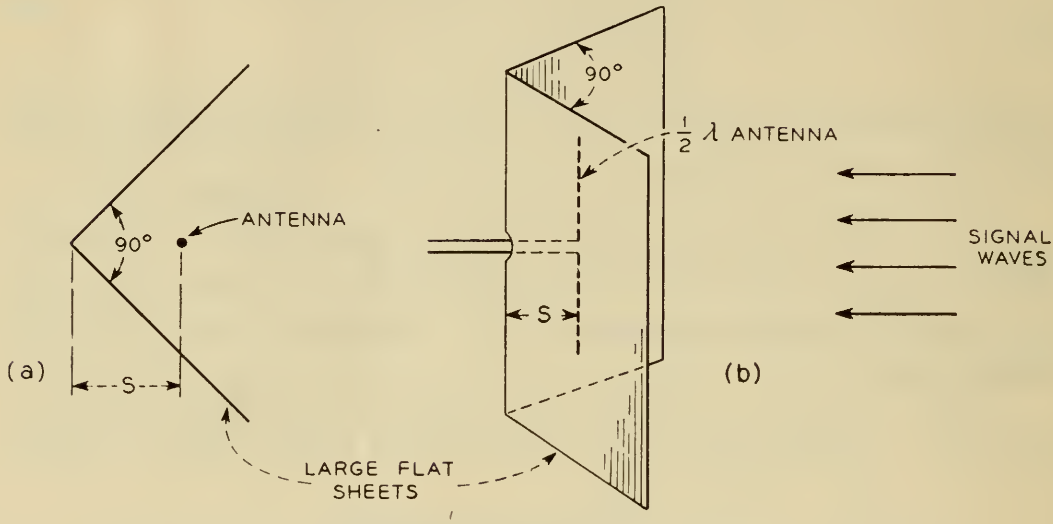


FIG. 18.6 A dihedral horn reflector: (*a*) top view; (*b*) perspective.

dihedral corner reflector (Fig. 18.7*a*), we may replace the reflector by three images of the primary antenna (Fig. 18.7*b*). This substitution is valid only for calculation of the major radiation lobe. To this order

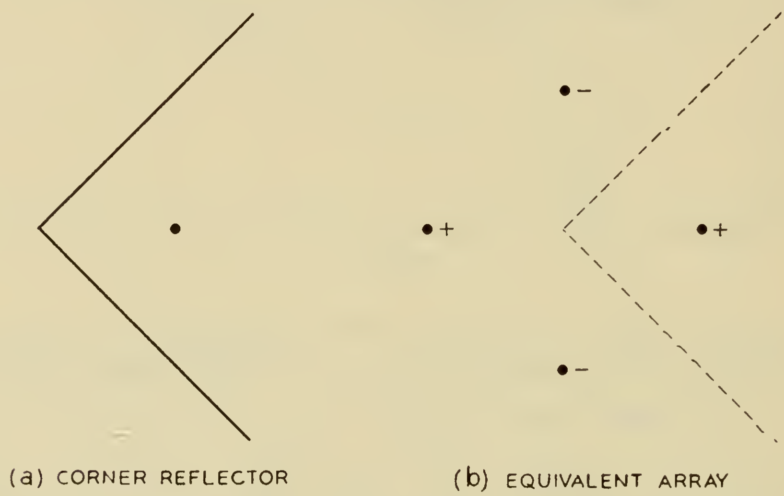


FIG. 18.7 (*a*) A large 90° dihedral corner reflector may be replaced for the purposes of analysis by (*b*) three images of the primary antenna.

of approximation there is no radiation except within one quarter of the total space. Better results may be obtained from the field in the aperture.

Dihedral corner reflectors are used in the 1- to 10-meter wavelength range. They are simple to construct and easy to transport, and the

dimensions are not critical. Tests indicate* that, in order to obtain the performance predicted for large dimensions, the over-all dimensions of the reflector need not exceed two wavelengths. The reflecting metal sheets may be replaced by grids of wires parallel to the primary antenna, with spacings about 0.1λ.

The 90° dihedral corner reflector (Fig. 18.6), increases the output of a half-wave antenna by about 10 db for spacings S varying from

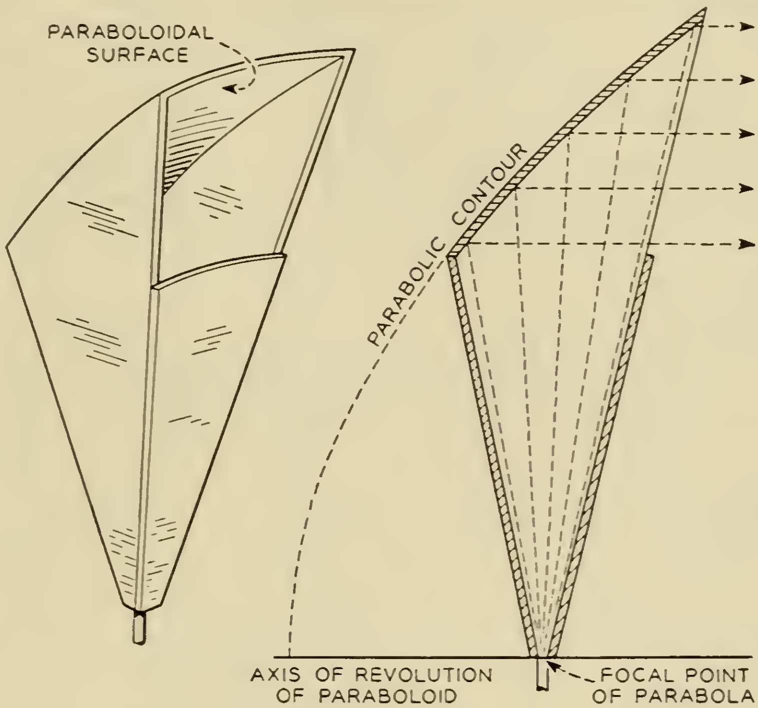


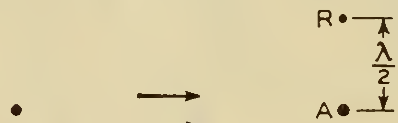
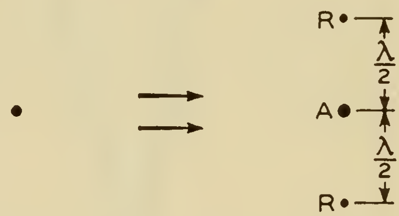
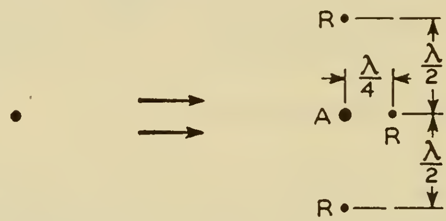
FIG. 18.8 A horn-reflector antenna.

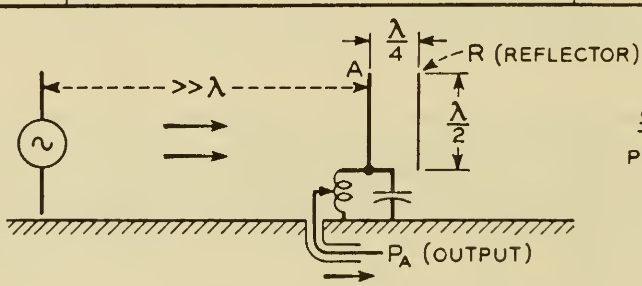
0.2λ to 0.5λ. That is, as far as gain is concerned, the spacing between the antenna and the reflector is not critical. The radiation resistance, however, increases gradually from 13 ohms for $S = 0.2\lambda$ to 120 ohms for $S = 0.5\lambda$.

Trihedral corner reflectors are used as targets for microwave radar navigation. Their main advantage over flat reflectors is that they need not be accurately oriented with respect to the incident waves. An analysis of such reflectors may be found in a paper by Sloan D. Robertson.†

* For detailed information see John D. Kraus, The corner-reflector antenna, *IRE Proc.*, **28**, November 1940, pp. 513–519.

† Targets for microwave radar navigation, *Bell Sys. Tech. Jour.*, **26**, October 1947, pp. 852–869.

REFLECTOR		$\frac{P_A \text{ (WITH REFLECTOR)}}{P_A \text{ (REFLECTOR REMOVED)}}$	
		EXPERIMENTAL	
POSITION	TOP VIEW	CLIFFWOOD	NAGY
(a) IN BACK OF ANTENNA		1.6 (2 DB)	2.3 (3.6 DB)
(b) IN FRONT OF ANTENNA		¼ (−6 DB)	
(c) SIDE REFLECTOR		1.9 (2.8 DB)	
(d) 2 SIDE REFLECTORS		2.7 (4.2 DB)	
(e) 1 BACK AND 2 SIDE REFLECTORS			3 (4.8 DB)



SIDE VIEW

POSITION (a)

FIG. 18.9 Reflecting wires.

18.6 Horn-reflector antennas

Horns and reflectors may be combined* as shown in Fig. 18.8. The horn operates as the feed for the reflector, and as such it is effectively "offset," so that the back reflection, so often present in reflecting antennas, is practically eliminated. Experiments showed that for one model the standing-wave ratio in the feed line (leading to the combination of the horn and the reflector) was only 0.1 db in a 10 per cent band of frequencies. Another important feature is that back-to-back cross-talk suppression of two such antennas is very high.

18.7 Experimental data on reflecting wires and plane sheets

Figure 18.9 presents some experimental gains obtained from reflecting wires in various arrangements. The first column gives experimental

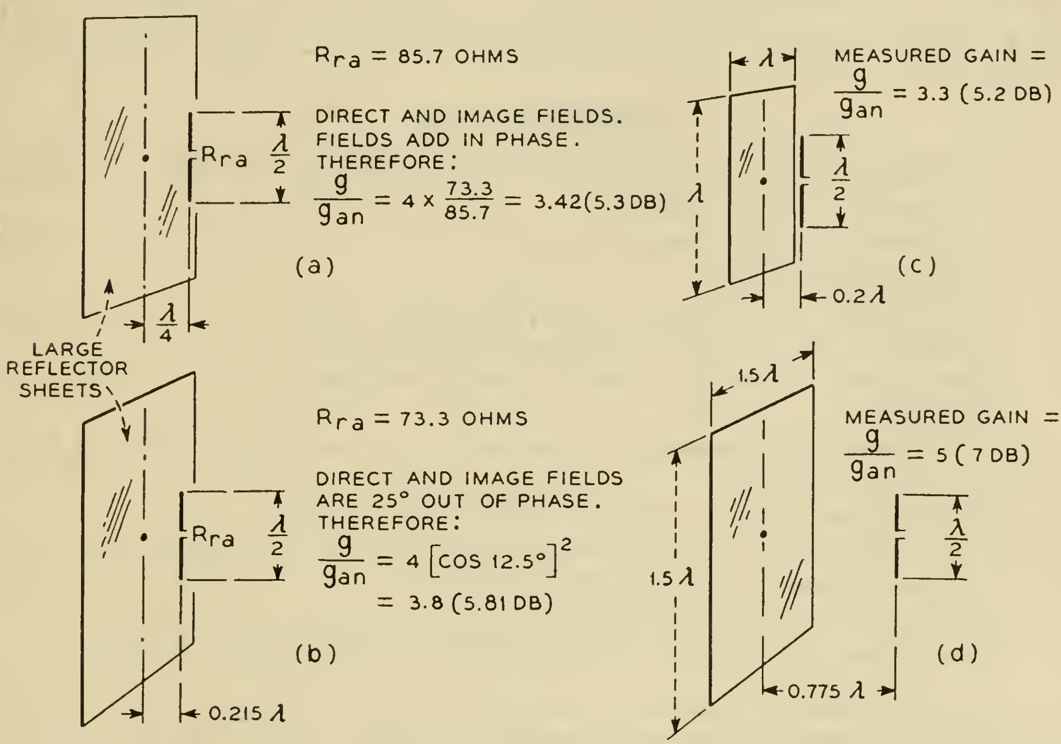


FIG. 18.10 Reflecting plane sheets.

data obtained in 1927 at Cliffwood, N. J., for $\lambda = 27$ meters; the figures in the last column are taken from A. Wheeler Nagy's paper† on parasitic wire reflectors. Greater gains are obtained by detuning the reflectors in such a way that the resulting phase difference between the currents in the antenna and the reflector produces a sharper pattern.

* H. T. Friis, Microwave repeater research, *Bell Sys. Tech. Jour.*, **27**, April 1948, pp. 183–246.

† An experimental study of parasitic wire reflectors on 2.5 meters, *IRE Proc.*, **24**, February 1936, pp. 233–254.

The entry *b* in Fig. 18.9 indicates that the field beyond the half-wave reflecting wire is greatly decreased. If, however, the length of the reflector is reduced to about 0.45λ, an increase in the field is obtained; that is, the reflecting wire becomes a director or a lens. Yagi arrays* are arrays consisting of one active antenna, one reflector, and several directors. Arrays of this type have found use in the meter-wavelength range. They are simple and inexpensive. Their main disadvantage is a limited bandwidth.

Figures 18.10*a* and *b* show calculated gains from large reflecting planes. Almost 6 db increase in directivity is obtained when the reflector is from 0.2λ to 0.25λ behind the antenna. The gain increases to 7 db for smaller spacings. When the reflecting sheet is small, its dimensions become important. In fact, the edge effect helps to increase the gain. Experimental work with $\lambda = 1$ meter by E. J. Sterba at Deal, N. J., in the summer of 1934 gave the results shown in Figs. 18.10*c* and *d*.

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* H. Yagi, Beam transmission of ultra short waves, *IRE Proc.*, **16**, June 1928, pp. 715–741; George H. Brown, Directional antennas, *IRE Proc.*, **25**, January 1937, pp. 78–145.

19

LENSES

19.1 Lenses

To obtain the maximum gain in the forward direction from a horn with an aperture of given area, we must have as nearly uniform distribution of phase over the aperture as possible; since differences in phase inevitably cause some destructive interference. This optimum condition

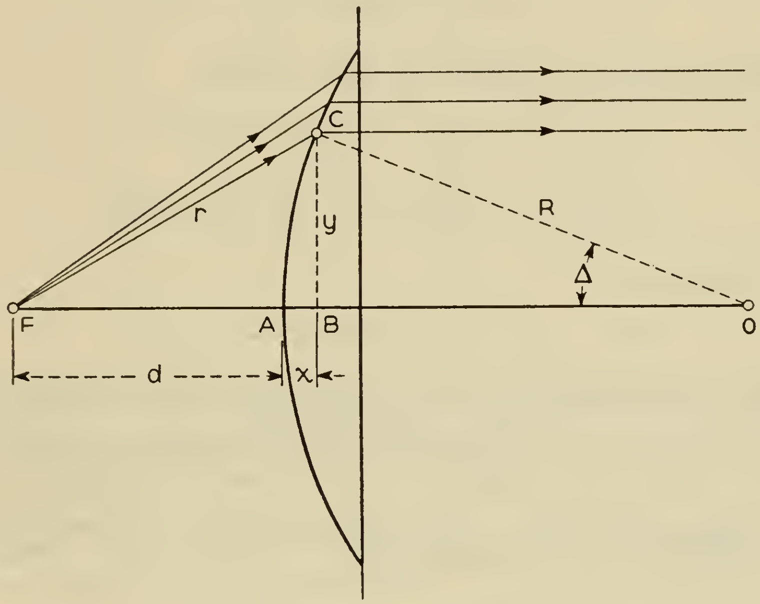

Fig. 19.1 A planoconvex lens.

may be approached when the length of the horn is large, and the curvature of the wavefront at the aperture of the horn is so small that the phase differences are negligible. For horns with large apertures, such lengths are impractical. It is more practical to insert a lens in the aperture and thus straighten the wavefront. Since the dimensions of the aperture are large compared with the wavelength, the design of such a lens may be based on geometrical optics; that is, we may neglect the

effect of the curvature of the interface between two media on the transmission of waves through the lens.

Let F be a source of spherical waves in front of a lens consisting of a segment of a dielectric sphere of radius R (Fig. 19.1). Consider a typical ray FC. In order that the wavefront be plane on passing through the lens, the phase at C must be equal to that at B; hence, if β_0 and β are the phase constants in free space and in the lens, we must have

$$\beta_0 r = \beta_0 d + \beta x. \tag{1}$$

The ratio n of the phase constants,

$$n = \frac{\beta}{\beta_0} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \frac{v_0}{v}, \tag{2}$$

is called the *index of refraction* of the lens. Dividing equation 1 by β_0, we thus obtain

$$r = d + nx. \tag{3}$$

From geometric considerations

$$r = \sqrt{(d + x)^2 + y^2}.$$

Substituting in equation 3 and squaring, we find

$$x^2 + y^2 = 2(n - 1)xd + n^2x^2. \tag{4}$$

Since

$$x = R(1 - \cos\Delta), \qquad y = R\sin\Delta,$$

equation 4 becomes

$$R = (n - 1)d + \tfrac{1}{2}n^2R(1 - \cos\Delta).$$

Since d and n are constants and Δ is variable, R must also be variable. Therefore, in the case of a spherical lens, this equation cannot be satisfied exactly; but, if Δ is small, $1 - \cos\Delta \simeq \frac{1}{2}\Delta^2$, and the last term is negligible when $n^2\Delta^2 \ll 4$. Thus, the radius of the *planoconvex lens* that will straighten spherical waves is

$$R = (n - 1)d. \tag{5}$$

Equation 4 gives the shape of the perfect (except for diffraction effects) planoconvex lens for any value of Δ. Transforming the equation to the standard form,

$$\left[\frac{(n + 1)x}{d} + 1\right]^2 - \frac{n + 1}{n - 1}\,\frac{y^2}{d^2} = 1, \tag{6}$$

we see that it is hyperboloidal.

If the index of refraction is less than unity, the radius is negative, and it seems likely that a planoconcave lens (Fig. 19.2) will straighten

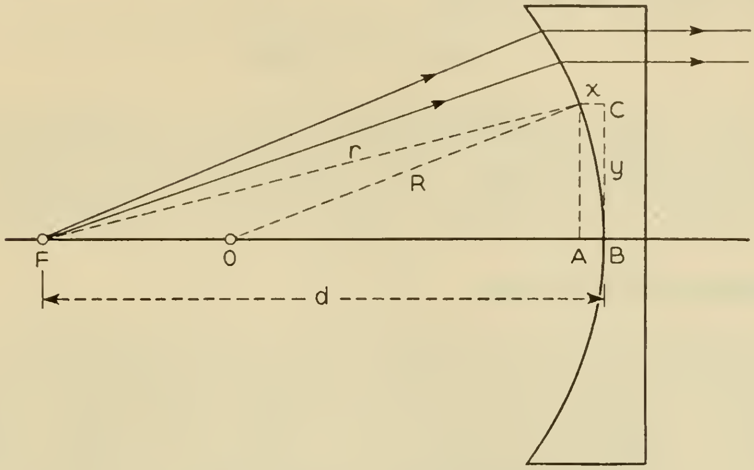

FIG. 19.2 A planoconcave lens.

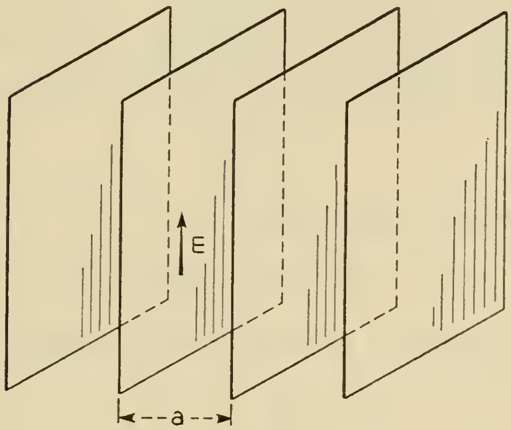

FIG. 19.3 Free space partitioned by parallel conducting planes is a nonisotropic dispersive medium. When E is perpendicular to the partitions, the index of refraction equals that of free space; but, when E is parallel to the partitions, the effective index of refraction is less than that of free space.

the wavefront. Repeating the calculations for the spherical case, we find that the radius of such a lens is*

$$R = (1 - n)d. \tag{7}$$

19.2 Waveguide lenses

When the electric vector is parallel to a set of conducting planes (Fig. 19.3), the phase velocity of the dominant waves is

$$v = \frac{v_0}{\sqrt{1 - (\lambda/2a)^2}}, \tag{8}$$

* For further details, see D. W. Fry and F. K. Goward, *Aerials for Centimetre Wave-Lengths*, University Press, Cambridge, 1950.

where v_0 is the velocity in free space and a is the separation between the planes. Hence, the effective index of refraction of free space partitioned by parallel conducting planes is

$$n = \sqrt{1 - (\lambda/2a)^2}. \tag{9}$$

If $a < \lambda < 2a$, such a medium may be used to construct converging planoconcave *waveguide lenses.* We should note that these lenses are frequency sensitive; nevertheless, they may have important uses.

19.3 Artificial dielectrics

Dielectric media whose refractive indices are large enough to make them suitable for microwave lenses are apt to be heavy. To obviate this difficulty, artificial dielectrics have been made out of light polystyrene foam in which are imbedded small metal objects. Under the influence of an impressed field, the electric charge on these objects is displaced, and the objects become dipoles. Let p_e be the moment of a typical electric dipole and N be the number of dipoles per unit volume; then the polarization of the medium due to the presence of the dipoles is

$$P = Np_e. \tag{10}$$

Hence, the displacement density is*

$$D = \varepsilon_0 E + P = \varepsilon_0 E + Np_e. \tag{11}$$

If ε is the effective dielectric constant of the medium, then

$$D = \varepsilon E,$$

and

$$\varepsilon = \varepsilon_0 + \frac{Np_e}{E}. \tag{12}$$

The moment of each individual dipole is proportional to the impressed field; thus,

$$p_e = \chi_e E, \tag{13}$$

where χ_e is the *electric polarizability* of the dipole. Therefore,

$$\varepsilon = \varepsilon_0 + N\chi_e, \qquad \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{N\chi_e}{\varepsilon_0}. \tag{14}$$

Under the influence of the magnetic field, circulating currents may be induced in the metal objects. Hence, in effect, these objects may

* *Electromagnetic Waves*, p. 91.

become magnetic dipoles. As above we find the effective permeability of the medium

$$\frac{\mu}{\mu_0} = 1 + \frac{N\chi_m}{\mu_0}, \tag{15}$$

where $\chi_m = p_m/H$, the ratio of the magnetic moment to the impressed magnetic intensity, is the *magnetic polarizability* of the object.

Thus, the index of refraction of a medium filled with metal objects is

$$n = \left[\left(1 + \frac{N\chi_e}{\varepsilon_0}\right)\left(1 + \frac{N\chi_m}{\mu_0}\right)\right]^{1/2}. \tag{16}$$

Whereas χ_e is always positive, χ_m may be either positive or negative. In making artificial dielectrics for microwave lenses we should therefore select objects with χ_m positive or at least nonnegative. Presently we shall find that both polarizabilities are affected not only by the shape of the objects but also by their orientation with respect to E and H.

To evaluate the index of refraction of an artificial dielectric we need to determine the polarizabilities of the objects with which it is loaded. First we shall calculate the polarizabilities χ_e^0, χ_m^0 of an isolated object; then we shall consider the effect of near neighbors. In approximate calculations of refractive indices (equation 16), we may let $\chi_e = \chi_e^0$ and $\chi_m = \chi_m^0$. All calculations will be based on the assumption that the *objects are small.* The polarizabilities are increased by proximity to resonance; but then they depend on the frequency and make artificial dielectrics more dispersive.

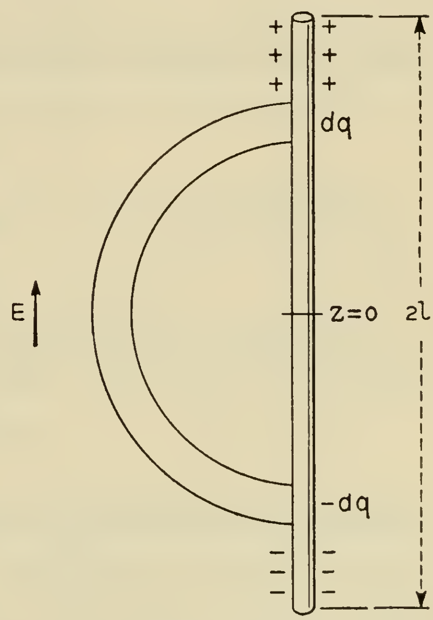

Fig. 19.4 A thin cylindrical rod in an electric field becomes an electric dipole.

19.4 Polarizability of isolated thin rods

To obtain the electric polarizability of an object, we must find first the distribution of electric charge on the object and then the moment of this distribution. As we shall presently see, this problem can be solved exactly for some shapes of objects. A straight cylindrical rod of finite length is not in this category, and the solution must be approximate.

Let the length of the rod (Fig. 19.4) be $2l$, and let the electric vector

be parallel to the rod. Under the influence of the impressed field, the charge on the rod is displaced until the total tangential electric intensity becomes equal to zero, that is, until the surface of the rod becomes an equipotential surface. If the impressed field is uniform, the impressed potential is

$$V^i = -Ez + \text{const.} \tag{17}$$

If V^r is the potential due to the displaced charge, then

$$V^r = Ez. \tag{18}$$

If $C(z)$ is the capacitance per unit length between two elements of the rod at distance z on opposite sides of the center, the charge on the upper element must be

$$dq = 2V^r\, C(z)\, dz = 2Ez\, C(z)\, dz, \tag{19}$$

and that on the lower element must be $-dq$. The moment of this doublet is the product of dq, and the distance between the charges,

$$dp_e = 2z\, dq = 4Ez^2\, C(z)\, dz. \tag{20}$$

Integrating we have,

$$p_e = 4E\int_0^l z^2\, C(z)\, dz. \tag{21}$$

Hence, the electric polarizability of a thin *isolated* rod of length $2l$ is

$$\chi_e^0 = 4\int_0^l z^2\, C(z)\, dz. \tag{22}$$

If the shape of the object is such that C is independent of z, then,

$$\chi_e^0 = \tfrac{4}{3}Cl^3. \tag{23}$$

In Chapter 10 we derived the following approximate expression for the capacitance per unit length,

$$C(z) = \frac{\pi\varepsilon}{\log[2z/a(z)]}, \tag{24}$$

where $a(z)$ is the radius at distance z from the center. In this expression the end effects are neglected. The evaluation of equation 22 is complicated even if the radius is constant. Noting that the integrand is small when z is small, we shall approximate equation 24 as follows. Let a_0 be some mean radius; then,

$$\begin{aligned} C(z) &= \frac{\pi\varepsilon}{\log(2l/a_0) + \log[za_0/l\, a(z)]} \\ &\simeq \frac{\pi\varepsilon}{\log(2l/a_0)}\left\{1 - \frac{\log[za_0/l\, a(z)]}{\log(2l/a_0)}\right\}. \end{aligned} \tag{25}$$

The error in this approximation is large when z is very small; but in the integrand of equation 22 it is multiplied by the square of z, and the product is small. The error becomes smaller as z increases. If we neglect the second term altogether, then, by equation 23,

$$\chi_e{}^0 = \frac{4\pi\varepsilon l^3}{3\log(2l/a_0)}. \tag{26}$$

If the second term is included, the integral 22 may be evaluated by parts. Thus, for a uniform rod we obtain

$$\chi_e{}^0 = \frac{4\pi\varepsilon l^3}{3\log(2l/a)}\left[1 + \frac{1}{3\log(2l/a)}\right] \simeq \frac{4\pi\varepsilon l^3}{3\log(2l/a) - 1}. \tag{27}$$

The end effect tends to increase $\chi_e{}^0$.

The magnetic polarizability of thin conducting rods is very small.

19.5 Electric polarizability of an isolated metal sphere

Let $E^i = E_0$ be a uniform electric field impressed on a conducting sphere of radius a (Fig. 19.5). The meridian component of this field is

$$E_\theta{}^i = -E_0 \sin\theta. \tag{28}$$

At the surface of the sphere, the reflected field due to the displaced charge must annihilate this component of the impressed field, and, hence, must vary as $\sin\theta$. The field that varies in this manner is the field (equation 4–67) of a doublet; hence,

$$E_\theta{}^r = \frac{p_e \sin\theta}{4\pi\varepsilon_0 r^3}, \tag{29}$$

where p_e is the electric moment of the doublet. The boundary condition gives

$$-E_0 + \frac{p_e}{4\pi\varepsilon_0 a^3} = 0, \quad p_e = 4\pi\varepsilon_0 a^3 E_0. \tag{30}$$

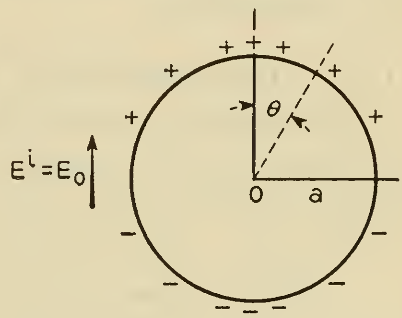

FIG. 19.5 A conducting sphere in an electric or magnetic field.

Hence, the electric polarizability of an isolated sphere is

$$\chi_e{}^0 = 4\pi\varepsilon_0 a^3. \tag{31}$$

19.6 Magnetic polarizability of an isolated metal sphere

Suppose now that the metal sphere (Fig. 19.5) is in a uniform magnetic field of intensity $H^i = H_0$. The normal component of the total magnetic field must vanish at the surface of the sphere. This means that circulating electric currents are induced in the sphere in such a direction that their magnetic field opposes the incident field in the region occupied

by the sphere. Hence, the induced magnetic moment is in the direction opposite to the impressed field, the polarizability is negative, and the effect of the metal spheres will be to make the effective permeability smaller than that of the surrounding medium.

To calculate the polarizability, we note that the radial component of the impressed field is

$$H_r{}^i = H_0 \cos \theta. \tag{32}$$

The field of a magnetic doublet is similar to the field (equation 4–67) of an electric doublet. We need only replace the electric moment p_e and the dielectric constant by the magnetic moment p_m and the permeability. Thus, the radial component of the reflected field is

$$H_r{}^r = \frac{p_m \cos \theta}{2\pi\mu_0 r^3}. \tag{33}$$

The total H_r must vanish at the surface $r = a$ of the sphere, and

$$H_0 + \frac{p_m}{2\pi\mu_0 a^3} = 0, \qquad p_m = -2\pi\mu_0 a^3 H_0. \tag{34}$$

Hence, the magnetic polarizability of the isolated metal sphere is

$$\chi_m{}^0 = -2\pi\mu_0 a^3. \tag{35}$$

Thus, the permeability is decreased due to the presence of spheres. As far as the index of refraction is concerned, half of the desirable effect due to the increased dielectric constant is annulled by the decrease in the permeability.

19.7 Polarizabilities of miscellaneous objects

The method used in the preceding sections to find the polarizabilities of a sphere may be applied to elliptic cylinders (Fig. 19.6), to prolate spheroids of revolution (Fig. 19.7), and to oblate spheroids of revolution (Fig. 19.8). The following are the results.

Elliptic cylinders (Fig. 19.6): In a field, E or H, parallel to the major axis $2a$, the polarizabilities *per unit length* are, respectively,

$$\bar{\chi}_e{}^0 = \pi\varepsilon_0 a(a + b), \qquad \bar{\chi}_m{}^0 = -\pi\mu_0 b(a + b), \tag{36}$$

where a and b are, respectively, the major and the minor semiaxes of of the elliptical cross section. In a field, E or H, parallel to the minor axis, we have, respectively,*

$$\bar{\chi}_e{}^0 = \pi\varepsilon_0 b(a + b), \qquad \bar{\chi}_m{}^0 = -\pi\mu_0 a(a + b). \tag{37}$$

* In a plane wave E and H are perpendicular, and the electric polarizability in equation 36 goes with the magnetic polarizability in equation 37; similarly, the electric polarizability in equation 37 goes with the magnetic polarizability in equation 36.

The dielectric constant and the permeability are then given by

$$\varepsilon = \varepsilon_0 + N\bar{\chi}_e{}^0, \qquad \mu = \mu_0 + N\bar{\chi}_m{}^0, \tag{38}$$

where N is the number of cylinders *per unit area* normal to their axes. For circular cylinders $b = a$, and for flat strips $b = 0$.

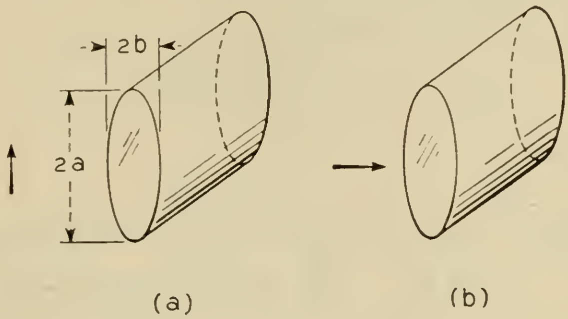

FIG. 19.6 Elliptic cylinders in an electric or magnetic field.

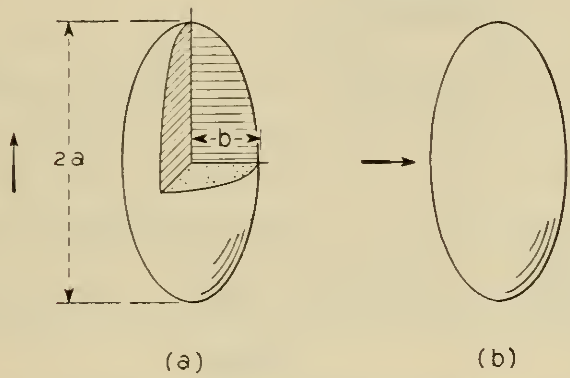

FIG. 19.7 Prolate spheroids of revolution in an electric or magnetic field.

Prolate spheroids (Fig. 19.7): In a field, E or H, parallel to the major axis, we have, respectively,

$$\chi_e{}^0 = \frac{4\pi\varepsilon_0 a l^2}{3\left(\dfrac{a}{l}\log\dfrac{a+l}{b} - 1\right)}$$

$$= \frac{4\pi\varepsilon_0 a(a^2 - b^2)}{3\left[\dfrac{a}{\sqrt{a^2-b^2}}\log\dfrac{a+\sqrt{a^2-b^2}}{b} - 1\right]},$$

$$\chi_m{}^0 = -\frac{4\pi\mu_0 b^2 l^3}{3\left(al - b^2\log\dfrac{a+l}{b}\right)}, \tag{39}$$

where a and b are the semimajor and semiminor axes, and l is the semifocal distance ($l^2 = a^2 - b^2$). If $b \ll a$,

$$\chi_e{}^0 = \frac{4\pi\varepsilon_0 a^3}{3\,[\log(2a/b) - 1]}, \qquad \chi_m{}^0 = -\frac{4\pi}{3}\mu_0 a b^2. \tag{40}$$

Similarly, in a field parallel to the minor axis,

$$\begin{aligned} \chi_e{}^0 &= \frac{8\pi\varepsilon_0 b^2 l^3}{3\left(al - b^2 \log \dfrac{a+l}{b}\right)}, \\ \chi_m{}^0 &= -\frac{8\pi\mu_0 a b^2}{3\left(1 - \dfrac{b^2}{l^2} + \dfrac{ab^2}{l^3}\log\dfrac{a+l}{b}\right)}. \end{aligned} \tag{41}$$

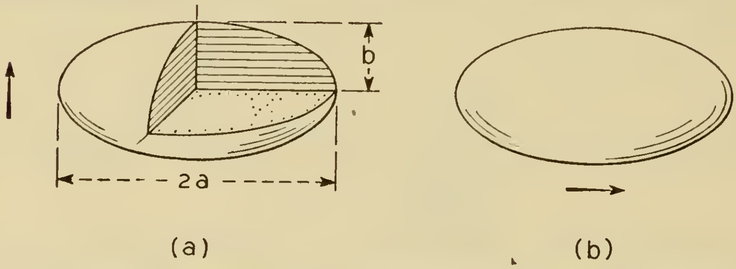

FIG. 19.8 Oblate spheroids of revolution in an electric or magnetic field.

Oblate spheroids (Fig. 19.8): In a field, E or H, parallel to the minor axis,

$$\begin{aligned} \chi_e{}^0 &= \frac{4\pi\varepsilon_0 b l^2}{3\left(1 - \dfrac{b}{l}\cot^{-1}\dfrac{b}{l}\right)}, \\ \chi_m{}^0 &= -\frac{4\pi\mu_0 l^3}{3\left(\cot^{-1}\dfrac{b}{l} - \dfrac{bl}{a^2}\right)}, \end{aligned} \tag{42}$$

where a, b, l are, respectively, the semimajor axis, the semiminor axis, and the semifocal distance. For a flat disk, $b = 0$, $l = a$, and

$$\chi_e{}^0 = 0, \qquad \chi_m{}^0 = -\tfrac{8}{3}\mu_0 a^3. \tag{43}$$

In a field parallel to the major axis,

$$\begin{aligned} \chi_e{}^0 &= \frac{8\pi\varepsilon_0 a^2 l^3}{3\,[a^2\cot^{-1}(b/l) - bl]}, \\ \chi_m{}^0 &= -\frac{8\pi\mu_0 a^2 b l^3}{3\,[l^3 + a^2 l - a^2 b\cot^{-1}(b/l)]}. \end{aligned} \tag{44}$$

In this case the polarizabilities for the flat disk are

$$\chi_e{}^0 = \tfrac{16}{3}\,\varepsilon_0 a^3, \qquad \chi_m{}^0 = 0. \tag{45}$$

19.8 Polarizing and depolarizing action of close neighbors

Near neighbors (Fig. 19.9) react on each other and thus change their polarizabilities. We assume that the polarizability of a metal object in any given environment is defined as the coefficient of proportionality in the equation

$$p_e = \chi_e E^i \tag{46}$$

for the moment induced by the primary field E^i. On the other hand, if we add to the primary field the field of the neighbors, the moment may be expressed as

$$p_e = \chi_e{}^0(E^i + E_1), \tag{47}$$

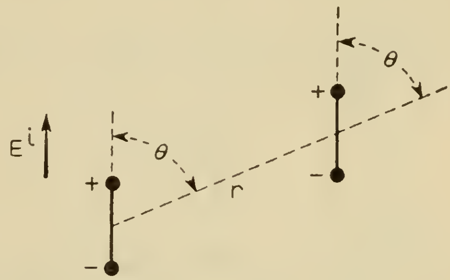

FIG. 19.9 Polarizing and depolarizing action of near neighbors depends on their relative positions.

where $\chi_e{}^0$ is the polarizability of the object when it is isolated and E_1 is the field of the neighbors. This is true because $\chi_e{}^0$ is the coefficient in the expression for the moment induced by the *total* impressed field.

In the case of two elements,

$$E_1 = E_r \cos\theta - E_\theta \sin\theta = \frac{p_e \cos^2\theta}{2\pi\varepsilon_0 r^3} - \frac{p_e \sin^2\theta}{4\pi\varepsilon_0 r^3}$$

or

$$E_1 = \frac{p_e(2\cos^2\theta - \sin^2\theta)}{4\pi\varepsilon_0 r^3}. \tag{48}$$

Substituting in equation 47, we find

$$p_e = \frac{\chi_e{}^0 E^i}{1 - (2\cos^2\theta - \sin^2\theta)(\chi_e{}^0/4\pi\varepsilon_0 r^3)}. \tag{49}$$

Hence, the polarizability as defined by equation 46 is

$$\chi_e = \frac{\chi_e{}^0}{1 - (2\cos^2\theta - \sin^2\theta)(\chi_e{}^0/4\pi\varepsilon_0 r^3)}. \tag{50}$$

The polarizability is diminished when $\theta = 90°$ and increased when $\theta = 0$.

In a regular lattice of elements, we should add the effects of all near neighbors; the value of χ_e so obtained should then be used in computing the total induced moment per unit volume: that is, the polarization (equation 10) of the artificial dielectric and the dielectric constant (equation 14).

Similar corrections for the influence of neighbors on the magnetic polarizability should be included if the distances between the objects are small.

19.9 Reflection by a lens

The coefficient of reflection at the surface of a lens will depend on the ratio of the intrinsic impedance of the artificial dielectric to the intrinsic impedance of free space. Loading of a dielectric with metal spheres, for instance, has the effect of decreasing the intrinsic impedance,

$$\eta = \left(\frac{\mu}{\varepsilon}\right)^{1/2} = \eta_0 \left(1 + \frac{N\chi_m}{\mu_0}\right)^{1/2} \left(1 + \frac{N\chi_e}{\varepsilon_0}\right)^{-1/2}, \tag{51}$$

by decreasing the permeability and increasing the dielectric constant. This is particularly undesirable since the effects of the changes in μ and ε on the index of refraction are opposite. What we need is a way of *increasing* the permeability rather than decreasing it.

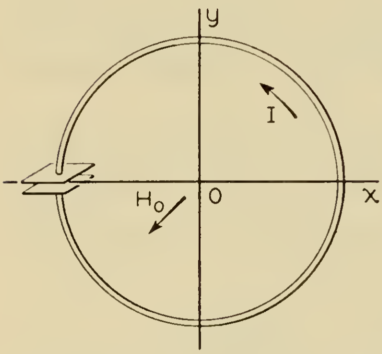

FIG. 19.10 A loop loaded with capacitance.

19.10 Methods for increasing the permeability of artificial dielectrics

Consider a loop with a capacitor (Fig. 19.10). Let the impressed magnetic intensity H_0 be in the positive z direction. The counterclockwise induced current is

$$I = \frac{-j\omega\mu_0 H_0 S}{j\omega L + (1/j\omega C)} = \frac{\omega^2 \mu_0 C S H_0}{1 - \omega^2 LC}, \tag{52}$$

where L is the inductance of the loop and C the capacitance in series with it. The moment of the magnetic doublet equivalent to the loop is

$$p_m = \mu_0 I S. \tag{53}$$

Hence, the magnetic polarizability is

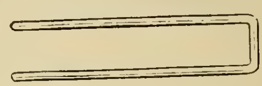

FIG. 19.11 A loop approaching resonance.

$$\chi_m{}^0 = \frac{\omega^2 \mu_0{}^2 C S^2}{1 - \omega^2 LC} = \frac{\omega^2 \mu_0 \varepsilon_0 (C/\varepsilon_0) S^2}{1 - \omega^2 \mu_0 \varepsilon_0 (LC/\mu_0 \varepsilon_0)} \mu_0. \tag{54}$$

The ratios C/ε_0 and L/μ_0 depend only on the geometry of the metal object; $\omega^2 \mu_0 \varepsilon_0 = 4\pi^2/\lambda^2$ where λ is the wavelength in free space corresponding to the given frequency.

The capacitance may be supplied by the loop itself (Fig. 19.11) if

it is made large enough to approach resonance. Unfortunately the polarizability of open loops depends on the frequency.

19.11 Artificial dielectrics with large dielectric constants

Figure 19.12a shows an artificial dielectric with a large effective dielectric constant. To calculate this constant, we may replace the dielectric

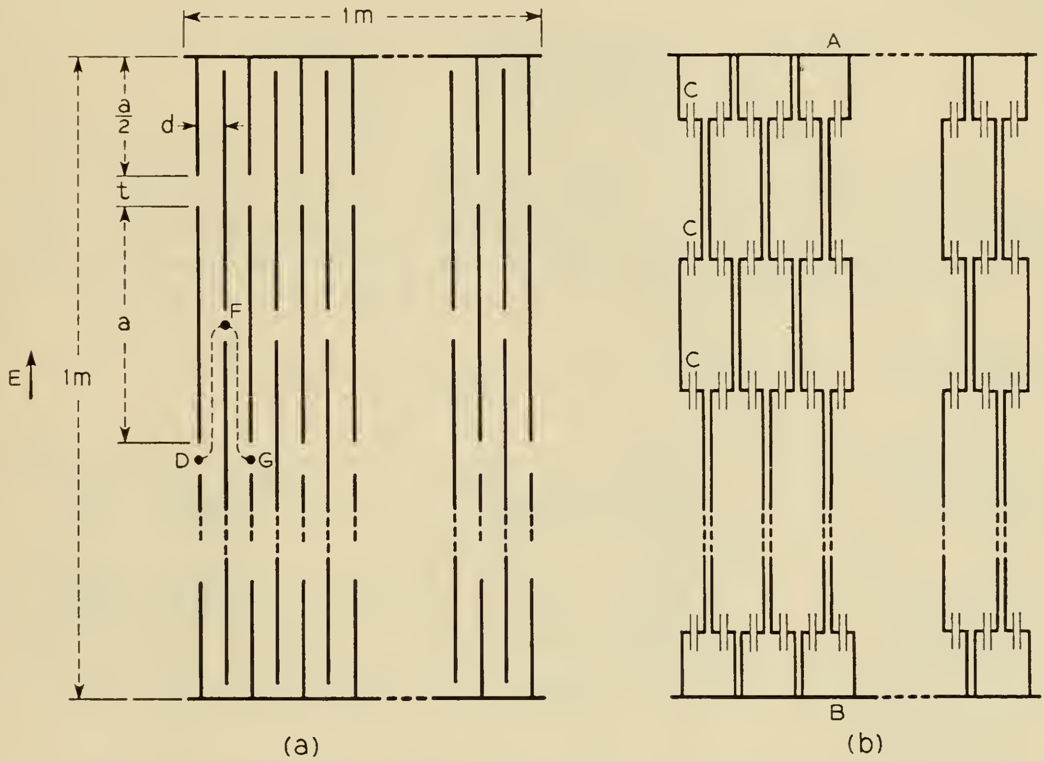

Fig. 19.12 Artificial dielectric with a large dielectric constant: (a) cross section of strip loaded materials; (b) its equivalent circuit with $2/(a+t)$ horizontal rows and $1/d$ vertical rows.

by an equivalent circuit (Fig. 19.12b). The capacitance of each capacitor is

$$C = \varepsilon_0 \frac{a-t}{2d} \cdot \tag{55}$$

The capacitance between the plates A and B one meter apart, per unit length in the direction of the wave, is

$$C_{AB} = C\frac{a+t}{2}\frac{1}{d} = \varepsilon_0 \frac{a^2-t^2}{4d^2} \simeq \varepsilon_0 \left(\frac{a}{2d}\right)^2 . \tag{56}$$

Without loading,

$$C_{AB} = \varepsilon_0; \tag{57}$$

hence, the relative dielectric constant of the loaded material is

$$\varepsilon_r \simeq \left(\frac{a}{2d}\right)^2 . \tag{58}$$

We can obtain this value more directly if we note that the strips force the waves to travel along the path DFG instead of the path DG; therefore,

$$\varepsilon_r = \left(\frac{\text{path } DFG}{\text{path } DG}\right)^2 \simeq \left(\frac{a}{2d}\right)^2 . \tag{59}$$

Figure 19.13a shows a second method of loading a dielectric with parallel strips. In view of the symmetry of the structure, we can insert conducting partitions and subdivide the medium into sections (Fig.

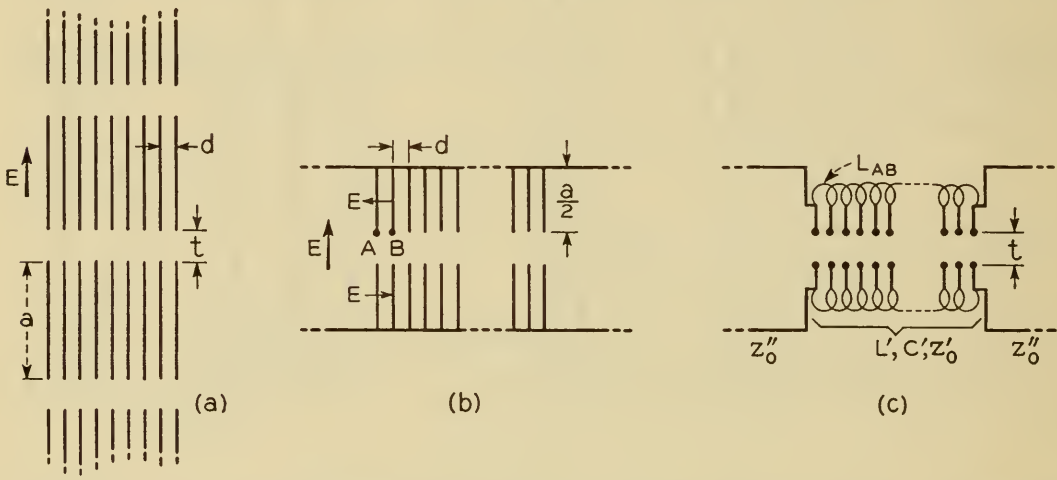

FIG. 19.13 (a) Artificial dielectric made with parallel conducting strips, (b) a section of it, and (c) the equivalent transmission line.

19.13b). The impedance looking upward from A, B equals that of a strip transmission line short-circuited at the far end; thus, we have a strip transmission line loaded with short-circuited strip transmission lines (Fig. 19.13c). The loading reactance per section is

$$X_{AB} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{d}{b}\tan \tfrac{1}{2}\omega\sqrt{\mu_0\varepsilon_0}\,a, \tag{60}$$

where b is the length of the strips. The series reactance of the loaded line (Fig. 19.13c) per unit length is

$$X' = 2\,\frac{1}{d}\,X_{AB} + \omega\mu_0\frac{t}{b} . \tag{61}$$

The capacitance per unit length is

$$C' = k\varepsilon_0 \frac{b}{t}, \tag{62}$$

where k is the "insufficient packing factor," which is less than unity and approaches unity as t/d increases. From equations 61 and 62, we find

$$Z_0' = \sqrt{\frac{X'}{\omega C'}}, \qquad v' = \frac{1}{\sqrt{L'C'}} = \frac{\omega}{\sqrt{X'\omega C'}}. \tag{63}$$

The effective dielectric constant is then obtained from

$$\varepsilon_r = \left(\frac{c}{v'}\right)^2. \tag{64}$$

If a is sufficiently small, $\tan x \simeq x$ and equation 60 becomes

$$X_{AB} = \omega L_{AB}, \qquad L_{AB} = \frac{\mu_0 ad}{2b}. \tag{65}$$

In this case,

$$\varepsilon_r \simeq k\,\frac{a+t}{t}, \tag{66}$$

and the standing wave ratio is

$$\text{SWR} = 10 \log_{10}\left(\frac{Z_0}{Z_0'}\right)^2 = 10 \log_{10} \frac{k(a+t)}{t} = 10 \log_{10} \varepsilon_r. \tag{67}$$

If t is large compared with d, the value of k is given by the formula

$$k = \frac{\pi t}{2d \cosh^{-1}(\exp \pi t/2d)}. \tag{68}$$

As $\pi t/2d$ increases, this becomes approximately

$$k \simeq \frac{1}{1 + 0.44d/t}. \tag{69}$$

From equation 64, using the formulas for v', X' and C', we find that ε_r is given by

$$\varepsilon_r = k\left[\frac{\lambda}{\pi t}\tan\frac{\pi a}{\lambda} + 1\right], \tag{70}$$

where k is defined by equation 68.

Even when t is not much larger than d, this formula gives values of ε_r which are in substantial agreement with measured values. Thus, for $a = \frac{1}{2}$ in., $t = \frac{1}{8}$ in., and $d = \frac{3}{32}$ in., the calculated value of ε_r

for $\lambda = 7$ cm is 4.14, whereas the value measured by W. M. Sharpless was 4.32. When d is increased to $\frac{3}{8}$ in., so that d is actually larger

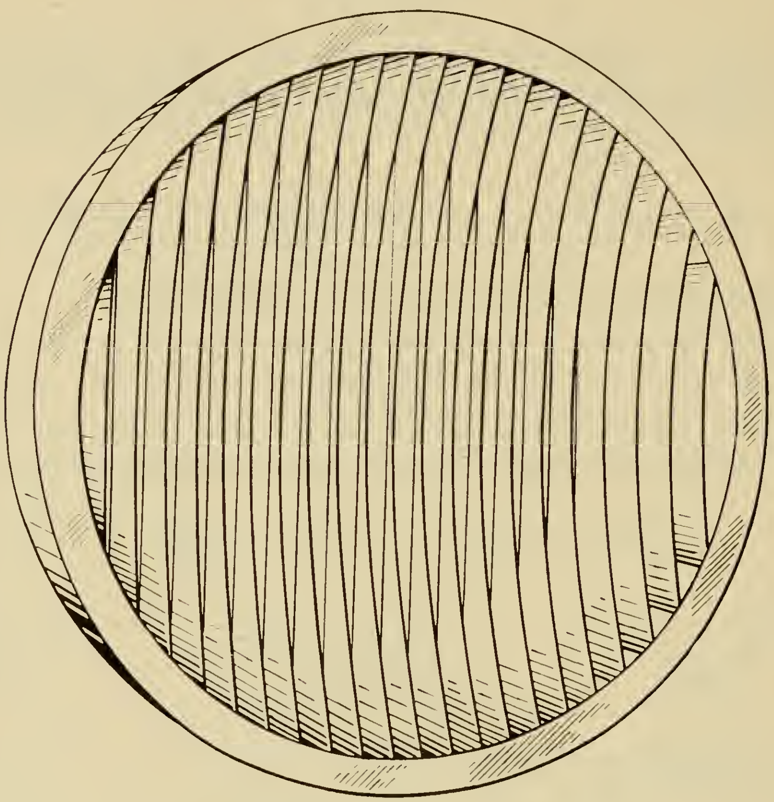

FIG. 19.14 A waveguide lens.

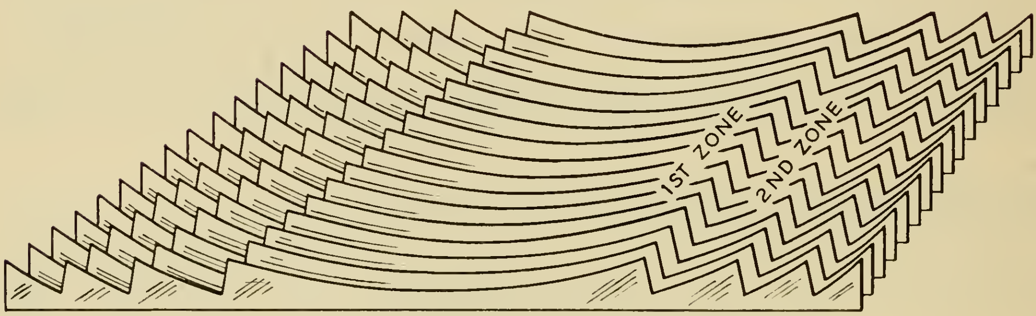

FIG. 19.15 A waveguide lens with "steps."

than t, use of equation 70 still does not lead to excessive errors, as shown by the following comparison with experimental data:*

$$\lambda = 7 \text{ cm}, \qquad \varepsilon_r(\text{calc}) = 2.58, \qquad \varepsilon_r(\text{meas}) = 2.45,$$

$$\lambda = 3.18 \text{ cm}, \qquad \varepsilon_r(\text{calc}) = 5.05, \qquad \varepsilon_r(\text{meas}) = 4.5.$$

* The measurements were actually made with closely spaced squares instead of with strips. Strips would have given slightly higher values for the measured dielectric constant.

In another set of measurements the dimensions of the structure were $a = 3/8$ in., $t = 1/4$ in., and $d = 3/16$ in. In this case, the following values were obtained:

$$\lambda = 7 \text{ cm}, \qquad \varepsilon_r(\text{calc}) = 1.95, \qquad (\varepsilon_r\text{meas}) = 1.68,$$

$$\lambda = 3.18 \text{ cm}, \qquad \varepsilon_r(\text{calc}) = 2.40, \qquad \varepsilon_r(\text{meas}) = 2.10.$$

Thus, our approximate analysis is fairly satisfactory, even when the effective dielectric constants are not large.

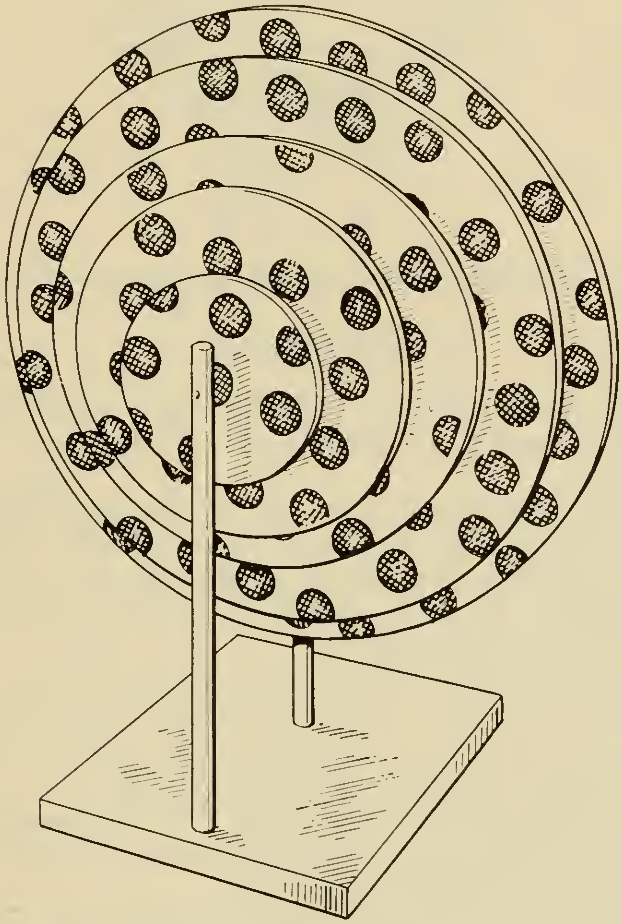

Fig. 19.16 An experimental lens of foam loaded with flat disks.

19.12 Representative lenses

Many types of metallic lenses have been developed by W. E. Kock, to whose papers* the reader is referred for more complete information. Several representative lenses are shown in Figs. 19.14–19.17.

* Metal-lens antennas, *IRE Proc.*, **34**, November 1946, pp. 828–836; Metallic delay lenses, *Bell Sys. Tech. Jour.*, **27**, January 1948, pp. 58–82.

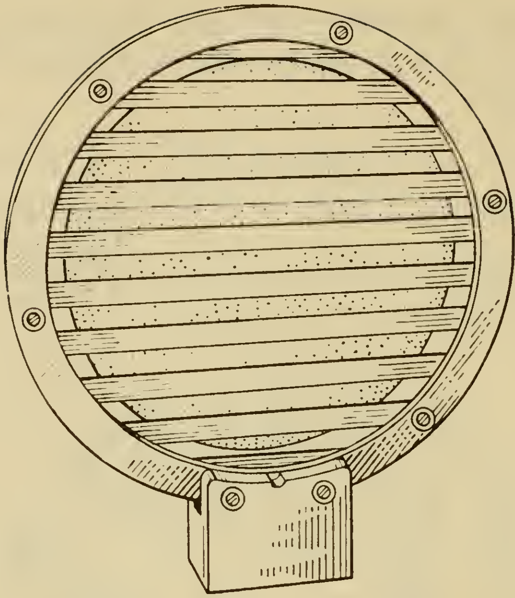

FIG. 19.17 An experimental lens of polystyrene sheets loaded with flat strips.

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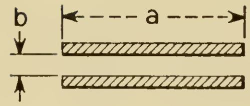
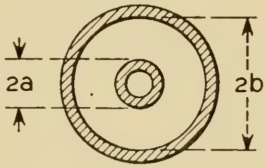
APPENDIX I

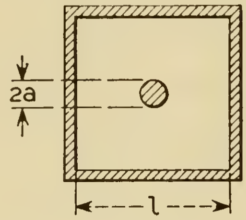
Characteristic Impedances of Transmission Lines



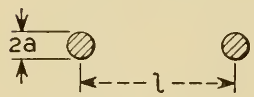
If $b \ll a$, $\qquad K = 120\pi \dfrac{b}{a}\cdot$



$$K = 60 \log \frac{b}{a} = 138 \log_{10} \frac{b}{a}\cdot$$



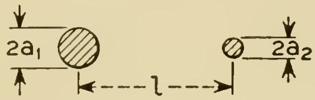
$$K = 60 \log \frac{1.078l}{2a} = 138 \log_{10} \frac{1.078l}{2a}\cdot$$



$$K = 120 \cosh^{-1} \frac{l}{2a}$$

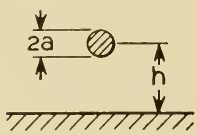
$$= 120 \log \left[\frac{l}{2a} + \sqrt{\left(\frac{l}{2a}\right)^2 - 1}\,\right]\cdot$$

If $2a \ll l$, $\quad K = 120 \log \dfrac{l}{a} = 276 \log_{10} \dfrac{l}{a}\cdot$



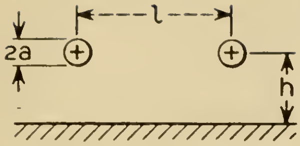
If $2a_1 \ll l$, and $2a_2 \ll l$,

$$K = 120 \log \frac{l}{\sqrt{a_1 a_2}} = 276 \log_{10} \frac{l}{\sqrt{a_1 a_2}}\cdot$$



$$K = 60 \cosh^{-1} \frac{h}{a}\cdot$$

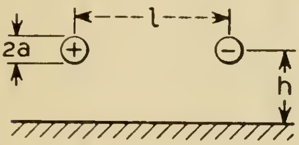
If $a \ll h$, $\quad K = 60 \log \dfrac{2h}{a} = 138 \log_{10} \dfrac{2h}{a}\cdot$



$$K = 30 \log \frac{2h}{a}\left[1 + \left(\frac{2h}{l}\right)^2\right]^{1/2}$$

$$= 69 \log_{10} \frac{2h}{a}\left[1 + \left(\frac{2h}{l}\right)^2\right]^{1/2}.$$

($2a$ is small.)

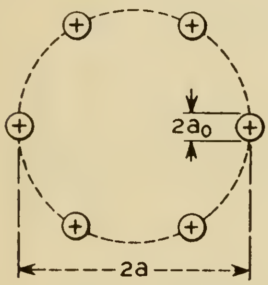


$$K = 120 \log \frac{l}{a}\left[1 + \left(\frac{l}{2h}\right)^2\right]^{-1/2}$$

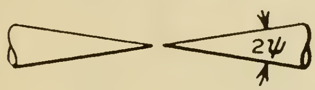
$$= 276 \log_{10} \frac{l}{a}\left[1 + \left(\frac{l}{2h}\right)^2\right]^{-1/2}$$

$$= 120 \log \frac{2h}{a}\left[1 + \left(\frac{2h}{l}\right)^2\right]^{-1/2}$$

$$= 276 \log_{10} \frac{2h}{a}\left[1 + \left(\frac{2h}{l}\right)^2\right]^{-1/2}.$$

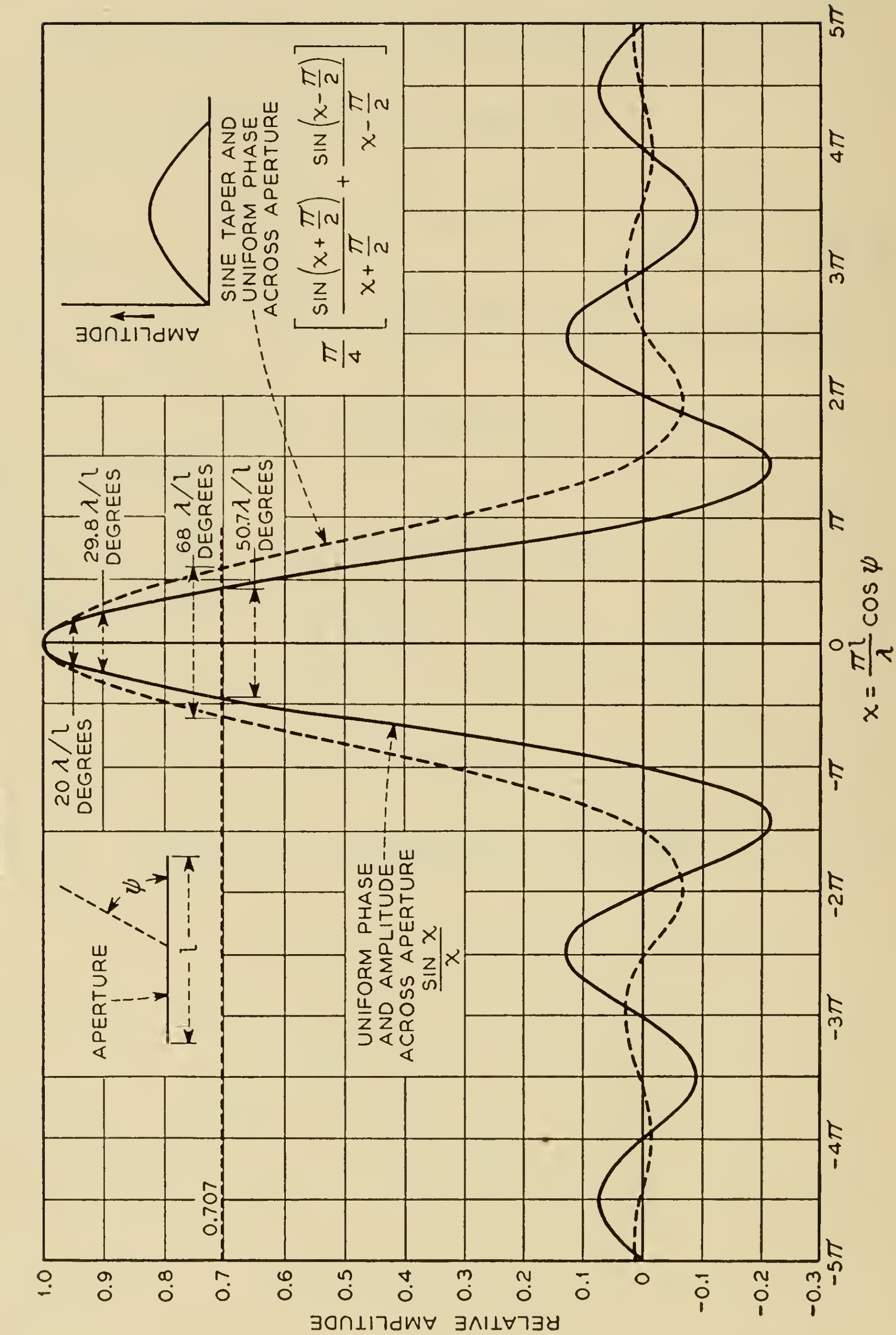


The radius of a conductor equivalent to a cage of n wires, $a_{\text{eff}} = a\left(\dfrac{na_0}{a}\right)^{1/n}$.



$$K = 120 \log \cot \tfrac{1}{2}\psi = 276 \log_{10} \cot \tfrac{1}{2}\psi.$$

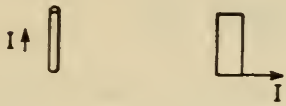
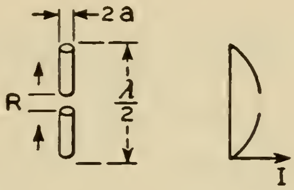
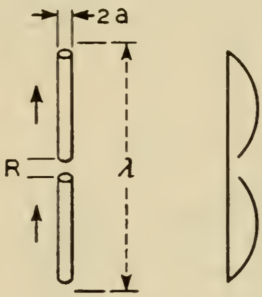
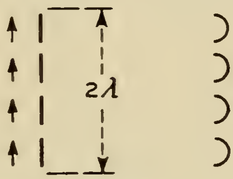
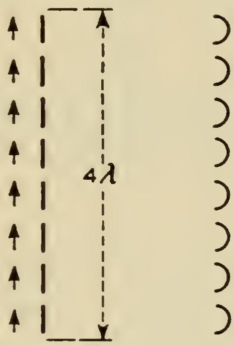
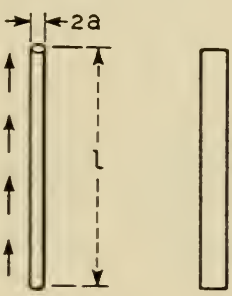
APPENDIX II



PATTERNS OF CONTINUOUS ARRAYS

APPENDIX III

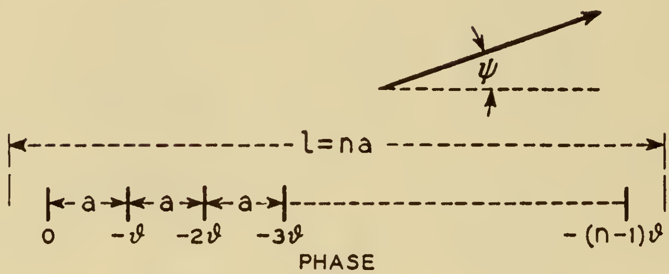
Radiation Resistance and Gain of Cylindrical Antennas

Current element ($l \ll \lambda$)		$G = 1.761$ db $R = 80\pi^2 \left(\frac{l}{\lambda}\right)^2$ ohms	$A \approx \frac{\lambda}{2} \times \frac{\lambda}{4}$
Half wave		$G = 2.151$ db $R \to 73.13$ ohms as $\frac{a}{\lambda} \to 0$	$A \approx \frac{\lambda}{2} \times \frac{\lambda}{4}$
Full wave		$G = 3.82$ db $R = \dfrac{\left(276 \log_{10} \frac{\lambda}{2a} - 110\right)^2}{199}$ ohms	$A = \lambda \times \frac{\lambda}{5.2}$
Two wave		$G = 6.41$ db	$A = 2\lambda \times \frac{\lambda}{5.7}$
Four wave		$G = 9.23$ db	$A = 4\lambda \times \frac{\lambda}{6.0}$
Long wire ($a < \lambda/4$) or radiating sources on long cylindrical surface (any diameter)		$G = 10 \log_{10} \frac{2l}{\lambda}$ $\theta(3 \text{ db}) = 50.7 \frac{l}{\lambda}$ degrees (width of major lobe)	$A = l \frac{\lambda}{2\pi}$

APPENDIX IV

Space Factors of Uniform Linear Arrays

General case



n current elements spaced distance a. $l = na$.

$$S = \frac{\sin\left[\frac{n}{2}\left(\frac{2\pi a}{\lambda}\cos\psi - \vartheta\right)\right]}{\sin\left[\frac{1}{2}\left(\frac{2\pi a}{\lambda}\cos\psi - \vartheta\right)\right]}.$$

Balanced couplet (Adcock antenna)

$$n = 2, \quad a \ll \lambda, \quad \vartheta = \pi: \qquad S = \frac{2\pi a}{\lambda}\cos\psi, \quad g = 3.75.$$

End-fire couplet

$$n = 2, \quad a = \frac{\lambda}{4}, \quad \vartheta = \frac{\pi}{2}: \qquad S = 2\cos\left[\frac{\pi}{4}\left(1 - \cos\psi\right)\right], \quad g = 3.$$

Broadside couplet (point-to-point)

$$n = 2, \quad \vartheta = 0: \qquad S = 2\cos\left(\frac{\pi a}{\lambda}\cos\psi\right). \qquad \text{For } a = \frac{\lambda}{2}, \quad g = 3.54.$$

Broadside array (point-to-point)

$$\vartheta = 0: \qquad S = \frac{\sin\left(\frac{n\pi a}{\lambda}\cos\psi\right)}{\sin\left(\frac{\pi a}{\lambda}\cos\psi\right)} = \frac{\sin\left(\frac{\pi l}{\lambda}\cos\psi\right)}{\sin\left(\frac{\pi l}{\lambda n}\cos\psi\right)}.$$

For $a \ll \lambda$ (continuous array), $S = n \dfrac{\sin\left(\dfrac{\pi l}{\lambda}\cos\psi\right)}{\dfrac{\pi l}{\lambda}\cos\psi}\cdot$ For $l \gg \lambda$, $g \to \dfrac{4l}{\lambda}$

End-fire array

A. ϑ = phase delay in plane wave traveling in direction of array: $\vartheta - \dfrac{2\pi a}{\lambda} = 0$:

$$S = \frac{\sin\left[\dfrac{\pi n a}{\lambda}(1-\cos\psi)\right]}{\sin\left[\dfrac{\pi a}{\lambda}(1-\cos\psi)\right]} = \frac{\sin\left[\dfrac{\pi l}{\lambda}(1-\cos\psi)\right]}{\sin\left[\dfrac{\pi l}{\lambda n}(1-\cos\psi)\right]}\cdot$$

For $a \ll \lambda$ (continuous array): $S = n \dfrac{\sin\left[\dfrac{\pi l}{\lambda}(1-\cos\psi)\right]}{\dfrac{\pi l}{\lambda}(1-\cos\psi)}\cdot$

For $l \gg \lambda$, $g \to \dfrac{4l}{\lambda}\cdot$

B. $\vartheta - \dfrac{2\pi a}{\lambda} \lessgtr 0$: $n\left[\vartheta - \dfrac{2\pi a}{\lambda}\right] = \vartheta'$:

$$S = \frac{\sin\left[\dfrac{\pi l}{\lambda}(1-\cos\psi) + \dfrac{\vartheta'}{2}\right]}{\sin\dfrac{1}{n}\left[\dfrac{\pi l}{\lambda}(1-\cos\psi) + \dfrac{\vartheta'}{2}\right]}\cdot$$

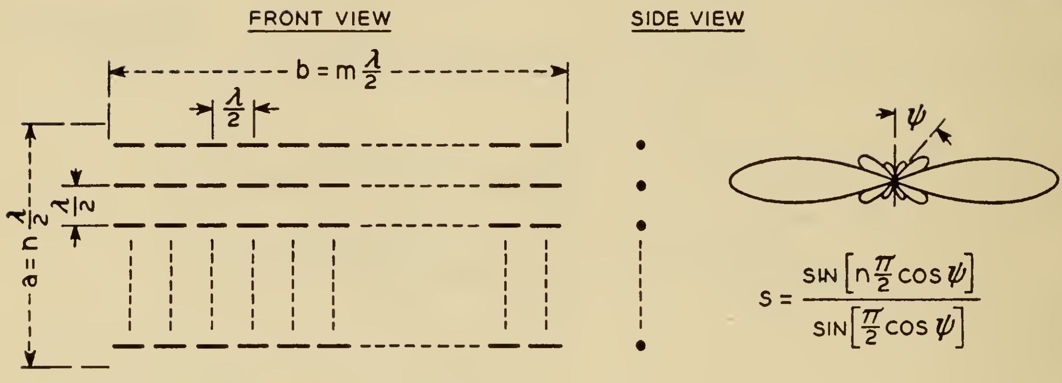
For $a \ll \lambda$ (continuous array, fishbone antenna) and $\vartheta' \ll n$

$$S = n\frac{\sin\left[\dfrac{\pi l}{\lambda}(1-\cos\psi) + \dfrac{\vartheta'}{2}\right]}{\dfrac{\pi l}{\lambda}(1-\cos\psi) + \dfrac{\vartheta'}{2}}\cdot$$

For $l \gg \lambda$, the gain is maximum when $\vartheta' = \pi$: $g_{\max} = 7.2\dfrac{l}{\lambda}\cdot$

APPENDIX V

Gain of Pine-Tree Antennas



Gain ***G*** (single curtain):

$n \backslash m$	1	2	3	4	5	6	7	8	9
1	2.15	3.82	5.33	6.41	7.31	8.05	8.68	9.23	9.71
2	5.97	8.05	9.71	10.83	11.77	12.54	13.17	13.75	14.24
3	7.85	9.77	11.33	12.44	13.35	14.07	14.73	15.27	15.77
4	9.20	11.24	12.88	14.03	14.95	15.67	16.33	16.88	17.41
5	10.25	12.22	13.80	14.90	15.83	16.57	17.17	17.76	18.25
6	11.07	13.10	14.72	15.85	16.77	17.53	18.18	18.75	19.22

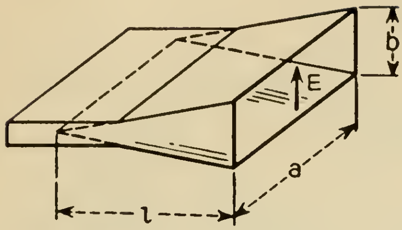
$n \backslash m$	10	11	12	13	14	15	16	17
1	10.16	10.55	10.93	11.26	11.58	11.87	12.15	12.40
2	14.67	15.08	15.47	15.79	16.12	16.41	16.67	16.95
3	16.20	16.61	16.97	17.33	17.65	17.92	18.20	18.46
4	17.85	18.26	18.63	18.97	19.27	19.58	19.85	20.12
5	18.70	19.09	19.47	19.80	20.13	20.35	20.70	20.95
6	19.67	20.10	20.45	20.77	21.10	21.39	21.68	21.92

For large antennas, $$A \approx \tfrac{1}{2}ab.$$

A reflecting curtain doubles A (3 db increase in gain); i.e. the effective area of a large pine-tree antenna with reflector is approximately equal to its actual area.

APPENDIX VI

Optimum horns | **Width of major lobes**



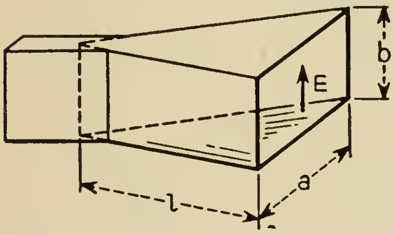
$$b \approx \sqrt{2l\lambda},$$
$$A \approx 0.65\,ab.$$

E plane:

$$\phi(3\text{ db}) \approx 53\frac{\lambda}{b}\text{ degrees,}$$

H plane:

$$\phi(3\text{ db}) = 68\frac{\lambda}{a}\text{ degrees.}$$



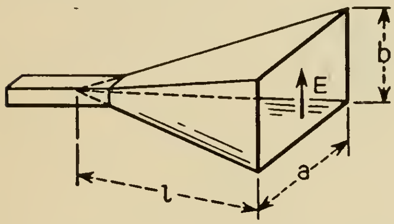
$$a \approx \sqrt{3l\lambda},$$
$$A \approx 0.63ab.$$

E plane:

$$\phi(3\text{ db}) = 51\frac{\lambda}{b}\text{ degrees,}$$

H plane:

$$\phi(3\text{ db}) \approx 80\frac{\lambda}{a}\text{ degrees.}$$



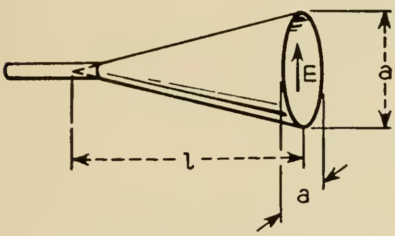
$$a \approx \sqrt{3l\lambda},$$
$$b = 0.81a,$$
$$A \approx 0.50ab,$$
$$\left(g \approx 15.3\frac{l}{\lambda}\right).$$

E plane:

$$\phi(3\text{ db}) \approx 53\frac{\lambda}{b}\text{ degrees,}$$

H plane:

$$\phi(3\text{ db}) \approx 80\frac{\lambda}{a}\text{ degrees.}$$



$$a \approx \sqrt{2.8l\lambda},$$
$$A \approx 0.52\frac{\pi}{4}a^2.$$

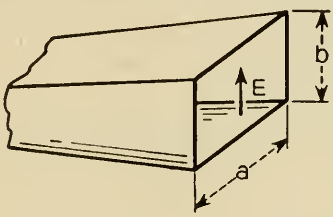
E plane:

$$\phi(3\text{ db}) \approx 60\frac{\lambda}{a}\text{ degrees,}$$

H plane:

$$\phi(3\text{ db}) \approx 70\frac{\lambda}{a}\text{ degrees.}$$

Long horns ($l \gg a$ or b)
(short horn with lens)



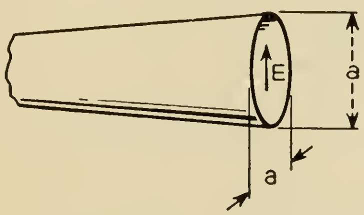
$$A = \frac{8}{\pi^2}ab$$
$$= 0.81ab.$$

E plane:

$$\phi(3\text{ db}) = 51\frac{\lambda}{b}\text{ degrees,}$$

H plane:

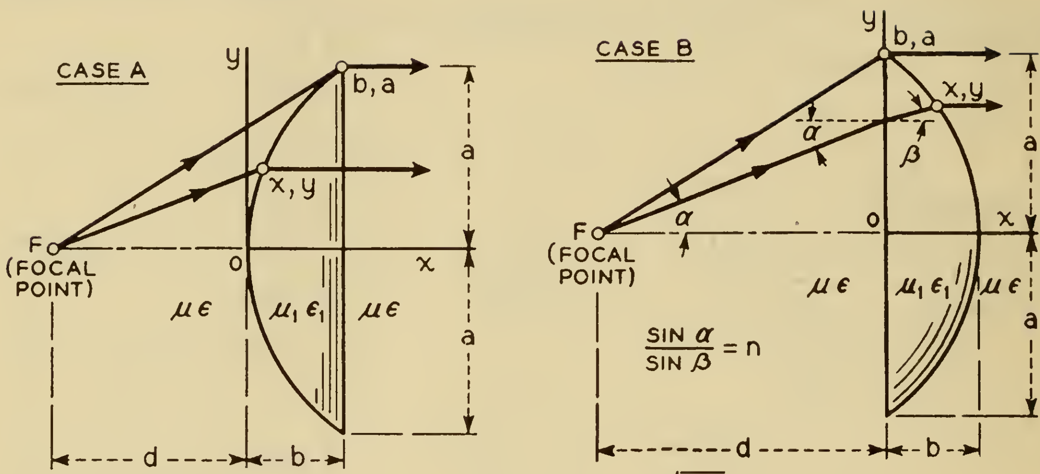
$$\phi(3\text{ db}) = 68\frac{\lambda}{a}\text{ degrees.}$$



$$A = \frac{2}{1.841^2 - 1}\,\frac{\pi}{4}a^2$$
$$= 0.84\frac{\pi}{4}a^2.$$

APPENDIX VII

Planoconvex lenses



Index of refraction $n = \sqrt{\dfrac{\mu_1\varepsilon_1}{\mu\varepsilon}}$

Case A

Shape of lens is hyperboloidal:

$$\left[\frac{(n+1)x}{d} + 1\right]^2 - \frac{n+1}{n-1}\,\frac{y^2}{d^2} = 1.$$

Radius of curvature at O is $R = d(n-1)$.

From equation of shape:

$$d + b = \frac{a^2 - (n-1)^2 b^2}{2(n-1)b},$$

$$d = \frac{a^2 - (n^2-1)b^2}{2(n-1)b}.$$

As

$$\frac{(n^2-1)b^2}{a^2} \to 0, \qquad d \to \frac{a^2}{2(n-1)b},$$

and point (b, a) approaches sphere of radius $R = d(n-1)$.

Case B

Shape of lens is given by:

$$x = d\,\frac{\left[1 + \left(\dfrac{a}{d}\right)^2\right]^{1/2} - \dfrac{1}{\cos\alpha}}{n\left(1 - \dfrac{\sin^2\alpha}{n^2}\right)^{-1/2} - 1}.$$

$$y = d\tan\alpha + x(n^2 - \sin^2\alpha)^{-1/2}\sin\alpha.$$

Hence,

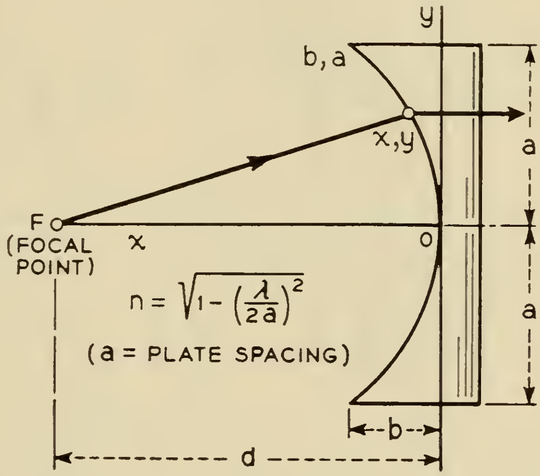
$$d = \frac{a^2 - (n-1)^2 b^2}{2(n-1)b}$$

(identical to equation for $d + b$ in case A). As

$$\frac{(n-1)^2 b^2}{a^2} \to 0, \qquad d \to \frac{a^2}{2(n-1)b},$$

and point (b, a) approaches sphere of radius $R = d(n-1)$.

Waveguide lenses



Shape of lens is ellipsoidal:

$$\left[\frac{1+n}{d}x - 1\right]^2 + \frac{1+n}{1-n}\frac{y^2}{d^2} = 1.$$

Radius of curvature at O is $R = d(1-n)$.
From equation of shape:

$$d = \frac{a^2 + (1-n^2)b^2}{2(1-n)b}.$$

As

$$\frac{(1-n^2)b^2}{a^2} \to 0, \qquad d \to \frac{a^2}{2(1-n)b},$$

and point (b, a) approaches sphere of radius $R = d(1-n)$.

APPENDIX VIII

Mutual Impedance of Parallel Antennas as Seen at the Input Terminals

$$Z_{12} = \frac{Z_{12}^a}{I_1 I_2} = \frac{R_{12}^a + j X_{12}^a}{I_1 I_2},$$

$$I_1 = \left[1 + \frac{M(l_1)}{K_a^{(1)}}\right] \sin \beta l_1 + \frac{N(l_1) - j Z_a(l_1)}{K_a^{(1)}} \cos \beta l_1; \quad I_2 = \left[1 + \frac{M(l_2)}{K_a^{(2)}}\right] \sin \beta l_2 + \frac{N(l_2) - j Z_a(l_2)}{K_a^{(2)}} \cos \beta l_2;$$

$$\begin{aligned}
R_{12}^a = {} & 30 \cos(L_1 + L_2)[2 \,\mathrm{Ci}\, \beta\rho - \mathrm{Ci}(R_{04} + L_2) - \mathrm{Ci}(R_{04} - L_2) + \mathrm{Ci}(R_{14} - L_1 - L_2) + \mathrm{Ci}(R_{14} + L_1 + L_2) - \mathrm{Ci}(R_{02}' - L_1) - \\
& \mathrm{Ci}(R_{02}' + L_1)] + 30 \cos(L_2 - L_1)[2 \,\mathrm{Ci}\, \beta\rho - \mathrm{Ci}(R_{04} + L_2) - \mathrm{Ci}(R_{04} - L_2) + \mathrm{Ci}(R_{24} - L_2 + L_1) + \mathrm{Ci}(R_{24} + L_2 - L_1) - \\
& \mathrm{Ci}(R_{02}' - L_1) - \mathrm{Ci}(R_{02}' + L_1)] + 30 \sin(L_1 + L_2)[\mathrm{Si}(R_{04} - L_2) - \mathrm{Si}(R_{04} + L_2) + \mathrm{Si}(R_{14} + L_1 + L_2) - \\
& \mathrm{Si}(R_{14} - L_1 - L_2) + \mathrm{Si}(R_{02}' - L_1) - \mathrm{Si}(R_{02}' + L_1)] + 30 \sin(L_2 - L_1)[\mathrm{Si}(R_{04} - L_2) - \mathrm{Si}(R_{04} + L_2) + \\
& \mathrm{Si}(R_{24} + L_2 - L_1) - \mathrm{Si}(R_{24} + L_1 - L_2) + \mathrm{Si}(R_{02}' + L_1) - \mathrm{Si}(R_{02}' - L_1)],
\end{aligned}$$

$$\begin{aligned}
X_{12}^a = {} & 30 \cos(L_1 + L_2)[\mathrm{Si}(R_{04} - L_2) + \mathrm{Si}(R_{04} + L_2) - 2 \,\mathrm{Si}\, \beta\rho - \mathrm{Si}(R_{14} - L_1 - L_2) - \mathrm{Si}(R_{14} + L_1 + L_2) + \mathrm{Si}(R_{02}' + L_1) + \\
& \mathrm{Si}(R_{02}' - L_1)] + 30 \cos(L_2 - L_1) [\mathrm{Si}(R_{04} - L_2) + \mathrm{Si}(R_{04} + L_2) - 2 \,\mathrm{Si}\, \beta\rho - \mathrm{Si}(R_{24} - L_2 + L_1) - \mathrm{Si}(R_{24} + L_2 - L_1) + \\
& \mathrm{Si}(R_{02}' + L_1) + \mathrm{Si}(R_{02}' - L_1)] + 30 \sin(L_1 + L_2)[\mathrm{Ci}(R_{04} - L_2) - \mathrm{Ci}(R_{04} + L_2) - \mathrm{Ci}(R_{14} - L_1 - L_2) + \\
& \mathrm{Ci}(R_{14} + L_1 + L_2) + \mathrm{Ci}(R_{02}' - L_1) - \mathrm{Ci}(R_{02}' + L_1)] + 30 \sin(L_2 - L_1)[\mathrm{Ci}(R_{04} - L_2) - \mathrm{Ci}(R_{04} + L_2) - \\
& \mathrm{Ci}(R_{24} - L_2 + L_1) - \mathrm{Ci}(R_{24} + L_2 - L_1) + \mathrm{Ci}(R_{02}' + L_1) - \mathrm{Ci}(R_{02}' - L_1)],
\end{aligned}$$

$$L_{1,2} = \beta l_{1,2}, \quad R_{04} = \beta\sqrt{\rho^2 + l_2^2}, \quad R_{14} = \beta\sqrt{\rho^2 + (l_1 + l_2)^2}, \quad R_{24} = \beta\sqrt{\rho^2 + (l_2 - l_1)^2}, \quad R_{02}' = \beta\sqrt{\rho^2 + l_1^2}.$$

APPENDIX IX

Constants of a Medium

Primary constants:

g = conductivity,

ε = dielectric constant, $\quad \varepsilon_r$ = dielectric constant relative to free space,

μ = permeability, $\quad \mu_r$ = permeability relative to free space.

$$Q = \frac{1}{\text{power factor}} = \frac{\omega\varepsilon}{g} = \frac{\text{displacement current density}}{\text{conduction current density}}.$$

Secondary constants:

$$\sigma = [j\omega\mu(g + j\omega\varepsilon)]^{1/2} = \alpha + j\beta = \text{intrinsic propagation constant.}$$

$$\eta = [j\omega\mu/(g + j\omega\varepsilon)]^{1/2} = \mathcal{R} + j\mathfrak{X} = \text{intrinsic impedance.}$$

$$\alpha = [\tfrac{1}{2}\omega\mu(\sqrt{g^2 + \omega^2\varepsilon^2} - \omega\varepsilon)]^{1/2} = \tfrac{1}{2}g\sqrt{\frac{\mu}{\varepsilon}}\left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{Q^2}}\right)\right]^{-1/2}$$

$$= \text{attenuation constant.}$$

$$\beta = [\tfrac{1}{2}\omega\mu(\sqrt{g^2 + \omega^2\varepsilon^2} + \omega\varepsilon)]^{1/2} = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{Q^2}}\right)\right]^{1/2}$$

$$= \text{phase constant.}$$

$$\mathcal{R} = \beta(g^2 + \omega^2\varepsilon^2)^{-1/2} = \frac{\beta}{\omega\varepsilon}\left(1 + \frac{1}{Q^2}\right)^{-1/2} = \text{intrinsic resistance.}$$

$$\mathfrak{X} = \alpha(g^2 + \omega^2\varepsilon^2)^{-1/2} = \frac{\alpha}{\omega\varepsilon}\left(1 + \frac{1}{Q^2}\right)^{-1/2} = \text{intrinsic reactance.}$$

$$v = \frac{\omega}{\beta} = \text{characteristic velocity.}$$

Free space:

$$g_v = 0, \qquad \varepsilon_v = \frac{10^{-9}}{36\pi}, \qquad \mu_v = 4\pi\cdot 10^{-7}, \qquad \eta_v = 120\pi,$$

$$\alpha_v = 0, \qquad \beta_v = \frac{\omega}{3.10^8}, \qquad v_v = 3.10^8.$$

Dielectrics ($Q > 10$):

$$\eta \simeq \frac{120\pi}{\sqrt{\varepsilon_r}}, \quad \alpha \simeq \frac{\omega\sqrt{\varepsilon_r}}{6Q}\cdot 10^{-8}, \quad \beta \simeq \frac{\omega\sqrt{\varepsilon_r}}{3}\cdot 10^{-8}, \quad v \simeq \frac{3.10^8}{\sqrt{\varepsilon_r}}.$$

Polystyrene, Polyethylene $\quad \varepsilon_r = 2.3, \quad Q = 3000.$

Hard rubber $\quad \varepsilon_r = 2.7, \quad Q = 200.$

Plywood $\quad \varepsilon_r = 1.8, \quad Q = 25.$

Conductors:

$$\sigma = (j\omega\mu g)^{1/2}, \qquad \eta = (j\omega\mu/g)^{1/2}, \qquad \alpha = \beta = (\tfrac{1}{2}\omega\mu g)^{1/2},$$

$$\mathcal{R} = \mathcal{X} = (\tfrac{1}{2}\omega\mu/g)^{1/2}. \qquad t = 1/\alpha = \text{skin depth}.$$

Copper:

$$g = 5.8\cdot 10^7, \quad \mu = 4\pi\cdot 10^{-7}, \quad \alpha = 15.1\sqrt{f}, \quad \mathcal{R} = 2.61\cdot 10^{-7}\sqrt{f}.$$

Other media:

Soil $\quad g = 0.001$ to $0.02, \quad \varepsilon_r = 10$ to $30.$

Sea water $\quad g = 5, \quad \varepsilon_r = 78.$

APPENDIX X

Summary of Maxwell's Equations for Propagation of Electromagnetic Waves

Maxwell's equations express the laws of interaction between electric and magnetic fields. They connect the spatial rates of propagation of these fields with the time rates. More specifically they express the propagation of two mutually perpendicular electric and magnetic field components in the direction perpendicular to their own directions. This propagation is affected by the field components in the direction of propagation.

If the z direction is chosen as the typical direction of propagation, then the equations of propagation of two pairs (E_x, H_y and H_x, E_y) of mutually perpendicular transverse field components are:

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y + \frac{\partial E_z}{\partial x}, \qquad \frac{\partial H_y}{\partial z} = -(g + j\omega\varepsilon)E_x + \frac{\partial H_z}{\partial y};$$

$$\frac{\partial H_x}{\partial z} = (g + j\omega\varepsilon)E_y + \frac{\partial H_z}{\partial x}, \qquad \frac{\partial E_y}{\partial z} = j\omega\mu H_x + \frac{\partial E_z}{\partial y}.$$

The components E_z, H_z in the direction of propagation depend on the transverse rates of propagation of the transverse field components:

$$(g + j\omega\varepsilon)E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \qquad j\omega\mu H_z = -\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}.$$

In spherical coordinates the equations of radial propagation of two pairs (E_θ, H_φ and H_θ, E_φ) of mutually perpendicular transverse components are:

$$\frac{\partial}{\partial r}(rE_\theta) = -j\omega\mu(rH_\varphi) + \frac{\partial E_r}{\partial \theta},$$

$$\frac{\partial}{\partial r}(rH_\varphi) = -(g + j\omega\varepsilon)(rE_\theta) + \frac{\partial H_r}{\sin\theta\,\partial\varphi};$$

$$\frac{\partial}{\partial r}(rH_\theta) = (g + j\omega\varepsilon)(rE_\theta) + \frac{\partial H_r}{\partial \theta},$$

$$\frac{\partial}{\partial r}(rE_\varphi) = j\omega\mu(rH_\theta) + \frac{\partial E_r}{\sin\theta\,\partial\varphi}.$$

The radial components depend on the transverse rates of propagation of the transverse components:

$$(g + j\omega\varepsilon)r^2E_r = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(rH_\varphi \sin\theta) - \frac{\partial}{\partial\varphi}(rH_\theta)\right],$$

$$j\omega\mu r^2H_r = \frac{1}{\sin\theta}\left[-\frac{\partial}{\partial\theta}(rE_\varphi \sin\theta) + \frac{\partial}{\partial\varphi}(rE_\theta)\right].$$

Note that the expressions on the left side of these equations represent the densities of radial electric and magnetic currents per unit solid angle.

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SUBJECT INDEX